Chapter 1

Utility Theory

1.1 Introduction

St. Petersburg Paradox (gambling paradox) the birth to the utility function http://policonomics.com/saint-petersburg-paradox/

The St. Petersburg paradox, is a theoretical game used in economics, to represent a classical example were, by taking into account only the expected value as the only decision criterion, the decision maker will be misguided into an irrational decision. This paradox was presented and solved in Daniel Bernoullis Commentarii Academiae Scientiarum Imperialis Petropolitanae (translated as Exposition of a new theory on the measurement of risk), 1738, hence its name, St. Petersburg. He solved it by making the distinction between expected value and expected utility, as the latter uses weighted utility multiplied by probabilities, instead of using weighted outcomes. However, since then, alternative approaches have been used by different researches to answer this paradox.

Suppose there is a fair game with 50/50 chance of winning/losing. The bet is \$1 and you can repeatedly bet at any amount. Hence the probability of winning the game is asymptotically 1. And hence the price to pay to enter the game is also infinity.

The key of the paradox is to determine the value someone would be willing to pay in order to play a lottery game that works as follows: a fair coin is tossed, if tail appears the player is paid \$2 (in case the amount paid to play is \$1), if not, the coin is tossed again, until tail appears, doubling the initial gain every time the coin is tossed. For example, for toss number 3 ($n = 3$), the payoff would be 8 (2^n) and the expected value, which here equals the payoff multiplied by the probability (here, 1).

The probability that the first tail appears in the toss number n is equal to $p_n = 1/2^n$, being 2^n the payoff. Therefore, the expected value for n tosses would be:

$$\sum\nolimits_{n=1}^{\infty} p_n 2^n = \sum\nolimits_{n=1}^{\infty} 1 = \infty \tag{1.1}$$

If we use the expected value as the decision criterion, the player should be willing to pay \$ in order to play. However, no rational individual would accept this.

1.2 Basic Properties/Axioms of Utility Function Behaviors of Human Being

- Prefer more to less – non decreasing utility function
- Satiability – the fact that any single want is satiable leads to the law of diminishing marginal utility
- Scarcity – increasing marginal cost of production

1.3 Risk Aversion

1.3.1 Basics

Jensens inequality

If $G(x)$ is concave in x, then:

$$\mathbb{E}[G(x)] < G(\mathbb{E}[x]) \tag{1.2}$$

The proof of the equality is really simple.

$$G(x) = G(\bar{x}) + G'(\bar{x})(x - \bar{x}) + \frac{1}{2}G''(x^*)(x - \bar{x})^2 \tag{1.3}$$

where $x^* \in [x, \bar{x}]$. As a result,

$$\begin{aligned}\mathbb{E}[G(x)] &= G(\bar{x}) + G'(\bar{x})(\mathbb{E}[x] - \bar{x}) + \frac{1}{2}G''(x^*)\mathbb{E}[(x-\bar{x})^2] \\ &= G(\mathbb{E}[x]) + \frac{1}{2}G''(x^*)\mathbb{E}[(x-\bar{x})^2] \\ &< G(\mathbb{E}[x])\end{aligned} \tag{1.4}$$

because $G''(x^*) < 0$ (slope $G'(x)$ decreasing) due to a concave function.

Risk Adverse Utility

For state-independent utility function of wealth (such as $W = \mathbb{E}[W] + \varepsilon$), the utility function is risk-averse if

$$\begin{aligned}U(\mathbb{E}[W]) &> \mathbb{E}[U(W)] \\ U(\mathbb{E}[W]) &> \mathbb{E}[U(\overline{W} + \varepsilon)]\end{aligned} \tag{1.5}$$

where

$$\mathbb{E}[\varepsilon] = 0$$

[Definition] An individual is risk-averse is defined iff his utility function of wealth is strictly concave at the relevant wealth levels.

□

Consequently, using Jensen's inequality, we have:

$$\begin{aligned}\mathbb{E}[U(\overline{W} + \varepsilon)] &< U(\mathbb{E}[\overline{W} + \varepsilon]) \\ &= U(\mathbb{E}[\overline{W}]) \\ \mathbb{E}[U(W)] &= U(\overline{W})\end{aligned} \tag{1.6}$$

[An Example] Consider a simple gamble,

$$\varepsilon = \begin{cases} \lambda a & 1-\lambda \\ -(1-\lambda)a & \lambda \end{cases} \tag{1.7}$$

Then,

$$\mathbb{E}[\varepsilon] = (1-\lambda)\lambda a - \lambda(1-\lambda)a = 0 \tag{1.8}$$

and

$$\begin{aligned}\mathbb{E}[\varepsilon^2] &= (1-\lambda)\lambda^2 a^2 + \lambda(1-\lambda)^2 a^2 \\ &= a^2\lambda(1-\lambda)(\lambda + (1-\lambda)) \\ &= a^2\lambda(1-\lambda)\end{aligned} \tag{1.9}$$

Let $W = \overline{W} + \varepsilon$ and as a result,

$$\mathbb{E}[\overline{W} + \varepsilon] = \overline{W} \tag{1.10}$$

Given that $\mathbb{E}[\overline{W} + \varepsilon] < U(\overline{W})$, we obtain $\overline{W} < U(\overline{W})$ for all (regardless of) a and λ.

Now, assume that there exists a W^* such that:

$$\mathbb{E}[U(\overline{W} + \varepsilon)] = U(W^*) \tag{1.11}$$

so that $W^* = \overline{W} - \pi_I$ represents a "certainty equivalent" wealth amount to $\overline{W} + \varepsilon$ for the individual that matches his expected utility and π_I is the monetary compensation of the uncertainty, or known as the **risk premium**.

Or alternatively,, we could define a π_C so that:

$$\mathbb{E}[U(\overline{W} + \varepsilon + \pi_C)] = U(\overline{W}) \tag{1.12}$$

where π_C represents the compensation to the individual for taking the gamble.

1.3.2 Pratt-Arrow Measure of RA

By Taylor's series expansion on the expected utility $\mathbb{E}[U(W)]$:

$$\begin{aligned}\mathbb{E}[U(\overline{W} + \varepsilon)] &= \mathbb{E}\left[U(\overline{W}) + U'(\overline{W})\varepsilon + \frac{1}{2}U''(\overline{W})\varepsilon^2 + o(\varepsilon)\right] \\ &\sim U(\overline{W}) + \frac{1}{2}U''(\overline{W})\mathbb{E}[\varepsilon^2]\end{aligned} \tag{1.13}$$

At the same time,

$$\begin{aligned}U(\overline{W} - \pi_I) &= U(\overline{W}) + U'(\overline{W})(-\pi_I) + o(\pi_I) \\ &\sim U(\overline{W}) + U'(\overline{W})(-\pi_I)\end{aligned} \tag{1.14}$$

Hence,

$$\begin{aligned} \frac{1}{2}U''(\overline{W})\sigma^2 &\sim U'(\overline{W})(-\pi_I) \\ \pi_I &= \left[-\frac{U''(\overline{W})}{U'(\overline{W})}\right]\frac{\sigma^2}{2} \end{aligned} \tag{1.15}$$

and

$$A(\overline{W}) = -\frac{U''(\overline{W})}{U'(\overline{W})} \tag{1.16}$$

is called the Pratt-Arrow measure of risk aversion. The higher is the risk aversion, the higher is the risk premium π_I. We can also have the alternative form:

$$-\frac{U''(\overline{W})}{U'(\overline{W})} = -\frac{d\ln U'(\overline{W})}{d\overline{W}} \tag{1.17}$$

[Theorem] If we double-integrate (see proof) the Pratt-Arrow risk aversion, we obtain a linear function in utility.

□

[Proof] Let

$$\begin{aligned} B &= -\int_{w\in\Omega} A(w)dw \\ &= \int_{w\in\Omega} \frac{d\ln U'(w)}{dw}dw \\ &= \ln U'(w) + \ln b \\ &= \ln\left[U'(w)b\right] \end{aligned} \tag{1.18}$$

Then,

$$\begin{aligned} \int e^B &= \int U'b \\ &= bU + a \end{aligned} \tag{1.19}$$

□

1.3.3 Absolute vs. Relative RA

1. $\frac{dA(w)}{dw} > 0$ means higher is wealth, higher is π_I (compensation for risk or risk premium)

2. $\frac{dA(w)}{dw} = 0$ means risk aversion has nothing to do with amount of wealth

3. $\frac{dA(w)}{dw} < 0$ means higher is wealth, lower π_I (lower the need for risk compensation)

Clearly, (3) is most unreasonable. This is because the marginal utility (of wealth) must be decreasing (axiom). Hence wealth has become less valuable as one becomes richer (has occupied more of it). (2) is also unreasonable because it implies a linear utility function. So only (1) is reasonable.

If the "absolute" risk aversion is diminishing with respect to wealth, then the "relative" risk aversion $R(W) = WA(W)$ is constant to wealth:

$$\begin{aligned} \frac{dR(w)}{dw} &= 0 = A(w) + w\frac{dA(w)}{dw} \\ \frac{dA(w)}{dw} &= -\frac{A(w)}{w} \end{aligned} \tag{1.20}$$

which is, we want the absolute risk aversion to be proportional to wealth.

1.3.4 Useful Utility Functions

In this section, we demonstrate some useful and popular utility functions.

Exponential $U(W) = -\frac{1}{a}e^{-aW}$

The absolute and relative risk aversion are:

$$\begin{aligned} U' &= e^{-aW} \\ U'' &= -ae^{-aW} \\ A &= -\frac{U''}{U'} = a \\ R &= aW \end{aligned} \tag{1.21}$$

This is constant and positive absolute risk aversion (situation 1) which is bad.

Quadratic $U(W) = a(W-b)^2$

The absolute and relative risk aversion are:

$$\begin{aligned} U' &= 2a(W-b) \\ U'' &= 2a \\ A &= -\frac{U''}{U'} = \frac{1}{b-W} \\ R &= \frac{W}{b-W} \end{aligned} \tag{1.22}$$

For A to be less than 0, W must be greater than b. So the easiest case is the set $b = 0$. That is:

$$\begin{aligned} A &= -\frac{1}{W} \\ \frac{dA}{dW} &= \frac{1}{W^2} > 0 \end{aligned}$$

which is not good (higher is wealth, higher is risk aversion).

The relative risk aversion is close to constant (under $b = 0$):

$$\begin{aligned} R(W) &= WA(W) = -1 \\ \frac{dR(W)}{dW} &= 0 \end{aligned}$$

so it does have the desirable relative risk aversion.

Log $U(W) = a + b \ln W$

The two risk aversions are:

$$\begin{aligned} U' &= \frac{b}{W} \\ U'' &= -\frac{b}{W^2} \\ A &= -\frac{U''}{U'} = \frac{1}{W} \\ R &= 1 \end{aligned} \tag{1.23}$$

This is opposite to quadratic utility function. Now both absolute and relative risk aversion is desirable. $dA(w)/dw < 0$ is now negative which is consistent with diminishing risk aversion.

HARA (hyperbolic absolute risk aversion)

This is a general utility function that can be made into any of the above special cases:

$$U(W) = \frac{1-r}{r}\left(\frac{aW}{1-r} + b\right)^r \tag{1.24}$$

The two risk aversions are:

$$\begin{aligned} U' &= a\left(\frac{aW}{1-r} + b\right)^{r-1} \\ U'' &= -a^2\left(\frac{aW}{1-r} + b\right)^{r-2} \\ A &= -\frac{U''}{U'} = a\left(\frac{aW}{1-r} + b\right)^{-1} \end{aligned} \tag{1.25}$$

For $r < 1$,

$$W\uparrow \quad \Rightarrow \quad A\downarrow$$

For $r > 1$,

$$W\uparrow \quad \Rightarrow \quad A\uparrow$$

The HARA class of utility functions can be made into the following special cases:

1. $r = 1$, which leads to $U = aW + b$ which is linear
2. $r = 2$, which leads to $U = -\frac{1}{2}(-aW + b)^2$ which is quadratic: $U = -e^{-aW}$
3. $r = \infty$ (and $b = 1$) which leads to exponential
4. $b = 0$ and $r < 1$ which leads to $U = \frac{W^r}{r}$ which is power
5. $b = 0$ and $r = 0$ which leads to $U = \ln W$ which is log

1.4 Exercises

1. Derive $dA(w)/dw$ for the HARA utility function.
2. Draw $R(w)$ (using W as x-axis) with $r = 0.5, a = 1, b = 1$.