VALUING BOND FUTURES AND THE QUALITY OPTION

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ABSTRACT

This paper develops a model for determining Treasury bond futures prices when the short position has a quality option. The model is developed in a general equilibrium economy where futures prices are driven by one or two factors. The main advantage of a factor based model over the exchange option based model is the ability to permit a realistic number of bonds in the deliverable set. The two factor quality option model is tested against two popular models which ignore the quality option, namely the CIR model (1981) and the cost of carry model using the cheapest to deliver bond.

1 INTRODUCTION

The Chicago Board of Trade's US Treasury Bond futures contract is one of the most actively traded securities in history. Given its volume, it is not surprising that there has been considerable interest in developing accurate pricing models for these contracts.

Historically, the futures price has been below that predicted by the traditional cost-of-carry model.[1] This deficiency has been attributed to several delivery options which the short retains. Kane and Marcus (1986) categorize these options as either timing options or quality options. Timing options have value because the short may deliver on any business day in the delivery month. Additional value arises because trading in the cash market continues after the futures price has settled.[2]

The quality option gives the short position the opportunity to deliver any US Treasury bond that has at least fifteen years to maturity or first call. Currently, more than thirty bonds, widely varying in coupon, callability and maturity, meet these criteria. The empirical evidence suggests that it is extremely difficult to predict which bond will be optimally delivered by the short. A system employing conversion factors for the various bonds has been developed by the Chicago Board of Trade (CBOT) in an effort to mitigate the scope of the short's option. However, the profitability of each position still depends heavily on which bond is delivered, after accounting for the effect of conversion factors. For this reason, the long position in the futures contract is said to face conversion factor risk.

Models for the quality option in Treasury bond futures can be divided into those which assume a stochastic process directly on bond prices and those which model interest rates instead. The former class of models includes Cheng (1985), Chowdry (1986), Hemler (1990) and Boyle (1989). The latter class of models includes Cox, Ingersoll, and Ross (1981), Ritchken and Sankarasubramanian (1992), Cherubini and Esposito (1996), and Bick (1996). We also develop and test a model of the quality option based upon the term structure of interest rates.

We develop tractable procedures in a general equilibrium economy where futures prices are driven by one or two factors, rather than by the prices of over thirty different bonds. In a series of papers, Vasicek (1977), Richard (1978), Brennan and Schwartz

[1] The traditional cost-of-carry model relates the futures price to the spot price of the underlying asset, which in this case is the bond which is currently cheapest to deliver.

[2] For the first 15 or 16 days of each delivery month, there is a 6 hour period during which the cash market remains open after the futures price has been settled. Futures trading ceases entirely for the last 7 trading days. These options have been termed the wild card option and the end-of-the-month option respectively.

(1979), Courtadon (1982) and Cox, Ingersoll and Ross (1985, hereafter CIR) have developed a methodology for pricing interest rate dependent claims. The methodology has been applied to determining bond spot and futures prices. This paper extends this methodology towards determining bond futures prices when the short position has a quality option.

In the one factor case, a closed form solution is developed which requires evaluating only univariate distribution functions. A more general two factor model can be extended from the one factor model in a number of ways. CIR (1985) develop several two and three factor models of the term structure. Using the same methodology as CIR, Longstaff and Schwartz (1992) derive a two factor term structure model in which the short rate and the volatility move randomly. Chen and Scott (1992) also follow the CIR framework and decompose the spot rate into two factors. Jamshidian (1993) and Duffie and Kan (1992) extend the Chen and Scott model to an arbitrary number of factors.

Aside from increased tractability, a term structure approach has several other advantages over models which specify bond price diffusions directly. One advantage is the paucity of inputs required to calculate the futures price, when there are a realistic number of bonds in the deliverable set. For our model, one needs only the values of at most two interest rates rather than the prices of over thirty different bonds. In contrast, when over thirty bonds are deliverable, over 465 bond return variances and covariances must be estimated[3] to implement models which specify bond price diffusions. Furthermore, those models generally do not preclude negative spot or forward rates of interest.[4] Moreover, the models employing bond price diffusions must assume that the coupon is a constant fraction of the bond price in order to obtain analytical solutions. Since bond coupons are constant, there is an internal inconsistency in these models unless the coupon rate is assumed to be zero. In contrast, term structure approaches allow for constant coupons. Finally, generalizing the analysis to allow for the callability feature of several of the underlying bonds is much more easily handled in a term structure framework. Including the call feature is particularly important because it increases the likelihood that the bond is optimally delivered.

The plan of this work is as follows. In Section II, we develop a model where the short rate of interest is taken as the single state variable. Analytic valuation formulae are obtained for the futures price with the quality option. Since the CBOT quality option is

[3] Imposing a single factor asset pricing model such as the CAPM reduces the number of parameters to be estimated.

[4] An important exception is Flesaker and Hughsten (1996). It would be interesting to apply their approach to quality options.

over long term bonds, the state space is augmented in Section III with a second factor that takes into account the effect of the long rate of interest. Section IV provides an empirical study of the two factor model. Section V concludes the paper.

2 ONE FACTOR MODEL

2.1 Assumptions and Notation

The following assumptions describe the structure of the model:

A1) Perfect Spot and Futures Markets

There are no transactions costs, margin requirements, limit moves, indivisibilities, differential taxes or short selling restrictions. Trading and marking-to-market occur continuously in time.

The import of this assumption is to rule out frictions of any sort and to cast the analysis in continuous time and space. In reality, marking-to-market occurs daily[5] and prices move in thirty seconds of a point subject to a three point daily price limit.

A2) Delivery Process

Delivery occurs at the contract expiration date. The short receives the futures price times the relevant conversion factor.

This assumption simplifies the analysis by excluding the short's timing option from consideration. The timing option will have smaller relative value when considering long maturity futures contracts.

A3) Deliverable Set

The coupon rate and maturity of all bonds in the deliverable set is known on the valuation date.

This assumption eliminates uncertainty as to the terms of any bonds issued between the valuation date and expiration. Since Treasury bonds are issued quarterly, this assumption is less valid for longer maturity contracts. If an eligible bond is callable, the assumption also eliminates uncertainty as to the call date.

A4) Mean Reverting Square Root Process

In the one factor model, the following dynamics is assumed for the instantaneous rate:

Eq 1 $$dr = \kappa(\mu - r)dt + \sigma\sqrt{r}dz$$

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[5] Flesaker (1988) and Chen (1991) have shown that the effect of daily marking to market on the futures price differs very little from that of continuous marking to market.

where κ, μ, and σ are positive constants. The increment of a standard Brownian motion is represented by dz. In a two factor model, the instantaneous short rate is decomposed into two independent factors, each of which follows a square root process:

Eq 2 $r = y_1 + y_2$

where

Eq 3 $dy_i = \kappa_i(\mu_i - y_i)dt + \sigma_i dz_i$

and i = 1, 2.

This assumption rules out jump processes for the spot rate and disallows its dependence on variables other than its current level *r* and calendar time *t*.

A5) Market Price of Spot Rate Risk

For any interest rate contingent claim, the excess return per unit of volatility risk is $\lambda\sqrt{r}\,/\,\sigma$ where λ is a constant.

This assumption is consistent with log utility function for the representative agent in the economy.

2.2 Valuation Results

CIR (1985) show that the value of a unit bond with time to maturity T_p is:

Eq 4 $P(r,T_p) = A(T_p)e^{-rB(T_p)}$

where

$$A(T) = \left[\frac{2\gamma e^{(\kappa+\lambda+\gamma)T/2}}{(\kappa+\lambda+\gamma)(e^{\gamma T}-1)+2\gamma}\right]^{2\kappa\mu/\sigma^2}$$

$$B(T) = \frac{2(e^{\gamma T}-1)}{(\kappa+\lambda+\gamma)(e^{\gamma T}-1)+2\gamma}$$

$$\gamma = \sqrt{(\kappa+\lambda)^2 + 2\sigma^2}$$

Let $F(r,T)$ be the futures price of a claim with time to expiration *T*. Using the CIR result (1985), the solution can be calculated by taking the following expectation:

Eq 5 $F_p(r, T_f; T_p) = \hat{E}[P(\tilde{r}, T_p)]$

In the above equation, $\tilde{r}$ is the terminal spot rate and $\hat{E}[\cdot]$ denotes expectations under the following risk-neutralized process:

Eq 6 $dr = [\kappa(\mu - r) - \lambda r]dt + \sigma\sqrt{r}d\hat{z}$

It can be shown that the expectation of Eq 5 under Eq 6 leads to CIR's formula for the futures price of a pure discount bond (1981):

Eq 7 $F_p(r, T_f, T_p) = C(T_f)e^{-rD(T_f)}$

where

$$C(T_f) = \left[\frac{\eta}{\eta + B(T_p)}\right]^{2\kappa\mu/\sigma^2} A(T_p)$$

$$D(T_f) = \frac{\eta e^{-(\kappa+\lambda)T_f}}{\eta + B(T_p)} B(T_p)$$

$$\eta = \frac{2(\kappa+\lambda)}{\sigma^2(1 - e^{-(\kappa+\lambda)T_f})}$$

Both the spot and futures prices of the pure discount bond are decreasing, convex functions of the spot rate *r*. While the spot price of the bond is declining in its time to maturity T_p, the futures price may be increasing or decreasing in the time to expiration T_f, ceteris paribus. Since the futures price is nothing more than an expectation under a certain process, there is no reason why changing the expiration date should unambiguously affect the futures price.

By CIR (1981), the futures price of a coupon bond, $F_b(r, T_f)$, can be specified as follows:

$$F_b(r, T_f) = \hat{E}[B(\tilde{r}, T_b)]$$

Eq 8 $$= \sum_{j=1}^{b} C_j \hat{E}[P(\tilde{r}, T_j)]$$

$$= \sum_{j=1}^{b} C_j F_p(\tilde{r}, T_f; T_j)]$$

where

Eq 9 $$B(\tilde{r}, T_b) = \Sigma_{j=1}^{b} C_j P(\tilde{r}, T_j)$$

and C_j is the coupon for $j = 1, \cdots, b-1$ and the coupon plus face value for $j = b$.

Next, we calculate the futures price, $F_b^{(2)}(r, T_f)$, when the short can deliver either of two Treasury bonds. Of course, the short selects the bond which maximizes his profit, after accounting for the conversion factors q_i,[6] which leads to the following terminal condition:

Eq 10 $$F_b^{(2)}(r,0) = \min\left[\frac{B(\tilde{r}, T_{b1})}{q_1}, \frac{B(\tilde{r}, T_{b2})}{q_2}\right]$$

The terminal futures price is just the smaller of the two "adjusted" bond prices, where the adjustment occurs by dividing each bond's price by its conversion factor. Depending on the coupons and maturities of the two bonds, one bond may be cheaper to deliver for all terminal interest rates, or one bond may be preferred for certain rates, while the other bond is preferred for different rates. In the former case, Eq 8 applies for the cheaper to deliver bond. In the latter case, the adjusted price of the cheaper bond must be determined for each level of the short rate at expiration. Suppose for simplicity that bond 1 is cheaper for interest rates below some critical level r^* while bond 2 is cheaper for higher rates. Then:

Eq 11
$$\begin{aligned}
F_b^{(2)}(r, T_f) &= \hat{E}\left[\min\left\{\frac{B(\tilde{r}, T_{b1})}{q_1}, \frac{B(\tilde{r}, T_{b2})}{q_2}\right\}\right] \\
&= \int_0^{\infty} \min\left\{\frac{B(\tilde{r}, T_{b1})}{q_1}, \frac{B(\tilde{r}, T_{b2})}{q_2}\right\} \phi(\tilde{r}) d\tilde{r} \\
&= \int_0^{r^*} \frac{B(\tilde{r}, T_{b1})}{q_1} \phi(\tilde{r}) d\tilde{r} + \int_{r^*}^{\infty} \frac{B(\tilde{r}, T_{b2})}{q_2} \phi(\tilde{r}) d\tilde{r}
\end{aligned}$$

[6] The conversion factor is the fraction of par value for which the bond would sell, if it were priced to yield 8%.

where $\phi(\tilde{r})$ is the probability density function of the terminal interest rate under *the drift-adjusted* mean-reverting square root process Eq 6. The Appendix presents this density function *en route* to proving the following result which is used to evaluate Eq 11:

Eq 12 $$\int_0^{r^*} \frac{B(\tilde{r},T_b)}{q}\phi(\tilde{r})d\tilde{r} = \sum_{j=1}^{b} \frac{C_j}{q} F_p(r,T_f;T_j)\chi^2[2(\eta + B(T_j))r^*;v,\Lambda]$$

where $\chi^2[x;v,\Lambda]$ is the non-central chi-square distribution function evaluated at x with $v = 4\kappa\mu/\sigma^2$ degrees of freedom and noncentrality parameter $\Lambda = \frac{2\eta^2\exp(-(\kappa+\lambda)T_f)}{\eta+B(T_j)}r$. Applying this result to Eq 11 implies that the current futures price is:

Eq 13 $$F_b^{(2)}(r,T_f) = \sum_{j=1}^{b1} \frac{C_{1j}}{q_1} F_p(r,T_f;T_j)\chi^2(r^*) + \sum_{j=1}^{b2} \frac{C_{2j}}{q_2} F_p(r,T_f;T_j)\chi^2(r^*)$$

where $\chi^2(r^*) = \chi^2[2(\eta + B(T_j))r^*;v;\Lambda]$.

2.3 Multiple Crossover Points

The pricing equation Eq 13 is predicated on there being at most one crossover point for the two underlying bond prices at expiration. Theoretically, there is no reason why the prices of the two bonds cannot cross themselves more than once. Furthermore, in any delivery month, there are 20 to 30 bonds deliverable against the CBOT Treasury bond futures contract. Fortunately, the biggest problem in extending our results to an arbitrary number of crossover points and deliverable bonds in mainly notational. The terminal condition when *n* bonds are deliverable is:

Eq 14 $$F_b^{(n)}(\tilde{r},0) = \min\left[\frac{B(\tilde{r},T_{b1})}{q_1},\cdots,\frac{B(\tilde{r},T_{b2})}{q_2}\right]$$

Employing the same technique as used to derived Eq 12, the current futures price generalizes to :

Eq 15 $$F_b^{(n)}(r,T_f) = \sum_{i=1}^{n}\sum_{j=1}^{b} \frac{C_{ij}}{q_i} F_p(r,T_f;T_j)w_{ij}$$

where $w_{ij} = \Sigma_{k=1}^{m} I_{ik}[\chi_j^2(r_k^*) - \chi_j^2(r_{k-1}^*)] > 0$, $i = 1,\cdots n$, $j = 1,\cdots bi$, are weights summing to unity across bonds. i.e. $\Sigma_{i=1}^{n} w_{ij} = 1$ for all *j*, and where $r_k^*, k = 0,1,\cdots,m$ are crossover points arranged in increasing order as $0 = r_0^* < r_1^* <\cdots< r_{m-1}^* < r_m^* = \infty$. I_{ik} is an indicator variable equal to one if bond *i* is cheapest in the interval and 0 otherwise, *i* =1,2.

Eq 15 is easier to use than its formal form might suggest. Figure 1 illustrates that if there are only three bonds which are cheapest to deliver in their respective subinterval, then Eq 15 will consist of only three terms, each of which is the futures price of the bond integrated against the probability kernel over the relevant spot rate range.

Figure 1: Three Adjusted Bond Prices Crossing Twice

The CBOT futures price is a weighted average of the futures prices of the component cash flows of each deliverable bond. The weight applied reflects[7] the probability (risk neutral) that the cash flow belongs to the cheapest deliverable bond at expiration. The futures price is a decreasing convex function of the current short rate *r*. It may be decreasing or increasing in the time to expiration, holding the underlying bond maturities constant.

Eq 15 can be used to develop the cost-of-carry relationship between the futures price and the underlying bond prices. Recall that B_i is the bond price of the *i*-th bond. Then:

Eq 16 $$F_b^{(n)}(r, T_f) = \sum_{i=1}^{n} x_i B_i$$

where $x_i = \Sigma_{j=1}^{b} \frac{C_{ij} F_p(r, T_f, T_j) w_{ij}}{q_i B_i}$.

Unlike the traditional cost-of-carry model, our model relates the futures price to several underlying bond prices. Eq 16 also indicates that the long position can eliminate conversion factor risk by always shorting x_i units of the *i*-th bond. At maturity, the long will only be shorting the cheapest deliverable bond. He can then use the bond received from his long position to cover this short.

[7] Note that the weight does not equal this probability but reflects it in that w_{ij} is increasing in the probability and equals 0 and one when the probability does.

Eq 15 can also be used to express the CBOT futures price in terms of the bond futures prices F_{bi} that obtain when no quality option exists:[8]

$$\text{Eq 17} \quad F_b^{(n)}(r,T_f) = \sum_{i=1}^{n} y_i F_{bi}$$

$$\text{with } y_i = \frac{\sum_{j=1}^{b} \frac{c_{ij}}{q_i} F_p(r,T_f;T_j) w_{ij}}{\sum_{j=1}^{b} \frac{c_{ij}}{q_i} F_p(r,T_f;T_j)} > 0,\; i = 1,\cdots,n$$

Thus the futures price may be written as a positively weighted average of each of the futures prices in the absence of the quality option. Each weight reflects the total probability that the associated bond is cheapest to deliver at expiration. Notice that Eq 17 reduces to the futures price of a single bond if the bond is cheapest with probability one. If the individual futures contracts exist or can be synthesized, then Eq 17 also indicates how a long position can eliminate conversion factor risk. By always shorting y_i contracts on each bond, the long will only be short the contract on the cheapest deliverable bond at maturity. He can then deliver the bond received from his long position to close out his short.

Eq 17 can be used to determine the impact of the quality option on the futures price. Suppose that the futures contract is nominally written on bond *n* but that the short owns the option to substitute any other bond if so desired. Of course, the short exercises this exchange option if one of the other bonds is less costly. The payoff of the option at expiration is:

$$\text{Eq 18} \quad \max\left\{\frac{B_n}{q_n} - \min\left[\frac{B_1}{q_1},\cdots,\frac{B_{n-1}}{q_{n-1}}\right],0\right\} = \frac{B_n}{q_n} - \min\left[\frac{B_1}{q_1},\cdots,\frac{B_n}{q_n}\right]$$

Consequently, the current *futures* price of this option is given by:

$$\text{Eq 19} \quad \hat{E}\left[\frac{B_n}{q_n} - \min\left\{\frac{B_1}{q_1},\cdots,\frac{B_n}{q_n}\right\}\right] = F_n - F_b^{(n)} = F_n(1-y_n) - \sum_{i=1}^{n-1} F_i y_i$$

[8] For simplicity, we assume that the conversion factor system still exists in that the invoice price resulting from delivery of bond *i* is $q_i F_{bi}$

Eq 19 shows that the bond futures price is affected by the *futures* price of the quality option rather than its spot price. As in most option pricing modes, Eq 19 shows that the (futures) price of the option is a probability-weighted difference in the (futures) prices of the optioned asset and the exercise asset. The weight on the futures price of the nominal bond is related to the probability that this bond is not cheapest to deliver. This weight measures the likelihood that the quality option finishes in the money. The weights on the futures prices of the other bonds reflect their respective probabilities of delivery.

3 TWO FACTOR MODEL

A disadvantage of single factor models is that bond returns of different maturities are perfectly correlated locally. Another disadvantage that arises when the single factor is the spot rate is that insufficient variance is generated for long term bond prices. In fact, as the time to maturity tends to infinity, the bond yield approaches a constant. As the CBOT quality option is defined over long term bonds and its value is sensitive to the underlying bond variances, a second factor is needed in order to induce the variability observed in long interest rates.

To empirically examine the need for the a factor, we use the parameter estimates of the single factor model by Chen and Scott (1993):

$$\kappa = 0.6248, \mu = 0.09304, \sigma = 0.10540, \lambda = -0.09235$$

Using these parameters, we determine the crossover rates for the one factor model for our sample period from March, 1989 till December, 1991. There are 20 contracts in this sample period. We calculate the adjusted bond prices, i.e. B/q, for all bonds that are eligible for delivery at each expiration date for all levels of interest rates.[9] The best bond to deliver for the short is the minimum of all adjusted bond prices at expiration. The results are shown in Table 1. With a single exception,[10] there exists no crossover rate for any other contract under the one factor model. As a result, Eq 13 reduces to the standard CIR futures price, Eq 8, and the quality option will have no value.

Table 1: Crossover Rates for the One Factor Model

[9] We use an increment of 0.001 for interest rates.

[10] The exception was for December 1990 where 8.75, May, 2020 is the cheapest when r is less than 1.5% and 7.25, May, 2016 is the cheapest when r is higher than 1.5%.

This problem is solved in this section by augmenting the state space with a second factor. We use the framework laid out by Cox, Ingersoll, and Ross (1985) in their equations (56)-(60). CIR develop a linear decomposition of the nominal rate r into two factors.[11] With an orthogonal transformation of the two factors, the nominal rate r can be expressed as the sum of the two orthogonal factors y_1 and y_2.

Chen and Scott (1993) find that the first factor covaries with the short term interest rate while the second factor reflects the long term rate impact. This decomposition is similar in spirit to the Brennan-Schwartz model.

Given $r = y_1 + y_2$, the closed form solution for the discount bond is clearly the product of two CIR bond prices:

Eq 20 $$P(y_1, y_2, T_p) = P(y_1, T_p)P(y_2, T_p)$$

where $P(y, T)$ is defined in Eq 4. Similarly, the futures price of a pure discount bond separates into the product of two single factor futures prices:

Eq 21 $$F_p(y_1, y_2, T_f; T_p) = F_p(y_1, T_f; T_p)F_p(y_2, T_f; T_p)$$

where $F_p(y, T_f; T_p)$ is defined in Eq 7. The CBOT futures price with an embedded quality option will therefore become:

Eq 22
$$\begin{aligned} F_b^{(n)}(y_1, y_2, T_f) &= \int_0^\infty \int_0^\infty \min\left\{ \frac{B(\tilde{y}_1, \tilde{y}_2, T_{b1})}{q_1}, \cdots, \frac{B(\tilde{y}_1, \tilde{y}_2, T_{bn})}{q_n} \right\} \phi(\tilde{y}_1, \tilde{y}_2) d\tilde{y}_1 d\tilde{y}_2 \\ &= \iint_{\tilde{y}_1, \tilde{y}_2 \in A_1} \frac{B(\tilde{y}_1, \tilde{y}_2, T_{b1})}{q_1} \phi(\tilde{y}_1, \tilde{y}_2) d\tilde{y}_1 d\tilde{y}_2 + \cdots \\ &\quad \iint_{\tilde{y}_1, \tilde{y}_2 \in A_n} \frac{B(\tilde{y}_1, \tilde{y}_2, T_{bn})}{q_n} \phi(\tilde{y}_1, \tilde{y}_2) d\tilde{y}_1 d\tilde{y}_2 \end{aligned}$$

where $\tilde{y}_i$ is the terminal factor level, i = 1,2, A_k indicates the region of the state space where bond k is cheapest to deliver, and where $A_1 \cup A_2 \cup \cdots \cup A_n = \Re_+^2$ is the whole space.

[11] See the original paper pp. 404-405 for details.

Using the same technique as in the Appendix, we can write the integrals in terms of bivariate non-central chi-squared distributions:

Eq 23
$$F_b^{(n)}(y_1, y_2, T_f) = \sum_{j=1}^{b1} \frac{C_{ij}}{q_1} F_p(y_1, y_2, T_f; T_j) \iint_{\tilde{x}_1, \tilde{x}_2 \in A_1} \phi(\tilde{x}_1)\phi(\tilde{x}_2) d\tilde{x}_1 d\tilde{x}_2 + \cdots$$
$$\sum_{j=1}^{bn} \frac{C_{ij}}{q_n} F_p(y_1, y_2, T_f; T_j) \iint_{\tilde{x}_1, \tilde{x}_2 \in A_n} \phi(\tilde{x}_1)\phi(\tilde{x}_2) d\tilde{x}_1 d\tilde{x}_2$$

where $\tilde{x}_i = 2(\eta + B(T_j))\tilde{y}_i$, for i = 1,2 and $\phi(\tilde{x}_i)$ is a non-central chi-squared density function. A closed form solution to this integral is difficult to obtain because the domain of each integral is a complicated region. Although the double integrals can be computed numerically, we actually implemented a lattice model which Longstaff and Schwartz (1992) advocate as equally efficient.

4 EMPIRICAL STUDY

In this section, we empirically examine our model, Eq 23, by comparing it against the CIR futures pricing model which does not incorporate the quality option and the popular cost of carry model (COC) assumes that the cheapest to deliver (CTD) bond at maturity is the current CTD bond.

4.1 Methodology

The model used in the empirical test is the two factor model of Eq 23. The model is a sum of two dimensional integrations. These integrals require CTD regions to be identified first. To identify the relevant regions, we need parameter values. There have been drastic developments in estimation techniques in recent years.[12] In this paper, we use the two factor model estimated by Chen and Scott (1993) with a weekly data set from 1980 to 1988. Their two factor model fits the yield curve reasonably well (for both in sample, 1980-88, and out of sample, 1989-91, periods). Three month, six month, five year, and the longest maturity Treasury issues are used to estimate the parameters for the two factor model as follows:

[12] For maximum likelihood estimation, see Chen and Scott (1993) and Pearson and Sun (1993); for Generalized method of moments, see Gibbons and Ramaswamy (1993) and Heston (1989); for the state space model with Kalman filtering, see Lund (1994), Chen and Scott (1995) and Duan and Simonato (1995).

$$\kappa_1 = 1.8341, \mu_1 = 0.05148, \sigma_1 = 0.1543, \lambda_1 = -0.1253$$
$$\kappa_2 = 0.005212, \mu_2 = 0.03083, \sigma_2 = 0.06689, \lambda_2 = -0.0665$$

With these parameter values, we can then calculate the regions where the cheapest bonds are delivered. Then bivariate integrations are implemented via a lattice model.

To implement Eq 23, in addition to the 8 parameter values, we also need two initial factor values. Chen and Scott (1993) recommend that initial factor values can be determined by matching two bonds from the yield curve — one short maturity and one long maturity. To satisfy our needs, we solve for the two factor values so that the long maturity bond is exactly the current CTD bond suggested by the COC method. In other words, we set the factor values so that the yield of the CTD bond is fitted perfectly. This result is then compared with the COC model.

The model we use, i.e., Eq 23, assumes away timing options. In order to investigate the importance of the timing options, we looked at the actual deliveries of the Treasury bonds in our sample period. The ex-post evidence presented by Table 2 shows that the daily timing option (including the wild card) has virtually no value. Table 2 presents actual deliveries in cumulative percentages. The "Last day" column presents the deliveries in the last day of the delivery month as a percentage of total deliveries in the delivery month. The "day -1" column is the deliveries for the last two days of the delivery month as a percentage of the total. Following this logic, Table 2 shows the cumulative (backwards) quantities of actual deliveries in the delivery month. For example, for contract 8703, the delivery month is March, 1987. The last day in that month is the 31st, a Tuesday. On that day, 7 issues are delivered, totaling 11014 bond contracts and representing 70.10% of total deliveries in March, 1987, which is 14428. On day -1, which is the 30th, 4 issues of 2210 contracts are delivered. Since this accounts for 15.32% of the total, the cumulative percentage for two days is 85.42%. There are only 3 out of 20 contract months where most deliveries occur prior to the last week.[13] In contrast, more than three quarters of the contract months are completely delivered in the last week of the delivery month. There is some flexibility in timing delivery during the last week of the delivery month when the futures market closes, the so called end-of-month timing option. But it is hard to find any significance for this value. Most deliveries in the last week concentrate on the last two days of the week. With the exception of the 3 contracts

[13] These three contracts are March 1989, June 1989, and December 1990.

previously mentioned, all the contracts had more than 70% of the deliveries occurring during the last two days.[14]

Table 2: Deliveries in Last Week

4.2 Data

In this empirical study, we use daily settlement futures prices from January, 1987 through December, 1991. We select this sample period because we would like to conduct both in-sample and out-of-sample tests. The Chen-Scott parameter values are estimated for the period 1980-1988. Therefore our sample period covers the in-sample period, 87-88, and the out-of-sample period, 89-91. The data set was acquired directly from the CBOT. We select prices of contracts that have maturities between 6 weeks and 4.5 months because of their liquidity and interest to traders.[15] Figure 2 presents a plot of futures prices. Since prices are taken from contracts that have 6 weeks to 4 and half months to maturity, there is little overlapping of any two consecutive contracts. In other words, for every trading day, we have only price. Table 3 breaks the time series down by the maturity of the contract. For a given contract month, the fluctuation is smaller and we can discern a definite pattern in the prices. Futures prices start out high, plunge in the middle of the sample period, and then climb back by the end of the sample period. It is also seen that prices are more volatile in 1987 and 1988 and stabilize after 1989.

Table 3: Summary Statistics of Daily Futures Prices

Figure 2: Daily Futures Prices

4.3 Results – Daily Testing

To calculate theoretical futures prices using Eq 23, we need the two factor values, y_1 and y_2 daily. To calculate daily factor values for our study, we would need all deliverable bond prices daily, which is difficult to handle. Fortunately, Chen and Scott (1993) have shown that the sum of the two factors which should yield the theoretical instantaneous rate approximates extremely well the 3 month T Bill rate. As a result, we shall use daily 3 month Treasury bill rates to back out the second factor value with weekly updating the first factor. This approximation should have little effect on our model futures prices

[14] The data of actual deliveries are obtained from *CBOT Financial Updates* and *CBOT Financial Futures Professional*.

[15] This selection of maturities was based upon a conversation with CBOT.

because the first factor contributes very little (less than 1%, see Chen and Scott (1993)) to the long term bond variability. The 1275 daily factor values for the second factor are calculated from January 1987 through December 1991.[16]

Before using Eq 23 to calculate theoretical prices, we need to identify all possible deliverable bonds at maturity. This is accomplished by simulating all bond prices at delivery dates. We select all deliverable bonds for every delivery month in the sample period (8703 through 9112), calculate their theoretical prices using Eq 9 and Eq 20, and use their conversion factors to identify the cheapest bonds for various interest rate regions. The results are reported in Table 4.

Table 4: Deliverable Bonds in the Two Factor Model

Table 4 presents a sharp contrast to Table 1. All contracts now have at least three possible bonds to deliver. For some contracts, there are 5 possible bonds for delivery. If the quality option is valued under the exchange-option framework, we need 2 to 4 dimensional integrations. Under the two factor model, we need just 2 dimensional integrations for any number of deliverable bonds. As we have pointed out, this is one of the major advantages of using our model.

With parameter and factor values, we can then use Eq 23 to calculate theoretical futures prices. Mean squared errors (MSEs) are calculated for each contract and reported in Table 5. For comparisons, we also compute the MSEs of all potentially deliverable bonds reported in Table 4 as well as a benchmark bond with an 8% coupon and 20 years to maturity. In Table 5, we report the MSEs of our model, labeled Eq 23, the benchmark bond, and the minimum and maximum MSEs over all potentially deliverable bonds.

Table 5: MSE's for Daily Testing

The magnitude of these MSEs is based upon a $100 face value. Other than the first two contracts (8703 and 8706), the combined MSE for the whole sample period is 3.14 (or the root MSE is 1.77). It is interesting to note that the performance of our model is close to the minimum MSE over all potentially deliverable bonds. This is in spite of the fact that the bond that generates the minimum MSE is not known at valuation date. Also the maximum MSE over all bonds is large. This indicates that the consequences of selecting

[16] The 3- month Treasury bill rates are the average of the bid and ask. We thank G. Hardouvelis for this data set.

the wrong bond in a single deliverable model are severe.[17] The large difference between the maximum MSE and the minimum MSE suggests that the quality option value is potentially large.

It is widely accepted that the most important input for an option is the volatility of the underlying asset and unfortunately this parameter is nonstationary. Although there have been models that deal with stochastic volatility option pricing problems, the most popular method adopted by practitioners is still the "implied volatility" method. Analogously, we update σ_2^2 daily. σ_2^2 computed at current date is used for the next day. With this adjustment, the performance of our model is indeed significantly improved. Except for March, 1987, all contracts are priced within 1% error. The results are presented in Table 6.

Table 6: MSE's for Daily Testing with Constantly Updated σ_2^2

4.4 Data and Results – Weekly Testing

The cost of carry model uses the current cheapest bond as the assumed only deliverable bond and uses the cost of carry formula to determine the futures price:

Eq 24 $$F_{coc} = \frac{(B_{ctd} + AI_1)(1 + R_f) - AI_2}{q_{ctd}}$$

where B_{ctd} is the cheapest bond on the transaction date (after accounting for its conversion factor), and where AI_1 is the accrued interest on the trade date, AI_2 is the accrued interest on the delivery date, and R_f is the short term risk free rate that matches the time to maturity of the futures contract.

Since the test of the COC model requires data on all deliverable bonds, the test is done on weekly data. We select Thursday's prices to be consistent with the Chen and Scott study (1993).[18] We find the cheapest to deliver bond by minimizing B/q for every given week and then use Eq 24 to compute futures prices. The results of this calculation as well as the weekly futures prices from the CBOT are plotted in Figure 3. The MSE is

[17] It should be kept in mind that all bonds considered here will potentially be the cheapest if the interest rates at maturity settle in its relevant region.

[18] If Thursday prices are not available, we use Friday's prices. This selection is also consistent with Chen and Scott (1993) so that the factor values for the model are consistent with the COC model.

calculated as 7.8838 for the whole period. The breakdown of each contract is given in Table 7.

Figure 3: Weekly Futures Prices

Table 7: MSE's for the Cost of Carry Model

To test our model against the COC model, we need to compute Eq 23 once a week. We repeat the procedure used to generate Table 5 and the results are given in Table 8. The MSE of our model for the whole sample period is 1.7197, representing an 80% reduction in error over the COC model.

Table 8: MSE's for weekly testing

5 SUMMARY

This paper derives and empirically tests a quality option model which overcomes several disadvantages of previous models. First, it is consistent with a general equilibrium theory of the term structure. Second, it avoids the use of multi-dimensional bond price processes which are difficult to implement. Third, it provides closed form solutions in the single factor case.

The empirical work in this paper is very supportive of the two factor model. The evidence suggests that the magnitude of the quality option in T-Bond futures contracts is not trivial. It also shows a significant difference between the two factor CIR model and the cost of carry model. Further research should focus on parameter estimation of the two factor model.

APPENDIX

This Appendix finds an expression for the first truncated moment of the terminal adjusted bond price:

$$\int_0^{r^8} \frac{B(\tilde{r}, T_b)}{q}\phi(\tilde{r})d\tilde{r} = \sum_{j=1}^{b} \frac{C_j}{q} F_p(r, T_f; T_j)\chi^2[2(\eta + B(T_j))r^*; \nu, \Lambda]$$

We begin by substituting the coupon bond definition into the integral on the left hand side of the above equation:

$$\int_0^{r^8} \frac{B(\tilde{r},T_b)}{q}\phi(\tilde{r})d\tilde{r} = \int_0^{r^8}\sum_{j=1}^{b} \frac{C_j}{q}P(\tilde{r},T_j)\phi(\tilde{r})d\tilde{r}$$
$$= \sum_{j=1}^{b} \int_0^{r^8}\frac{C_j}{q}P(\tilde{r},T_j)\phi(\tilde{r})d\tilde{r}$$
$$= \sum_{j=1}^{b} \frac{C_j}{q}A(T_j)\int_0^{r^8} e^{-\tilde{r}B(T_j)}\phi(\tilde{r})d\tilde{r}$$

The integral here is recognized as the truncated Laplace transform of the density function $\phi(\tilde{r})$, evaluated at the point $B(T_j)$. Using the density function provided in CIR (1985) under the square root process:

$$\phi(\tilde{r}|r) = \eta e^{-u-v}\left(\frac{v}{u}\right)^{q/2} I_q(2\sqrt{uv})$$

where:

$\eta = \frac{2(\kappa+\lambda)}{\sigma^2\left(1-e^{-(\kappa+\lambda))T_f}\right)}$

$u = \eta e^{-(\kappa+\lambda)T_f} r$

$v = \eta\tilde{r}$

$q = \frac{2\kappa\mu}{\sigma^2} - 1$

$I_q(\cdot)$ is the modified Bessel function of the first kind of order q:

$$I_q(x) = \sum_{j=0}^{\infty} \frac{(x/2)^{q+2j}}{j!\Gamma(q+1+j)}$$

Substituting this expression back to the integral yields:

$$A(T_j)\int_0^{r^*} e^{-\tilde{r}B(T_j)}\phi(\tilde{r})d\tilde{r} = A(T_j)\int_0^{r^*} e^{-\tilde{r}B(T_j)}\eta e^{-u-v}\left(\frac{v}{u}\right)^{q/2}\sum_{j=0}^{\infty}\frac{(uv)^{q/2+j}}{j!\Gamma(q+1+j)}d\tilde{r}$$
$$= A(T_j)\eta\int_0^{r^*} e^{-u-(\eta+B(T_j))\tilde{r}}\left(\frac{v}{u}\right)^{q/2}\sum_{j=0}^{\infty}\frac{u^j(\eta\tilde{r})^{q+j}}{j!\Gamma(q+1+j)}d\tilde{r}$$

In order to represent the integral as the distribution function of a non-central chi-squared distribution, consider the affine transformation $y = 2(\eta + B(T_j))\tilde{r}$:

$$A(T_j)\eta\int_0^{2(\eta+B(T_j))r^*} e^{-u-v/2}\sum_{j=0}^{\infty}\frac{u^j\left[\frac{\eta y}{2(\eta+B(T_j))}\right]^{q+j}}{j!\Gamma(q+1+j)}\cdot\frac{1}{2(\eta+B(T_j))}dy$$

$$= A(T_j)\left(\frac{\eta}{\eta+B(T_j)}\right)^{q+j}\int_0^{2(\eta+B(T_j))r^*}\frac{e^{-u-v/2}}{2}\sum_{j=0}^{\infty}\frac{u^j\left[\frac{\eta u}{\eta+B(T_j)}\right]^j\left[\frac{y}{2}\right]^{q+j}}{j!\Gamma(q+1+j)}dy$$

$$= C(T_f)e^{\frac{-B(T_j)u}{\eta+B(T_j)}}\int_0^{2(\eta+B(T_j))r^*}\frac{e^{-u-v/2+\frac{B(T_j)u}{\eta+B(T_j)}}}{2}\sum_{j=0}^{\infty}\frac{u^j\left[\frac{\eta u}{\eta+B(T_j)}\right]^j y^{q+j}}{j!\Gamma(q+1+j)2^{q+j}}dy$$

$$= F_p(r,T_f;T_j)\int_0^{2(\eta+B(T_j))r^*}\frac{e^{-v/2-\frac{\eta u}{\eta+B(T_j)}}}{2^{q+1}}\sum_{j=0}^{\infty}\frac{u^j\left[\frac{\eta u}{\eta+B(T_j)}\right]^j y^{q+j}}{j!\Gamma(q+1+j)2^j}dy$$

$$= F_p(r,T_f;T_j)\int_0^{2(\eta+B(T_j))r^*}\frac{e^{-\frac{(y+\Lambda)}{2}}}{2^{v/2}}\sum_{j=0}^{\infty}\frac{\Lambda^j y^{v/2+j-1}}{j!\Gamma(v/2+j)2^{2j}}dy$$

$$= F_p(r,T_f;T_j)\chi^2[2(\eta+B(T_j))r^*;v,\Lambda]$$

where v (degrees of freedom) and Λ (non-centrality parameter) are defined in the text. We complete the proof by substituting this result into the original equation.

Table 1
Crossover Rates for the One Factor Model

Contract	Cheapest Deliverable Bonds		Crossover Rate
	Bond 1	Bond 2	
8703	9.25, Feb, 2016	N.A.	N.A.
8706	7.25, May, 2016	N.A.	N.A.
8709	8.875, May, 2018	N.A.	N.A.
8712	7.25, May, 2016	N.A.	N.A.
8803	8.875, May, 2018	N.A.	N.A.
8806	7.25, May, 2016	N.A.	N.A.
8809	8.875, May, 2018	N.A.	N.A.
8812	7.25, May, 2016	N.A.	N.A.
8903	8.875, May, 2018	N.A.	N.A.
8906	7.25, May, 2016	N.A.	N.A.
8909	8.125, Aug, 2019	N.A.	N.A.
8912	7.25, May, 2016	N.A.	N.A.
9003	8.125, Aug, 2019	N.A.	N.A.
9006	7.25, May, 2016	N.A.	N.A.
9009	8.125, Aug, 2019	N.A.	N.A.
9012	8.75, May, 2020	7.25, May, 2016	0.015
9103	7.875, Feb, 2021	N.A.	N.A.
9106	8.125, May, 2021	N.A.	N.A.
9109	7.875, Feb, 2021	N.A.	N.A.
9112	8.125, May, 2021	N.A.	N.A.

Note: The parameter values used to simulate the crossover rate are the estimates of the one factor CIR model in Chen and Scott (1993). Eq 9 is used to calculate various bond prices. The cheapest bond is identified by minimizing B/q (where q is the conversion factor) across various bonds. Bond 1 is the cheapest bond in the interest rate region lower than the crossover rate and Bond 2 is the cheapest bond in the interest rate region higher than the crossover rate.

Table 2
Deliveries in Last Week

Contract	Total	Last day	day -1	day -2	day -3	day -4	day -5
8703	100.00%	70.10%	85.42%	99.98%	99.99%	100.00%	100.00%
8706	100.00%	99.85%	99.85%	99.93%	100.00%	100.00%	100.00%
8709	100.00%	99.44%	99.89%	99.89%	99.89%	99.91%	100.00%
8712	100.00%	65.00%	74.16%	74.16%	74.88%	74.88%	76.50%
8803	100.00%	93.18%	94.62%	97.21%	97.24%	97.24%	97.24%
8806	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%
8809	100.00%	99.87%	99.92%	99.92%	100.00%	100.00%	100.00%
8812	100.00%	51.41%	85.10%	85.67%	88.00%	93.59%	94.31%
8903	100.00%	0.01%	0.96%	2.95%	13.45%	15.31%	15.31%
8906	100.00%	21.35%	22.05%	22.05%	22.21%	27.77%	28.16%
8909	100.00%	70.55%	71.15%	73.29%	77.27%	79.89%	79.89%
8912	100.00%	49.24%	94.93%	94.93%	94.93%	96.18%	96.20%
9003	100.00%	99.89%	99.89%	100.00%	100.00%	100.00%	100.00%
9006	100.00%	97.42%	99.86%	99.86%	99.87%	99.93%	99.93%
9009	100.00%	87.14%	99.99%	100.00%	100.00%	100.00%	100.00%
9012	100.00%	1.92%	1.92%	4.05%	4.05%	4.13%	4.13%
9103	100.00%	96.58%	99.90%	99.94%	100.00%	100.00%	100.00%
9106	100.00%	98.29%	100.00%	100.00%	100.00%	100.00%	100.00%
9109	100.00%	99.83%	99.94%	99.94%	99.94%	99.96%	100.00%
9112	100.00%	99.13%	99.59%	99.83%	99.83%	99.83%	99.83%

Note: The percentage numbers are cumulative actual deliveries from the last day in the delivery month. The “Last day” column presents the deliveries in the last day of the delivery month as a percentage of total deliveries in the delivery month. The “day -1” column is the deliveries for the last two days of the delivery month as a percentage of the total. The remaining columns show the cumulative (backwards) quantities of actual deliveries in the delivery month. For example, for contract 8703, the delivery month is March, 1987. The last day in that month is 31st, a Tuesday. On that day, 7 issues are delivered, totaling 11014 bond contracts and representing 70.10% of total deliveries in March, 1987, which is 14428. On day -1, which is the 30th, 4 issues of 2210 contracts are delivered, representing 15.32% of the total. Therefore, the cumulative percentage for two days is 85.42%.

Table 3
Summary Statistics of Daily Futures Prices from 1/87 till 12/91

	N	Mean	Std Dev	Min	Max
All maturities	1275	92.65	4.7686	77.78	104.75
8703	21	100.62	0.6833	99.47	101.59
8706	63	97.23	3.2328	88.56	101.38
8709	64	90.72	1.5606	86.84	93.19
8712	65	84.59	3.3369	77.78	90.09
8803	63	88.23	2.1470	83.72	93.91
8806	64	91.19	1.9662	97.34	94.16
8809	64	86.68	1.2191	84.44	89.56
8812	65	87.53	2.1673	83.94	91.41
8903	63	89.09	1.1461	86.97	91.44
8906	64	88.73	1.0879	86.50	91.28
8909	64	95.23	3.1389	88.34	100.38
8912	65	97.42	1.1772	95.25	99.84
9003	63	98.29	1.9000	93.22	100.28
9006	64	92.26	1.6765	88.59	94.72
9009	64	93.15	1.1783	89.78	95.19
9012	65	89.60	1.2881	87.16	93.09
9103	63	94.91	1.6509	91.09	97.56
9106	64	95.95	1.1031	93.44	97.94
9109	63	93.83	0.9250	92.28	95.94
9112	64	98.17	1.4522	95.25	100.41

Note: The face value of the underlying bond is assumed to be $100.

Table 4
Possible Deliverable Bonds

Contract	Bond 1	Bond 2	Bond 3	Bond 4	Bond 5
8703	10.75,2,03	8.375,8,03	7.625,2,07	9.25,2,16	
8706	11.625,11,0	7.875,11,02	8.25,5,05	7.25,5,16	
8709	2	8.375,8,03	7.625,2,07	8.875,8,17	
8712	10.75,2,03	7.875,11,02	8.25,5,05	7.25,5,16	
8803	10.75,2,03	8.375,8,03	7.625,2,07	9.25,2,16	8.875,8,17
8806	11.125,8,03	7.875,11,02	8.25,5,05	7.25,5,16	
8809	11.875,11,0	8.375,8,03	7.625,2,07	9.25,2,16	8.875,8,17
8812	3	7.875,11,02	8.25,5,05	7.25,5,16	
8903	13.75,8,04	8.375,8,03	7.625,2,07	9.25,2,16	8.875,8,17
8906	12.3755,04	8.25,5,05	7.25,5,16		
8909	13.75,8,04	7.625,2,07	9.25,2,16	8.875,8,17	8.125,8,19
8912	7.875,11,02	8.25,5,05	7.25,5,16		
9003	8.375,8,03	7.625,2,07	9.25,2,16	8.125,8,19	
9006	7.875,11,02	10.375,11,0	7.25,5,16		
9009	10.75,2,05	7	9.25,5,16	8.125,8,19	
9012	12.75,11,05	7.625,2,07	7.25,5,16		
9103	9.375,2,06	10.375,11,0	9.25,2,16	8.875,8,17	8.125,8,19
9106	13.875,5,06	7	7.25,5,16		
9109	12,8,08	7.625,2,07	9.25,2,16	8.125,8,19	7.875,2,21
9112	14,11,06	10.375,11,0	7.25,5,16		
	12,8,08	7			
	14,11,06	7.625,2,07			
		10.375,11,0			
		7			

Note: These bonds are the possible deliverable bonds for each contract. The table is similar to Table 1 except that a two factor version of Eq 9, i.e., Eq 9 with Eq 20, is used. All bonds are reported by their coupons (first number) and maturity month (second number), and maturity year (third number).

Table 5
MSEs for Daily Testing

Contract month	Eq 23	Minimum MSE	Maximum MSE	Standard bond	No. of observations
8703	49.7002	50.7703	79.1000	78.7916	22
8706	26.8069	31.8355	42.1938	46.6872	63
8709	7.4268	8.1547	18.3405	17.3513	64
8712	3.4922	3.5766	13.4459	10.3625	65
8803	3.7274	4.0159	17.1279	12.7534	63
8806	3.2481	3.0677	13.7105	11.5997	64
8809	3.0263	3.1512	21.1838	11.6031	64
8812	1.3520	0.8502	15.9503	9.7288	65
8903	0.8782	0.9223	9.8478	7.3389	63
8906	3.4598	3.4136	5.1505	6.3542	64
8909	5.4144	2.1401	4.8693	5.5764	64
8912	3.2872	3.6892	3.9476	8.3660	65
9003	2.7328	0.5997	2.3080	5.1455	63
9006	2.8513	2.7955	5.3054	6.7327	64
9009	1.8728	1.7265	2.7025	6.4281	64
9012	3.3684	3.3324	8.8694	7.6758	65
9103	3.8978	1.9298	3.6592	6.1190	63
9106	2.4051	2.4568	7.3579	9.4754	64
9109	2.5173	2.7798	6.4933	9.0371	63
9112	1.5190	1.8175	3.2730	7.7540	64

Note: The standard bond is 8%, 20 years. Minimum MSE is the smallest MSE of all deliverable bonds (given in Table 4) and Maximum MSE is the largest MSE of all deliverable bonds. Except for the column labeled Eq 23, all futures prices are calculated using Eq 8. All numbers are based upon $100 face value. The parameter values used are the estimates of the two factor CIR model provided by Chen and Scott (1993).

Table 6

MSE's for Daily Testing with Constantly Updated σ_2^2

	March	June	September	December
87	13.74	0.77	0.85	1.27
88	0.68	0.48	0.53	0.27
89	0.31	0.72	0.91	0.58
90	0.29	0.82	0.71	1.18
91	0.55	0.23	0.24	0.32

Note: The MSE's reported are calculated the same way as Eq 23 in Table 5 except that the volatility parameter σ is updated daily. We use the implied σ for the next day's futures price calculation.

Table 7
MSE's for Cost of Carry Model

	March	June	September	December
87	0.50	9.46	14.17	7.49
88	3.73	19.36	3.87	0.08
89	3.02	15.57	5.10	11.69
90	8.66	2.64	11.79	2.97
91	6.42	1.45	9.07	8.35

Note: The cheapest to deliver bond on a given date is determined as the minimum value of B/q for all deliverable bonds at that date. The cost of carry model uses this calculation as an input to Eq 24 to compute the theoretical futures price for the that date.

Table 8
MSEs for Weekly Testing

Contract month	Eq 23	minimum MSE	maximum MSE	Standard bond	No. of observations
8703	0.7369	0.9160	1.8727	4.3480	5
8706	2.7926	4.5609	4.9210	11.4634	13
8709	0.8227	1.1015	9.1228	8.2017	13
8712	2.3235	1.6407	20.5019	11.7122	13
8803	2.9988	3.4097	18.7796	11.3540	13
8806	1.6132	1.1850	12.5321	10.3990	13
8809	4.8756	4.9762	16.6053	7.2182	13
8812	2.5149	1.5748	20.5235	11.8615	13
8903	2.2478	2.4881	15.1713	11.8820	13
8906	1.7584	1.3174	12.2034	10.5474	13
8909	0.5034	0.9473	5.2865	8.3225	14
8912	0.1722	0.3414	2.3993	6.8495	13
9003	0.2096	0.7002	1.7925	5.3077	12
9006	1.2687	1.0714	10.7484	11.5240	14
9009	1.4389	1.6704	4.5730	8.8719	13
9012	1.7394	1.4108	16.2757	12.2948	13
9103	1.1707	1.3718	3.4492	8.1110	13
9106	0.7994	0.8725	7.1878	9.1853	13
9109	1.8887	2.3356	7.0130	7.8561	13
9112	1.8911	2.2348	3.3466	7.6364	13

Note: The standard bond is 8%, 20 years. Minimum MSE is the smallest MSE of all deliverable bonds (given in Table 4) and Maximum MSE is the largest MSE of all deliverable bonds. Except for Eq 23, all futures prices are calculated using Eq 8. All numbers are based upon $100 face value. This table is similar to Table 5 except that weekly prices instead of daily prices are used. The parameters are taken from Chen and Scott (1993).

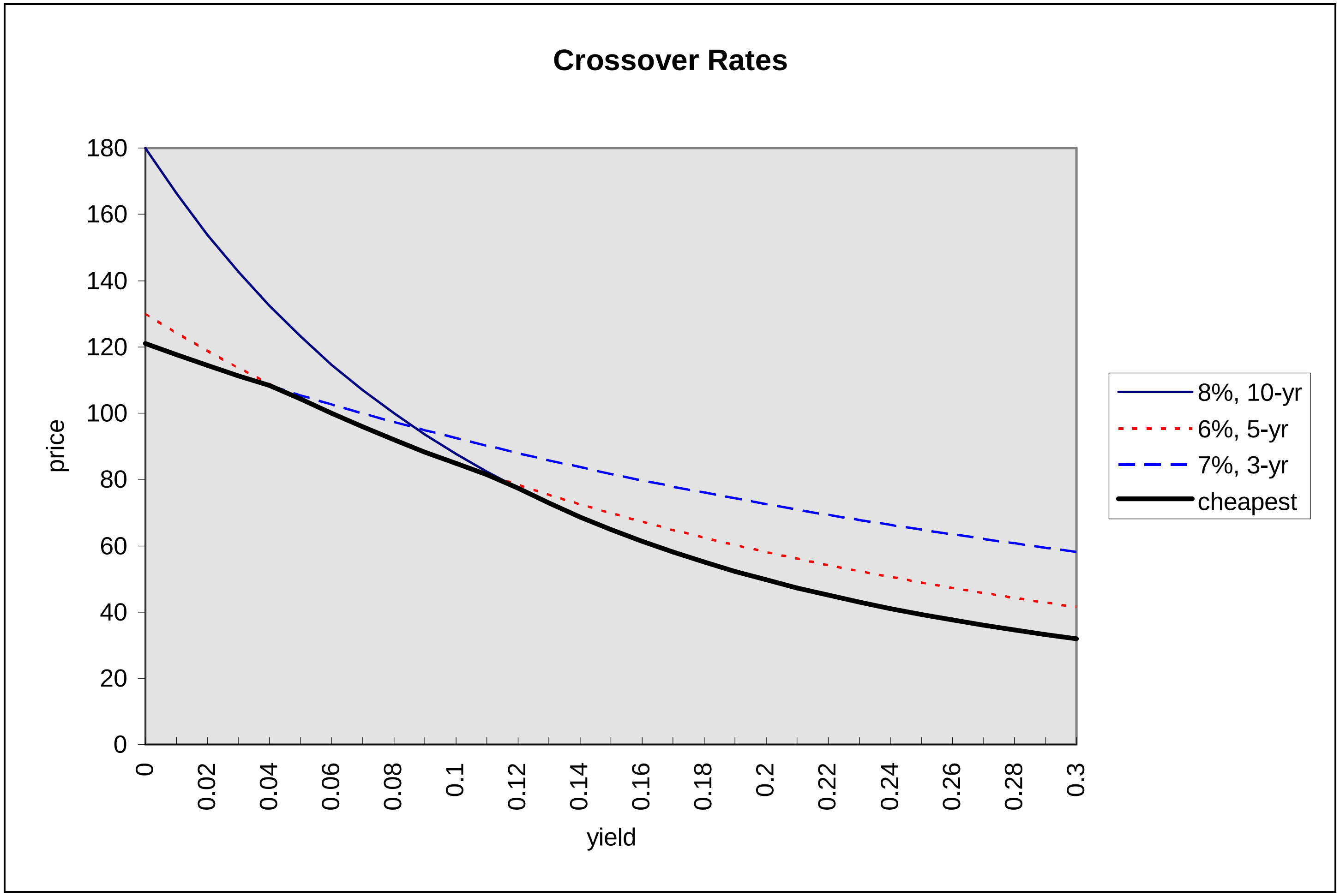

Figure 1: Three Adjusted Bond Prices Crossing Twice

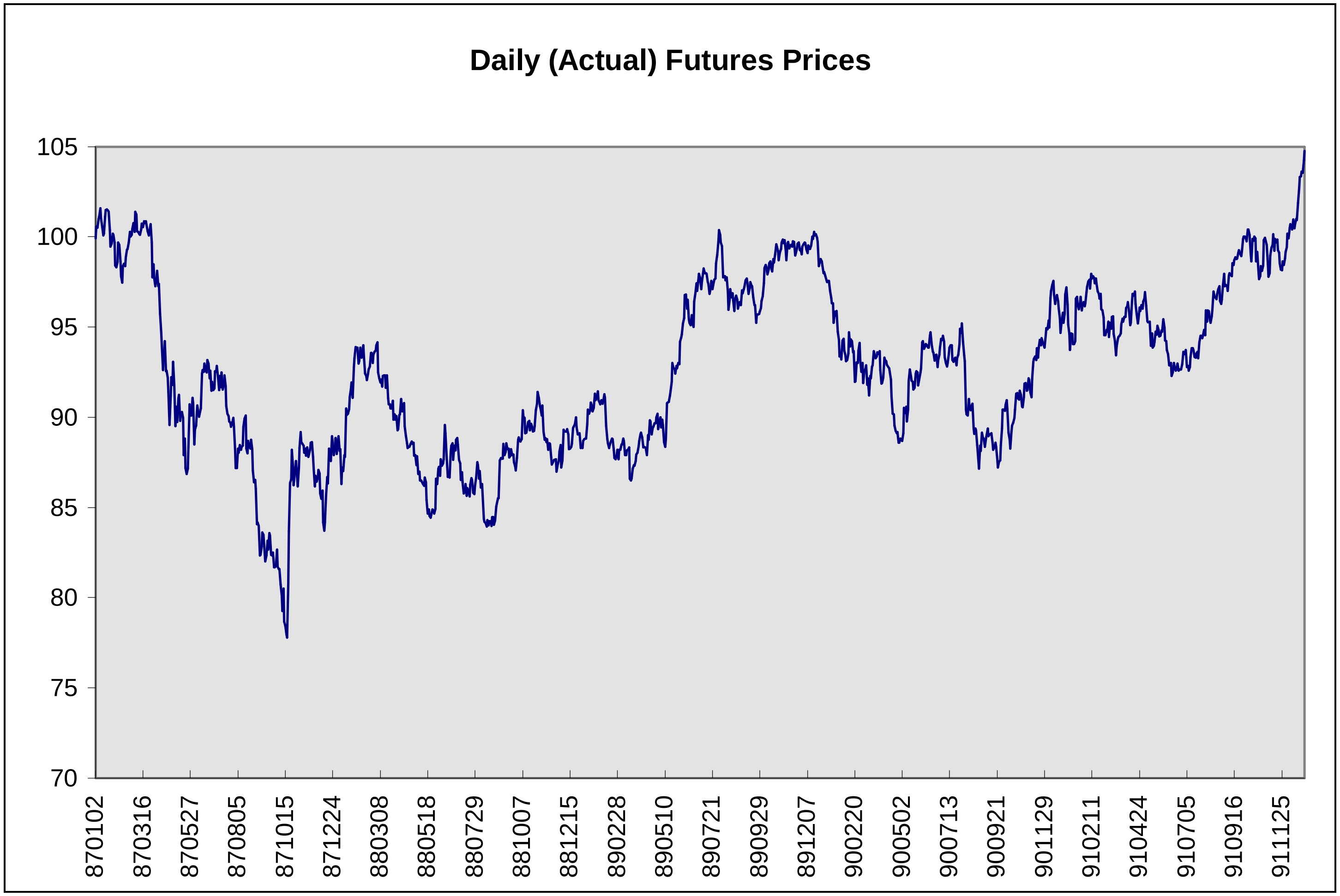

Figure 2: Daily Futures Prices — Actual

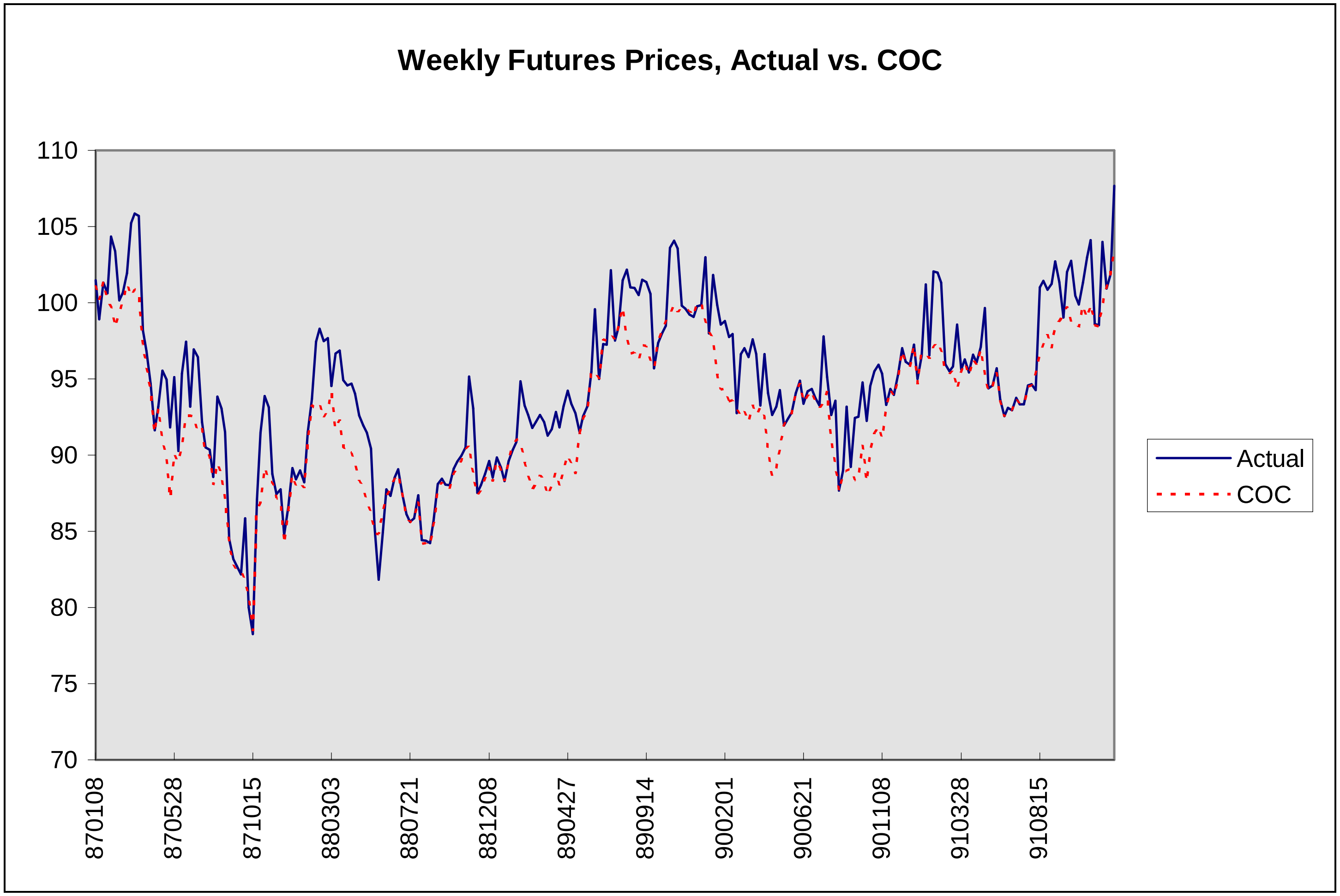

Figure 3: Weekly Futures Prices — Actual versus Cost of Carry

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