EMBEDDED OPTIONS IN TREASURY BOND FUTURES PRICES: NEW EVIDENCE

By

Ren-Raw Chen*
Graduate School of Business Administration
Fordham University

Hann-Shing Ju
Department of Electrical Engineering,
National Chung-Hsing University,

Shih-Kuo Yeh
Department of Finance
National Chung-Hsing University

January 31, 2009

* We would like to thank Peter Carr and Louis Scott for helpful comments.

EMBEDDED OPTIONS IN TREASURY BOND FUTURES PRICES: NEW EVIDENCE

ABSTRACT

The Treasury bond futures contract has known embedded options, namely the quality option that permits the short side to deliver the cheapest bond and the three timing options that permit the short side to delivery at the most favorable time. The literature has provided only partial solution to this pricing problem. Some have used the exchange option methodology that does not calibrate to the Treasury yield curve. Others have used a term structure model and yet ignored the fact that the futures price is not a *risk-neutral* expectation of the cheapest-to-deliver bond price but a *forward* expectation.

In this paper, we use a two-factor Cox-Ingersoll-Ross term structure model that is calibrated to the Treasury yield curve and compute the futures price with the forward pricing methodology. Using weekly futures prices from January 1992 till December 2000, we discover a substantial difference between the risk neutral expectation used in the literature and the forward expectation that requires a recursive algorithm. We find that the correctly estimated futures price with the quality option is 1% lower on average than the futures price estimated in the literature. We also estimate the end-of-month timing option to be 23 basis points on average. This indicates that the end-of-month timing option value has been over-estimated in the literature because of a wrongly estimated quality option value.

EMBEDDED OPTIONS IN TREASURY BOND FUTURES PRICES: THEORY AND EMPIRICAL EVIDENCE

I. INTRODUCTION

The delivery options in Treasury bond futures are generally known as the quality option and the three timing options. The *quality* option gives the short the right to deliver any eligible bond (no less than 15 years to maturity or first call) and the various timing options give the short the flexibility of making the delivery decision any time in the delivery month. The *end-of-month timing* option refers to the deliveries occurring at the last 7 business days in the delivery month when the futures market is closed to trading. For the remaining about 15 business days of the delivery month, the *wild card timing* option refers to the period from 2:00 p.m. to 7:00 p.m. (Chicago time) every day when the futures market is closed but the bond market is open while the *accrued interest timing* option refers to the period from 7:20 a.m. to 2:00 p.m. when both futures and its underlying bond markets are open.

Delivery options in Treasury bond futures are difficult to price. A recursive algorithm based upon a lattice model must be used for valuing such options accurately, as Boyle (1989) demonstrates that the futures price is effectively a forward price.[1] To this date, the literature has been approximating the forward price valuation with a futures price valuation due to the high computational cost. Aided with today's computing power, we can now compute accurately the Treasury bond futures price with the proper recursive algorithm. In this paper, we estimate the value of the quality option as well as the value of the end-of-month timing option. Using a two-factor Cox-Ingersoll-Ross model calibrated to the cheapest-to-deliver bond and weekly futures prices data from January 1992 till December 2000, we discover a substantial difference between the risk neutral expectation used in the literature and the forward expectation that requires a recursive algorithm. We find that the correctly estimated futures price with the quality option is 1% lower on average than the futures price estimated in the literature. We also estimate the futures price with both the quality option and the end-of-month timing option to be 23 basis points on average. This indicates that the end-of-month timing option value has been over-estimated in the literature because of a wrongly estimated quality option value.

---

[1] Furthermore, we note that the wild card timing option is a series of compound options on the forward price which cannot be priced accurately without a multi-recursive system. As a result, an accurate valuation of these delivery options is very expensive.

An early discussion of the valuation of the quality option appears in Cox, Ingersoll, and Ross (1981) in which they state that their valuation can be applied to futures with the quality option when the single spot bond price is replaced with the minimum from the deliverable set. Jones (1985) argues that although there are multiple bonds eligible, bonds with extremely high and low durations are the ones to be delivered. His argument holds if the yield curve is flat. If the yield curve is not flat, then durations of different maturity bonds are not directly comparable and therefore his extreme duration rule fails. Hemler (1988) uses Margrabe's (1978) exchange option formula to price the quality option but the pricing formula becomes intractable as the number of deliverable bonds increases. Carr (1988) was the first to use factor models to price the quality option and Carr and Chen (1996) extend the Carr model to include a second factor. Ritchken and Sankarasubramanian (1992) use the Heath-Jarrow-Morton (1992) framework to find the quality option value. Livingston (1987) analyzes the quality option on the forward contract.

Timing options in general have no closed form solutions and are therefore studied with lattice methods. Kane and Marcus (1986a) lay out a general framework for analyzing the wild card option. Broadie and Sundaresan (1987) develop a lattice model to value the end-of-month option. Their focus is strictly on the futures price in the end-of-month period. Boyle (1989) uses a two-period model to show that the timing option could have a significant impact. His analysis assumes constant interest rates and does not directly apply to Treasury bond futures. These papers remain theoretical and provide no empirical evidence.

Empiricists in general agree that the quality option and the end-of-month timing option have non-trivial values.[2] Yet, there is no consensus as to the magnitudes of these option values.[3] This is because (1) most studies do not distinguish between the quality option value and the value from the other timing options, let alone values among various timing options and (2) none of these models adopts a factor-based, general equilibrium term structure model to value these various embedded options. As a result, this paper contributes to the literature by using a two-factor Cox-Ingersoll-Ross term structure model (1985) to estimate the end-of-month timing option value.

The paper is organized as follows. The next section derives the proper pricing model for the quality option and discusses how each timing option should be estimated. We show how the futures price with the quality option is effectively a forward price when the futures market is

---

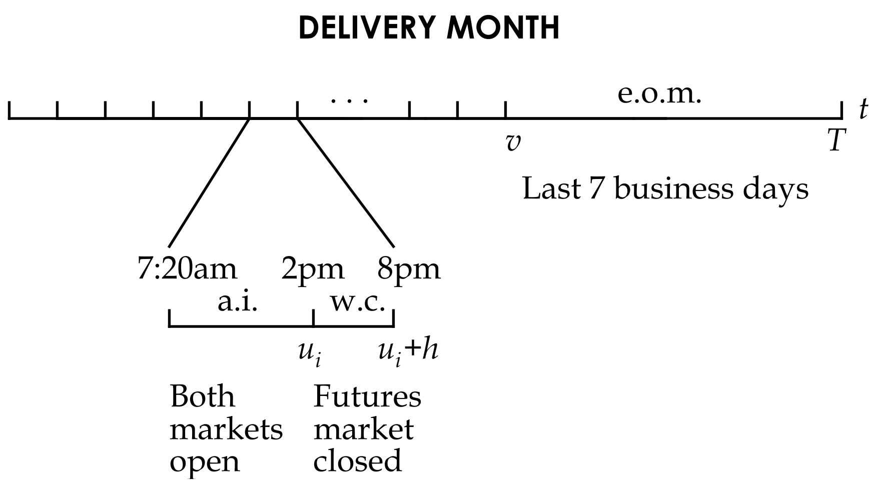
[2] See, for example, Carr and Chen (1996), Kilcollin (1982), Benninga and Smirloc (1985), Kane and Marcus (1986b), and Hegde (1990).

[3] See, for example, Arak and Goodman (1987), Hegde (1988, 1990), Gay and Manaster (1984, 1986), and Kane and Marcus (1986a, 1986b). The evidence from these studies is inconclusive.

closed and the bond market is open. Section III contains an empirical study where a two-factor equilibrium term structure model is estimated under the Chen and Scott (1993) technique. The quality option value and the end-of-month timing option are estimated using weekly futures prices from 1992 till 2000. Finally, the paper is concluded in Section IV.

II THE QUALITY OPTION AND THE FUTURES PRICE

The delivery option that has the most economic value is the quality option that gives the short of the futures contract the right to choose the cheapest bond to deliver at the delivery date. Other delivery options that are embedded in Treasury bond futures are known as the three timing options. The short can choose any time in the delivery month to make a delivery. The short can make a delivery even when the futures market is closed. At the end of the delivery month, for 7 business days, the futures market is closed but the short can still make a delivery. This is understood as the end-of-month timing option. For the remaining about 15 business days in the delivery month, the short can deliver either between 7:20 a.m. and 2:00 p.m. (Chicago time) when both the futures market and the underlying bond market are open or after 2:00 p.m. when the futures market is closed.[4] The former timing option is called accrued interest timing option and the latter timing option is also known as the daily wild card play. The following picture explains graphically various timing options.



[4] Treasury bond market is an over the counter market that has no official closing time, even though market practice adopts 3:00 p.m. Eastern time as a symbolic closing time. The futures market allows the short up to 8:00 p.m. Eastern time to make the delivery announcement, and hence theoretically there is a 5-hour window for the wild card.

The period of the last 7 business days is the end-of-month period. Throughout the paper we use v for the starting time and T for the ending time of this period. For the rest of the delivery month, there are two sections of each day, the accrued interest period and the wild card period. For a regular futures trading day i between 7:20 a.m. and 2 p.m. Chicago time, both bond and futures markets are open simultaneously. The futures market closes at 2 p.m. but there is no official closing time for the bond market. Since the short has till 8 p.m. to make the delivery decision, the wild card period is defined over 3 p.m. (u_i) to 8 p.m. ($u_i + h$) Eastern time.

The notation and symbols used in the paper are also summarized as follows:

$\Phi(t) =$ "quoted" futures price with all delivery options

$\Phi^{*}(t) =$ futures price with the quality option and continuous marking to market

$\Phi^{**}(t) =$ futures price with the quality option at the absence of continuous MTM

$\Phi_i(t) =$ futures price of the ith quoted bond price

$\Psi_i(t) =$ forward price of the ith quoted bond price

$a_i(t) =$ accrued interest of the ith bond

$P(t,T) =$ discount bond price at time t of \$1 at time T

$Q_i(t) =$ "quoted" coupon bond price of the ith bond

$q_i =$ conversion factor of the ith bond

$\delta(t,T) =$ random discount factor between t and T

Before we start our analysis, we need Jamshidian's separation theorem (1987) and his definition of the forward measure.[5]

Theorem 1 (Forward Measure)

Let $P(t,T)$ be the price of a pure discount bond delivering \$1 at some future date and it follows the dynamics as:

$$\frac{dP(t,T)}{P(t,T)} = r(t)dt + b(t,T)dW^{Q}(t)$$

where r is the instantaneous risk-free rate, b is maturity dependent bond volatility, and $dW^{Q}(t)$ is the standard Wiener process defined under the risk-neutral space. Then the forward measure is defined as:

[5] Also see Hull (2003).

$$\frac{dP(t,T)}{P(t,T)} = \left(r(t) - b(t,T)^2\right)dt + b(t,T)dW^{F(T)}(t)$$

where $dW^{F(T)}(t) = dW^{Q}(t) + b(t,T)dt$ *. Under this forward measure, all expected values taken will be forward prices, that is:*

$$\begin{aligned} E_t^Q[\delta(t,T)X(T)] &= E_t^Q[\delta(t,T)]E_t^{F(T)}[X(T)] \\ &= P(t,T)E_t^{F(T)}[X(T)] \end{aligned}$$

where $\delta(t,T) = \exp\left(-\int_t^T r(u)du\right)$ *and* $E_t^{F(T)}[X(T)]$ *computes the forward price of X.*

A simple proof of this theorem is given in an appendix although the original proof is available in Jamshidian (1987).

A. The Quality Option with Continuous Marking to Market

In the absence of all timing options, the quality option gives the short the right to deliver the cheapest bond only at maturity, T, and the short receives the following payoff:

(1) $$\max\left\{q_i\Phi(T) - Q_i(T)\right\}$$

Note that the accrued interests of both bond and futures contracts are equal and canceled. Since the delivery value of (1) has to be identically 0 for all states, we can solve for the futures price at maturity as:

(2) $$\Phi(T) = \min\left\{\frac{Q_i(T)}{q_i}\right\}$$

and today's futures price is merely a risk-neutral expectation of this payoff:

(3) $$\begin{aligned} \Phi^*(t) &= E_t^Q\left[\min\left\{\frac{Q_i(T)}{q_i}\right\}\right] \\ &= \frac{E_t^Q[Q_1(T)]}{q_1} - E_t^Q\left[\max\left\{\frac{Q_1(T)}{q_1} - \frac{Q_i(T)}{q_i}\right\}\right] \\ &= \frac{\Phi_1(t)}{q_1} - E_t^Q\left[\max\left\{\frac{Q_1(T)}{q_1} - \frac{Q_i(T)}{q_i}\right\}\right] \end{aligned}$$

Note $\Phi_1(t) = E_t^Q[Q_1(T)]$ is the futures price of the first bond with no option and $\Phi^*(t)$ is the futures price of the cheapest bond at maturity. This result has been shown previously by Carr (1988) and other authors. This equation says that the futures contract with the quality option is equivalent to a futures contract without the quality option (only bond 1 is eligible for delivery) with an exchange option held by the short. With a specific term structure model, equation (3) becomes an analytical solution.[6]

B. The Quality Option with no Marking to Market When the Futures Market Is Closed

Equation (3) is correct only if marking to market is applied continuously throughout the life of the futures contract. Unfortunately, in the last 7 business days of the delivery month, the futures market is not open and the futures contract is not marked to market. The futures price used for settlement in this period is the last settlement price at the beginning of the 7-day period. Since the futures price is already determined, the actual payoff at the last delivery day, T, is not necessarily 0. The short can actually gain or lose. To avoid arbitrage, the futures price at the beginning of the 7-day period should be set so that the expected present value of payoffs at maturity is 0. Under this circumstance, the futures price at the beginning of the 7-day period is a *forward price*, not a futures price. Formally, label the futures price as $\Phi^{**}(v)$ to represent the futures price at the beginning of the end-of-month period, v, should be so set that:

(4) $$E_v^Q[\delta(v,T)\max\{\Phi^{**}(v)q_i - Q_i(T)\}] = 0$$

where δ is the stochastic discount factor assumed to be strictly less than 1. Using Theorem 1, we can then rewrite (4) as:

(5) $$E_v^{F(T)}[\max\{\Phi^{**}(v)q_i - Q_i(T)\}] = 0$$

which can be expanded as follows:

[6] For example, the closed form solution under the one-factor Cox-Ingersoll-Ross model can be found in Carr (1988).

$$0 = E_v^{F(T)}[\max\{\Phi^{**}(v)q_i - Q_i(T)\}]$$

(6) $$0 = E_v^{F(T)}[\Phi^{**}(v)q_1 - Q_1(T) + \max\{\Phi^{**}(v)(q_i - q_1) - (Q_i(T) - Q_1(T)), 0\}]$$

$$0 = \Phi^{**}(v)q_1 - \Psi_1(v) + E_v^{F(T)}[\max\{Q_1(T) - Q_i(T) - \Phi^{**}(v)(q_1 - q_i), 0\}]$$

and the futures price at time v can be written as:

(7) $$\Phi^{**}(v) = \frac{\Psi_1(v)}{q_1} - \frac{1}{q_1} E_v^{F(T)}\left[\max\left\{Q_1(T) - Q_i(T) - K_i^{**}\right\}\right]$$

where $K_i^{**} = (q_1 - q_i)\Phi^{**}(v)$. Note that $\Psi_1(v) = E_v^{F(T)}[Q_1(T)]$ is the forward price of the first bond. The interpretation of this result is similar to that of (3), except that the risk neutral measure is replaced by the forward measure defined in Theorem 1 and the futures price becomes the forward price. However, unlike (3), the futures price at time v has no easy solution, because it appears on both sides of the equation. This futures price has to be solved recursively using a numerical method. In a lattice framework suggested by Boyle (1989), we first choose an initial value for the futures price at time v, calculate payoffs at various states at maturity T, and then work backwards along the lattice. We adjust the futures price until the discounted payoff computed from the lattice is 0. Once the futures price at time v is set, we can then travel back along the lattice and use the risk neutral probabilities till the end of the last wild card period, $u_n + h$. Then the similar procedure for the end-of-month period is repeated for the last wild card period to arrive at the futures price at the beginning of the wild card period u_n. Again, the risk neutral expectation is taken at $u_{n-1} + h$ and a recursive search is to compute the futures price at u_{n-1}. The process is repeated until the delivery month is over. Since the futures price becomes a forward price which cannot be obtained without a recursive search. The search for the "forward price" takes place at every node at all the times (i.e., u_1, u_2, $\cdots$, u_n, v). As a result, to compute the futures price with the quality option is prohibitively expensive.

C. The Timing Options

Besides the quality option, there are three timing options embedded in the Treasury bond futures price. The most valuable one is the end-of-month (EOM) timing option. Without the EOM timing option, we know that the futures price should be set according to (7). With the EOM timing option, deliveries can occur any time in the end-of-month period as long as the current delivery payoff is more than the present value of the expected payoff. This is similar to the early exercise decision of an American option. There is no closed form solution to compute American option prices. Precisely as Boyle (1989) has observed, the pricing of quality and timing options would need a lattice model.

Given that the two factors in the Cox-Ingersoll-Ross model are orthogonal, we use the method introduced in Hull and White (1990a) for both factors. As in an American option, early delivery (i.e. early exercise) is activated if the delivery payoff is larger than the continuation value (expected value of future payoffs). However, this delivery decision is intertwined with the recursive process in computing the quality option value. That is, every time we start with a trial value for the futures price at the beginning of the EOM period. This futures price will not change throughout the EOM period since the futures market is closed. We then work backwards from the end of the EOM period with an early delivery decision checked at every node until we reach the beginning of the EOM period. If the expected payoff computed via this backward induction at the beginning of the EOM period is not 0, then the trial futures price must be revised. The process continues until the payoff at the beginning of the EOM period is 0. The computation cost of such a recursive algorithm in a two-dimensional lattice is high.

In addition to the EOM timing option that refers the last 7 trading days of the delivery month, there are two other timing options in the rest 15 days of the delivery month – the accrued interest timing option and the wild card timing option. The accrued interest timing option refers to the flexibility for the short to deliver the cheapest bond any time in the delivery month when both futures and spot markets are open. This is everyday from 7:20 a.m. to 2:00 p.m. (Chicago time) from the first day of the delivery month to right before the end-of-month period. Since the futures market is open, the futures contract is marked to market and deliveries can take place any time. As a result, the futures price can never be greater than the cheapest-to-deliver bond price. If the futures price were greater than the cheapest bond price, then deliveries would take place instantly. The short will sell the futures, buy the cheapest bond, make the delivery, and earn an arbitrage profit. Formally, for $t < v$,

$$\Phi(t) > \min\left\{\frac{Q_i(t)}{q_i}\right\} \Leftrightarrow \max\left\{\Phi(t)q_i - Q_i(t)\right\} > 0 \tag{8}$$

Therefore, the futures price in the period where both markets are open must be less than the cheapest-to-deliver bond price to avoid arbitrage. On the other hand, if the futures price is lower, one can long futures and short spot but the delivery will not occur because the short position of the futures contract will lose money if he makes a delivery. Consequently, the delivery will be postponed and there is no arbitrage profit to be made. If the futures price is always less than the cheapest-to-deliver bond price (adjusted by its conversion factor), the delivery payoff now is negative as opposed to 0 at the end. As a result, the short will never deliver until the last day.

Consequently, the accrued interest timing option has no value. We restate this result in the following proposition.

Proposition 1

The accrued interest timing option without the wild card and end-of-month options has no value.[7]

□

The existence of the other timing options will lower the current futures price, further reducing the incentive for the short to deliver early. We state this result in the following Corollary.

Corollary 1-1

The accrued interest timing option with the end-of-month options has no value.

□

While the accrued interest timing option is worthless, the wild card timing option is not. When the futures market is closed, there is no marking to market in the futures market and the futures contract becomes a forward contract. Boyle (1989) has demonstrated that in a case of forward contracts timing options will have value. We shall extend Boyle's analysis to the wild card option in Treasury bond futures. Similar to the end-of-month option, the wild card option refers to the flexibility in delivery in a 5-hour period every day for about 15 days where the futures market is closed but the bond market is open. However, the wild card option is different from the end-of-month option in that the futures market will reopen after each wild card period and the futures contract will be marked to market. If bond prices drop in the wild card period, given that the futures price is fixed, the short can benefit from delivering the cheapest bond. However, the short can equally benefit from the marking to market when the futures market reopens on the next day. As a result, the incentive for the short to deliver in the wild card period is minimal. Delivery can take place in a wild card period only when the payoff from immediate delivery exceeds the expected present value of marking to market on the next day.

The modeling of the wild card period requires a very fine grid. To model the wild card option properly, we need at least two steps in each wild card period to allow for early exercise. As a result, it requires a minimum of four steps per day.[8] Given that practically the wild card

---

[7] The name "accrued interest" comes in because in the delivery month, the bond price increases due to accrued interests. Here, Q is a traded price that included accrued interests.

[8] If we ignore the 1 and ½ hour difference in the lattice between the accrued interest period that is 6 and ½ hours and the wild card period that is 5 hours.

option has very little value due to the next-day marking-to-market, we will not evaluate the wild card option in this paper.

III EMPIRICAL STUDY

In this section, we empirically examine the magnitude of each bound using a two-factor CIR model. We empirically estimate the weekly values of the quality and end-of-month options for the period from November 7, 1991 to November 2, 2000. These prices cover contracts from March 1992 to December 2000. Our term structure estimation employs weekly Treasury data from January 4, 1991 to December 29, 1998. Hence, we perform both in-sample (March 1992 ~ December 1998) and out-of-sample (March 1999 ~ December 2000) tests.

We start with the estimation of the term structure. Then we demonstrate how to value the quality option properly as a forward price and how it is intertwined the end-of-month timing option. Finally we show the empirical results.

A. Term Structure Model Estimation

In estimating the two-factor CIR term structure model, we use weekly (Friday) four Treasury interest rate series: the 3-month and 6-month Treasury-bills and the 5-year and 30-year Constant Maturity Treasury (CMT) interest rates to estimate the parameters of the model. The weekly data is from January 4, 1991 to December 29, 1998, which contains 416 observations in total. Data source is the Aremos USFIN databank. The estimation procedure is identical to that described in Chen and Scott (1993). In addition to our estimates, as a robustness comparison, we also use the results from Chen and Scott (1993) who use a weekly data set from 1980 to 1988 and the estimates from both estimations are reported in Table 1. We can see that the estimates do not change much from one period to another, while the new estimates do show slightly lower reverting level and slower mean reversion. The first factor remains strong mean reversion while the second remains to be close to a random walk.

In addition to estimating the parameters, we also estimate factor values. In Chen and Scott (1993), factor values are computed by fitting the long and short rates of the yield curve. For our purposes (that we need to price the cheapest-to-deliver bond correctly),[9] the factor values are solved for by matching the short rate and the cheapest-to-deliver bond price. In reality, the delivery options are priced off the cheapest-to-delivery bond and a series of exchange

---

[9] Treasury bond futures prices are affected by all bonds underlying the yield curve, and yet doubtlessly the cheapest-to-deliver bond has the most influence.

options to the next cheapest, the third cheapest, and so on. By calibrating the term structure model of Chen-Scott to the cheapest-to-delivery bond, we shall provide the most accurate valuation of the delivery options using the two-factor CIR model. It is generally understood that the two-factor CIR model does not fit the yield curve well.[10] In order to mitigate the concern of Jagannathan, Kaplin and Sun (2003), we must examine how good our term structure fit is for the set of deliverable bonds. We are not particularly concerned with the whole yield curve fit because the majority of the risk of the delivery options resides in the set of deliverable bonds. Furthermore, as a practical concern, we present the fitting performance of the three most relevant bonds – the cheapest, second cheapest, and third cheapest. The probability of other bonds become the cheapest is small and the impact of other deliverable bonds is believed to be negligible.

Theoretically, the cheapest bond at any point in time should be fitted perfectly by tweaking the second factor, since there is one equation and one unknown. However, there is no solution to the second factor at the following dates when we try to fit the cheapest bond: 980903, 980910, 980917, 980924, 981001, 981015, 981029, 981203, 981210, and 981217. Figure 1 plots the yield curves for a sub-period (January 2, 1998 ~ December 28, 2000) from our CMT dataset. It can be seen that the above dates where the second factor fails to coincide (CTD bond fails to fit) with the period when the yield curve is steeply sloped and the short rates are small. This is a problem already described in Chen and Scott (1993). Chen and Scott recommend a three-factor model to improve the fit. However, due to the reality that this problem is only present for 10 cases and the complexity of estimating a three-factor model, we decide to stay with the two-factor model.[11] Or alternatively, we can allow the first factor to be flexible until we are able to fit the CTD bond. But in order to maintain consistency, we allow the CTD bond to be not perfectly fitted for those 10 dates.[12] The following summary illustrates the cheapest bond that fails to be fitted and the difference between the market price and the model price.

---

[10] See, for example, Chen and Scott (1993) and Jagannathan, Kaplin and Sun (2003).

[11] Chen and Scott (1993) argue that the three-factor model does not necessarily dominate the two-factor model in that the three-factor model, although fits better the term structure, generates extra volatility. See Chen and Scott for details.

[12] The result of the alternative fitting is available upon request.

date	coupon	maturity	market price	model price	% diff
980903	11.250	150215	164.6250	159.2477	3.38%
980910	11.250	150215	167.9063	158.6218	5.85%
980917	11.250	150215	167.2500	163.3068	2.41%
980924	11.250	150215	168.3438	163.2337	3.13%
981001	11.250	150215	171.7188	163.7861	4.84%
981015	11.250	150215	169.3438	165.2365	2.49%
981029	11.250	150215	168.2813	164.6084	2.23%
981203	11.250	150215	168.7813	162.1672	4.08%
981210	11.250	150215	169.3438	161.4943	4.86%
981217	11.250	150215	167.9688	161.1123	4.26%

Note that other than these 10 dates, the CTD bond is fitted perfectly. In order to mitigate the criticism of Jagannathan, Kaplin and Sun (2003), we must also examine the fitting performance of the second cheapest and the third cheapest. Figure 2 presents the fitting performance of the two-factor model (with the 3-month short rate and the CTD bond perfectly fitted). The percentage fitting error (theoretical price ÷ market price − 1) is plotted. The second CTD bonds are fitted very well. The average percentage error (APE) is 10 basis points in the sample period (1992 ~ 2000). The root mean square errors (RMSE) is 1.04%.

The fitting performance of the third CTD bonds presents a very different profile. The third CTD bonds are fitted substantially poorly in the sample period (1992 ~ 2000). As opposed to 10 basis points APE and 1.04% RMSE, the errors of fitting the third CTD bond are 26 basis points APE and 1.61% RMSE.

B. Futures Data and Results

Daily futures prices are obtained from the Chicago Board of Trade (CBOT) between January 2, 1987 and December 29, 2000 and the summary statistics starting 1992 are given in Table 2. In this study, we choose weekly (Thursday) prices from November 7, 1991 through November 2, 2000 (470 observations). These are March 1992 through December 2000 futures contracts. We perform our empirical study on weekly prices as opposed to daily prices for the following reasons: (i) the computation cost is too high for daily estimation; (ii) the computation of the cheapest bond requires the information of all the bonds in the term structure and hence daily data collection of the entire term structure is impossible; and (iii) most importantly, the end-of-month timing option value is too costly to estimate daily.

We select futures prices that have time to maturity between 6 weeks to 4½ months to assure that there is no overlap between two consecutive contracts. Also the futures contract with time to maturity between 6 weeks to 4½ months is the most liquidly traded one. Note that the sudden drop in the futures price for the March 2000 contract is due to the change of the discount rate in the conversion factor (from 8% to 6%). In this study, we only report the empirical result between 1992 and 2000 using our estimates of the term structure model. The result using the Chen-Scott estimates for the period before 1992 are omitted due to space limitation and can be obtained upon request.

The main result of the paper is the comparison of the model price and the market price of the Treasury bond futures. The model price is computed by a two-factor CIR model with the quality and the end-of-month delivery options. The difference is plotted in Figure 3 (market price minus model price). We observe a few interesting results. First, the pricing errors are bigger when the times to maturity are longer. This is intuitive because as the contract approaches maturity, the uncertainty is smaller and the model predicts the price more accurately. Secondly, there is no specific pattern of errors for the in-sample period (1992 ~ 1998) and yet the model severely underestimates the market value for the out-of-sample period (1999 ~ 2000). Finally, the period that shows the most substantial errors (second half of 1998) is when the yield curve demonstrates the most curvature and the model provides the poorest fit (see Figure 2 and the previous section for a discussion).

A summary result is presented in Table 3. In this table, we group the prices by their contract month which is shown in the first column. The second column contains the number of observations (weekly) for the corresponding contract. For example, the contract that expires in March 1992 (i.e. 9203) has been collected 13 weekly historical prices from 11/07/1991 till 1/30/1992. Similarly, the June 1992 contract has 13 weekly prices from 2/6/1992 till 4/30/1992. The remaining columns present average market price of each contract, the average model price after taking into account the quality option with *risk-neutral* expectation valuation (no recursive algorithm), the average model price after taking into account the quality option with the *forward* expectation valuation (with recursive algorithm), and finally the average model price after taking into account of the end-of-month timing option.

In their seminal paper, Cox, Ingersoll, and Ross (1981) show that the forward price of an interest derivative must be strictly lower than the futures price of the same security.[13] The Treasury bond futures price is a futures price with a short exchange option. Under equation (3) which is a "risk neutral expectation" evaluation, the Treasury bond futures price is a futures price of a chosen bond with a short exchange option under the risk neutral expectation. Under

[13] See page 332 of the paper.

equation (7) which is a "forward expectation" evaluation, the Treasury bond futures price is a forward price of a chosen bond with a short exchange option under the forward expectation. According to Cox, Ingersoll, and Ross, both terms in equation (3) are greater than their counterparts in equation (7) and yet the difference of the second terms (exchange option) is larger than the difference of the first terms, resulting in a less value in equation (7) than in equation (3). Our numerical result confirms the observation made by Cox, Ingersoll, and Ross.

Indeed, equation (7) yields uniformly lower prices than equation (3). In the entire sample period, on average, the difference is roughly 1% which is larger than the end-of-month timing option value which is 23 basis points. This result demonstrates how the literature overestimates the end-of-month timing option value simply because it underestimates the quality option value. The quality option value should be computed by equation (7) via a forward expectation as opposed to equation (3).

Next we compare the model values with the market values. We find that the model underestimates the market value. However, this is due to largely the underpricing of the second half of 1998 where the two-factor CIR model does not calibrate well to the yield curve. We find that the model underestimates the market value by roughly 50 basis points. Three contracts 9809, 9812, and 9903 alone contribute 30 basis points of the underpricing and the rest 33 contracts contribute to only 20 basis points. This reveals the importance of the calibration of the term structure model to the yield curve.

Our final observation of the result is that the out-of-sample performance is substantially worse than the in-sample performance. The 8 out-of-sample contracts yield 213 basis points error while the in-sample contracts yield only less than 28 basis points. Even without the troubling 3 contracts the out-of-sample period still shows an error of 132 basis points. This demonstrates an important fact that the entire shape of the yield curve is important in pricing Treasury bond futures. Remember in the out-of-sample period, the cheapest-to-delivery bond is still perfectly calibrated (except for the troubling three contracts). Yet the model futures prices under-represent the market much more severely than the in-sample period. As a result, we learn that if the parameters of the model are not updated frequently, even though the underlying bond is priced perfectly, the Treasury bond futures contract can be significantly mispriced.

VI CONCLUSION

In this paper we first demonstrate that the literature mis-characterizes the values of the quality and end-of-month delivery options. The literature under-estimates the quality option value and over-estimates the end-of-month timing option value. This is because the past research computes the Treasury bond futures price as a risk neutral expectation of the cheapest-to-deliver bond price (as equation (2) demonstrates). We show in this paper that the Treasury bond futures price is in fact a forward price (as equation (7) demonstrates). This increases the value of the quality option substantially (lower the value of the Treasury bond futures).

We also compute the end-of-month timing option value via a two-dimensional lattice proposed by Hull and White (1990a). We find that the timing option value to be small.

The model performs very well in overall fitting. The average futures price over the entire sample period is 109.94 and the model value is 109.44. The model in general under-estimates the market price. The in-sample average model price is 111.31 versus the average market price of 111.68. The out-of-sample performance is worse. The average model price is 94.48 versus the average market price of 96.10. We discover two important results. First, we find that the calibration of the model to the underlying cheapest-to-deliver bond is important. We identify three contracts that are severely mispriced due to the failure of the calibration to the underlying bond. Second, we find that the fitting of the overall yield curve is also important. This can be seen from the in-sample versus out-of-sample performance comparison. In the out-of-sample period, the underlying cheapest-to-deliver bond is fitted perfect and yet the pricing error is still much larger than that of the in-sample period.

APPENDIX

From Theorem 1, we have:

$$\text{(A1)} \qquad \begin{aligned} E_t^Q\left[\delta(t,T)X(T)\right] &= E_t^Q[\delta(t,T)]E_t^{F(T)}[X(T)] \\ &= P(t,T)E_t^{F(T)}[X(T)] \end{aligned}$$

where δ is strictly less than 1. Due to the risk neutral pricing result we have, the LHS must equal $X(t)$, and hence:

$$\text{(A2)} \qquad X(t) = \frac{E_t^{F(T)}[X(T)]}{P(t,T)}$$

Note that the forward measure is maturity dependent. Clearly, the Radon-Nikodym Derivative (RND) is:

$$\text{(A3)} \qquad \eta(t,T) = \frac{\delta(t,T)}{P(t,T)}$$

Since the measure is T-dependent, so should be the RND (usually, RND is just $\eta(t)$.) Let the interest rate process be:

(A4) $$dr(t) = \hat{\mu}(r,t)dt + \sigma(r,t)dW^Q(t)$$

Applying Ito's lemma,

$$\begin{aligned}
0 = \ln P(T,T) &= \ln P(t,T) + \int_t^T \frac{1}{P(u,T)}\left[P_u(u,T)du + P_r(u,T)dr + \frac{1}{2}P_{rr}(u,T)(dr)^2\right]d\hat{W}(u) \\
&- \int_t^T \frac{1}{2}\left[\frac{\sigma(r,u)P_r(u,T)}{P(u,T)}\right]^2 du \\
&= \ln P(t,T) + \int_t^T \frac{1}{P(u,T)}\left[P_u(u,T)du + P_r(u,T)\hat{\mu}(r,u) + \frac{1}{2}P_{rr}(u,T)\sigma(r,u)^2\right]du \\
&+ \int_t^T \frac{1}{P(u,T)}P_r(u,T)\sigma(r,u)d\hat{W}(u) - \int_t^T \frac{1}{2}\left[\frac{\sigma(r,u)P_r(u,T)}{P(u,T)}\right]^2 du \\
&= \ln P(t,T) + \int_t^T r(u)du + \int_t^T \frac{1}{P(u,T)}P_r(u,T)\sigma(r,u)d\hat{W}(u) - \int_t^T \frac{1}{2}\left[\frac{\sigma(r,u)P_r(u,T)}{P(u,T)}\right]^2 du
\end{aligned} \tag{A5}$$

Letting:

(A6) $$\theta(t,T) = -\frac{\sigma(r,t)P_r(t,T)}{P(t,T)}$$

and moving the first two terms to the left:

$$\begin{aligned}
-\int_t^T r(u)du - \ln P(t,T) &= \int_t^T -\theta(u,T)d\hat{W}(u) - \int_t^T \frac{1}{2}\theta(u,T)^2 du \\
\frac{\delta(t,T)}{P(t,T)} = \eta(t,T) &= \exp\left(\int_t^T -\theta(u,T)d\hat{W}(u) - \int_t^T \frac{1}{2}\theta(u,T)^2 du\right)
\end{aligned} \tag{A7}$$

This implies the Girsanov transformation of the following:

$$\begin{aligned}
W^{F(T)}(t) &= W^Q(t) + \int_t^T \theta(u)dt \\
&= W^Q(t) - \int_t^T \sigma(r,u)\frac{P_r(u,T)}{P(u,T)}du
\end{aligned} \tag{A8}$$

The interest rate process under the forward measure henceforth becomes:

(A9) $$dr(t) = \left[\hat{\mu}(r,t) + \sigma(r,t)^2 \frac{P_r(t,T)}{P(t,T)}\right] dt + \sigma(r,t) dW^{F(T)}(t)$$

Note that the forward measure is quite general. It does not depend on any specific assumption on the interest rate process.

REFERENCES

Arak, M. and L. Goodman, "Treasury Bond Futures: Valuing the Delivery Option," *Journal of Futures Markets*, 1987.

Benninga, S. and M. Smirlock, "An Empirical Analysis of the Delivery Option, Marking to Market, and the Pricing of Treasury Bond Futures," *Journal of Futures Markets*, V. 7, No. 3, p. 269-286, 1985.

Boyle, P., "The Quality Option and Timing Option in Futures Contracts," *Journal of Finance*, V. 44, No. 1, March, 1989.

Broadie, M. and S. Sundaresan, "The Pricing of Timing and Quality Options: An Application to Treasury Bond Futures Markets," Working Paper, 1987.

Carr, P. and R. Chen, "Valuing Bond Futures and the Quality Option," Working paper, Rutgers University, November, 1996.

Carr, P., "Valuing Bond Futures and the Quality Option," Working Paper, University of California at Los Angeles, 1988.

Chen, R. and L. Scott, "Maximum Likelihood Estimation of a Multi-Factor Equilibrium Model of the Term Structure of Interest Rates," *Journal of Fixed Income*, 1993.

Chen, R. and S-K. Yeh, "Analytical Bounds for American Option Prices," Journal of Financial and Quantitative Analysis, March, 2002.

Chowdry, B., "Pricing of Futures with Quality Option," Working paper, University of Chicago, December, 1986.

Cohen, H., "Isolating the Wild Card Option," *Mathematical Finance*, V. 5, No. 2, p. 155-165, April, 1995.

Cox, J., J. Ingersoll, and S. Ross, "The Relation Between Forward Prices and Futures Prices," *Journal of Financial Economics*, p. 321-346, 1981.

Cox, J., J. Ingersoll, and S. Ross, "A Theory of the Term Structure of Interest Rates," *Econometrica*, March, 1985.

Fleming, J., and R. Whaley, "The Value of Wildcard Options," *Journal of Finance*, V. 49, No. 1, p. 215-36, March 1994.

Gay, G. and S. Manaster, "The Quality Option Implicit in Futures Contracts," *Journal of Financial Economics*, V. 13, p. 353-370, 1984.

Gay, G., and S. Manaster, "Implicit Delivery Options and Optimal Delivery Strategies for Financial Futures Contracts," *Journal of Financial Economics*, V. 16, p. 41-72, 1986.

Heath, D., R. Jarrow, and A. Morton, "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation," *Econometrica*, V. 60, No. 1, pp. 77-105, 1992.

Hegde, S., "An Empirical Analysis of Implicit Delivery Options in the Treasury Bond Futures Contract," *Journal of Banking and Finance*, V. 12, No. 3, p. 469-492, 1988.

Hegde, S., An Ex Post Valuation of the Quality Option in Treasury Bond Futures Contract, *Journal of Banking and Finance*, V. 14, No. 4, p. 741-760, 1990.

Hemler, M., "The Quality Delivery Option in Treasury Bond Futures Contracts," *Journal of Finance*, V. 45, No. 5, December, 1990.

Hull, J., and A. White, "Valuing Derivative Securities Using the Explicit Finite Difference Method," *Journal of Financial & Quantitative Analysis*, V. 25, p. 87-100, 1990a.

Hull, J. and A. White, "Pricing Interest-Rate-Derivative Securities," *The Review of Financial Studies*, V. 3, No. 4, p. 573-592, 1990b.

Hull, J., *Options, Futures, and Other Derivative Contracts*, Prentice Hall, 5th ed., 2003.

Jagannathan, R., A. Kaplin, and S. Sun, "An Evaluation of Multi-Factor CIR Models Using LIBOR, Swap Rates, and Cap and Swaption Prices," *Journal of Econometrics*, V. 116, p. 113-146, 2003 (September-October).

Jamshidian, F., "An Exact Bond Option Formula," *Journal of Finance*, 1989.

Jamshidian, F., "Pricing of Contingent Claims in the One-Factor Term Structure Model," Working Paper, Merrill Lynch Capital Markets, 1987.

Kane, A. and A. Marcus, "Conversion Factor Risk and Hedging in the Treasury Bond Futures Market," *Journal of Futures Markets*, V. 4, No. 1, p. 55-64, 1984.

Kane, A. and A. Marcus, "The Quality Option in the Treasury Bond Futures Market: An Empirical Assessment," *Journal of Futures Markets*, V. 6, No. 2, p. 231-248, 1986a.

Kane, A. and A. Marcus, "Valuation and Optimal Exercise of the Wild Card Option in the Treasury Bond Futures Market," *Journal of Finance*, V. 41, No. 1, p. 195-207, 1986b.

Kilcollin, T., "Difference Systems in Financial Futures Markets," *Journal of Finance*, V. 37, 1982.

Langetieg, T., "A Multivariate Model of the Term Structure," *The Journal of Finance*, V. 35, No. 1, p. 71-97, 1980.

Livingston, M., "The Delivery Option on Forward Contracts," *Journal of Financial and Quantitative Analysis*, V. 22, March, 1987.

Longstaff, F., and E. Schwartz, “Interest Rate Volatility and the Term Structure: A Two-Factor General Equilibrium Model,” *Journal of Finance*, V. 47, No. 4, p. 1259-1282, 1992.

Margrabe, W., “The Value of an Option to Exchange One Asset for Another,” *Journal of Finance*, V33, p.177-186, 1978.

Ritchken, P. and L. Sankarasubramanian, “Pricing the Quality Option in Treasury Bond Futures,” *Mathematical Finance*, V. 2, No. 3, p. 197-214, July, 1992.

Stulz, R., “Options on the Minimum or the Maximum of Two Risky Assets: Analysis and Applications,” *Journal of Financial Economics*, V. 10, p. 161-185, 1982.

Turnbull, S., and F. Milne, “A Simple Approach to Interest-Rate Option Pricing,” *The Review of Financial Studies*, V. 4, No. 1, p. 87-120, 1991.

Vasicek, O., “An Equilibrium Characterization of the Term Structure,” *Journal of Financial Economics*, November, 1977.

Table 1: Parameter Estimates of the Two-factor Cox-Ingersoll-Ross Model

	Chen-Scott Estimation					New Estimation			
	factor 1	std.err.	factor 2	std.err.		factor 1	std.err.	factor 2	std.err.
α	1.834100	0.222800	0.005212	0.115600	α	0.879967	0.001014	0.004423	0.000014
μ	0.051480	0.005321	0.030830	0.683300	μ	0.043822	0.000009	0.029555	0.000097
σ	0.154300	0.005529	0.066890	0.002110	σ	0.097855	0.001429	0.095974	0.000018
ς	-0.125300	0.180600	-0.066500	0.115400	ς	-0.146140	0.000151	-0.178846	0.000361
	likelihood function = 7750.82					likelihood function = 11722.81			
	# of obs. 470					# of obs. 416			

Note:

Chen-Scott estimates are taken from Exhibit 2, Panel B on page 21 of Chen and Scott (1993) who take Thursday weekly prices of 13-week, 26-week, 5-year, and longest maturity Treasuries. The period of study is January 1980 to December 1988. The new estimates use Friday weekly T-Bill rates of 3 months and 6 months and CMT rates of 5 years, and 30 years. The period of study is January 1991 to December 1998. The new estimates are estimated with RATS where the number of usable observations in the estimation is 387.

Table 2: Summary Statistics of Daily Futures Prices

Contract Month	N	Mean	Std. Dev	Min	Max
All maturities	3537	103.69	11.4274	77.78	134.66
9203	62	101.25	2.106	97.78	105.25
9206	62	98.94	0.8243	97.28	100.31
9209	64	100.79	2.0015	97.31	105.16
9212	64	104.46	1.1772	102.31	106.91
9303	61	103.9	1.7221	100.28	107.22
9306	64	109.79	1.9499	105.69	112.66
9309	64	112.3	2.3068	108.44	115.97
9312	64	118	2.83	102.63	121.94
9403	62	110.33	0.9456	113.34	117.44
9406	64	108.69	3.4739	103.25	115.34
9409	64	103.02	1.2617	100.31	105.44
9412	64	100.08	1.9172	97.06	103.81
9503	60	98.64	1.5588	95.44	101.47
9506	64	103.57	1.4609	100.5	106.31
9509	64	112.34	2.2275	106.97	115.75
9512	61	113.51	2.6347	108.69	117.44
9603	63	119.33	1.4683	116.75	121.56
9606	65	112.94	3.5877	106.75	120.22
9609	62	108.1	1.0822	105.88	111.84
9612	62	109.72	1.7013	106.41	113
9703	59	113.06	1.9931	109.78	120.06
9706	61	109.45	1.9678	106.63	113.44
9709	62	111.83	0.3393	108.31	116.75
9712	62	114.62	1.7645	112.06	118.47
9803	59	120.13	1.8508	117.03	123.72
9806	61	120.56	0.8548	118.66	122.44
9809	63	122.1	1.3907	118.88	124.16
9812	62	127.7	2.8354	122.97	134.66
9903	58	127.75	1.4533	124.72	130.63
9906	64	121.89	1.4661	119.47	126.19
9909	63	116.3	1.4175	113.63	119.38
9912	62	113.38	1.2839	110.84	116.16
0003	60	92.38	1.9398	89.22	95.66
0006	63	95.8	1.7785	92.47	99.34
0009	63	96.46	1.8667	92.66	99.38
0012	62	99.54	0.8819	97.63	101.22

Note:

Daily futures prices between 1/2/1992 and 12/31/2000 are taken with maturity between 6 weeks and 4½ months for each contract. Such a selection enjoys high liquidity and rare overlapping between contracts.

Table 3: Empirical Result of the Quality and the End-of-month Options

contract month	# of obs.	Quoted Futures Price	Q.O. - fut	Q.O.-fwd	E.O.M.
9203	13	101.216	102.959	101.594	101.467
9206	13	98.875	102.161	101.231	101.021
9209	13	100.930	103.408	102.041	101.924
9212	13	104.577	106.505	105.482	105.271
9303	13	103.926	106.158	104.976	104.759
9306	13	110.099	111.952	110.638	110.41
9309	13	112.274	112.481	111.313	111.068
9312	13	118.125	118.106	116.968	116.714
9403	13	115.250	116.262	114.708	114.435
9406	13	108.777	110.551	109.441	109.2
9409	13	103.132	104.899	103.711	103.484
9412	13	100.277	103.04	102.799	102.584
9503	13	98.438	98.8111	97.7802	97.5432
9506	13	103.394	105.749	104.506	104.244
9509	14	112.212	113.245	112.576	112.314
9512	12	113.485	115.895	114.164	113.842
9603	14	119.299	118.216	117.756	117.477
9606	13	112.681	115.37	113.624	113.321
9609	13	108.375	110.081	109.328	109.06
9612	13	109.630	112.551	110.975	110.681
9703	13	112.834	113.692	113.061	112.774
9706	13	109.301	112.319	110.749	110.453
9709	13	111.875	112.642	111.931	111.662
9712	13	114.690	116.781	114.958	114.653
9803	13	120.329	119.323	118.996	118.679
9806	13	120.625	121.672	119.524	119.208
9809	13	122.120	120.582	120.238	119.934
9812	13	127.772	124.971	122.746	122.428
9903	13	127.916	123.796	123.731	123.418
9906	13	121.709	122.628	120.398	120.081
9909	13	116.298	116.12	115.521	115.237
9912	13	113.397	114.498	112.764	112.459
0003	13	92.378	92.3245	91.6551	91.4675
0006	13	95.856	95.1406	94.3988	94.2028
0009	14	96.574	96.0491	95.2603	95.0649
0012	13	99.606	98.272	97.3958	97.1698
all maturities	470	109.940	110.811	109.693	109.436

Note: The theoretical values are computed using the new estimates (right panel of Table 1). The period of 92 ~ 98 is in-sample and of 99 ~ 00 is out-of-sample (shaded). The futures price with the quality option valued as the risk neutral expectation (Q.O.-fut; equation (2)) is reported in the 4th column. The futures price with the quality option valued as the forward expectation

(Q.O.-fwd; equation (7)) is reported in the 5th column. The future price with both quality option and the end-of-month timing option (E.O.M.) is reported in the last column.

Figure 1: Yield Curves for the Selected Period

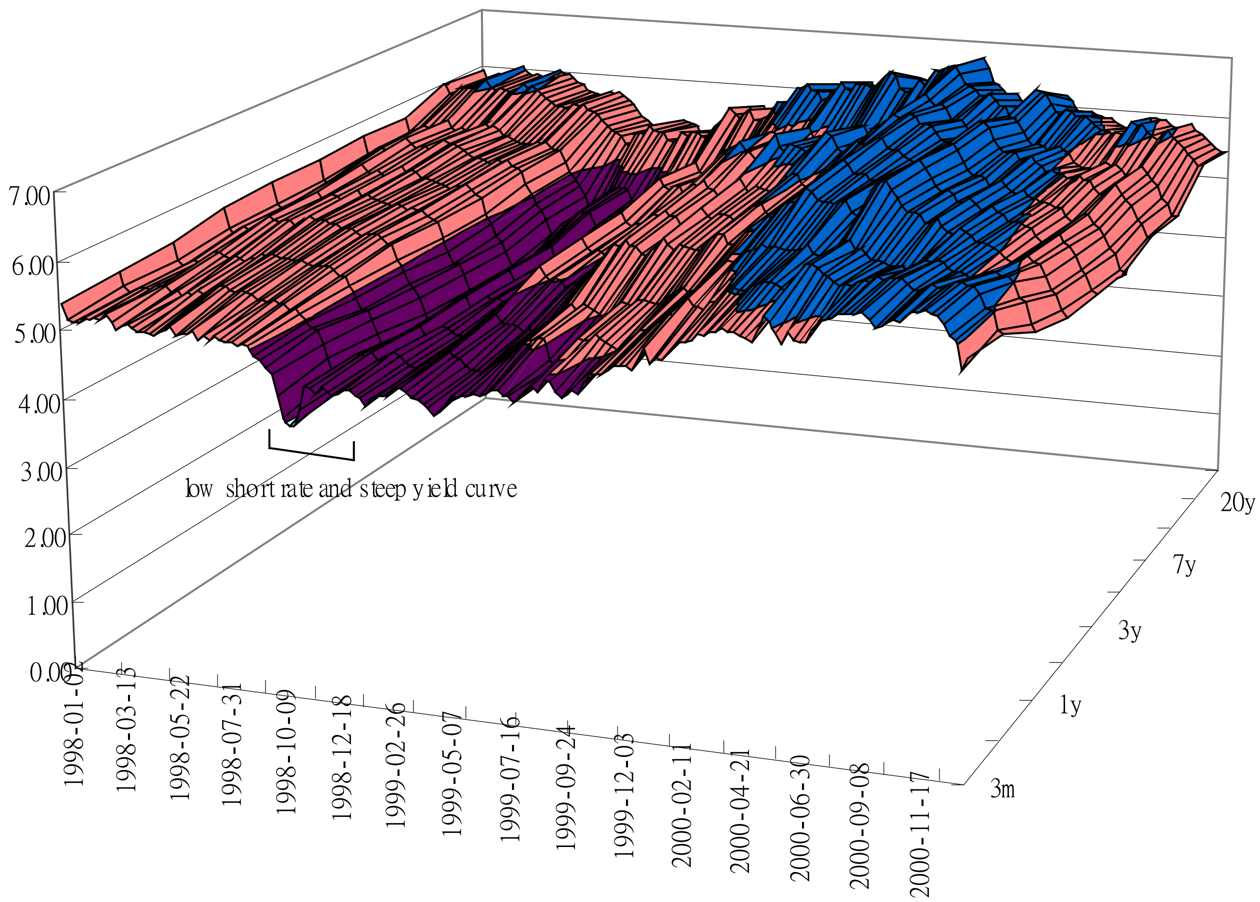
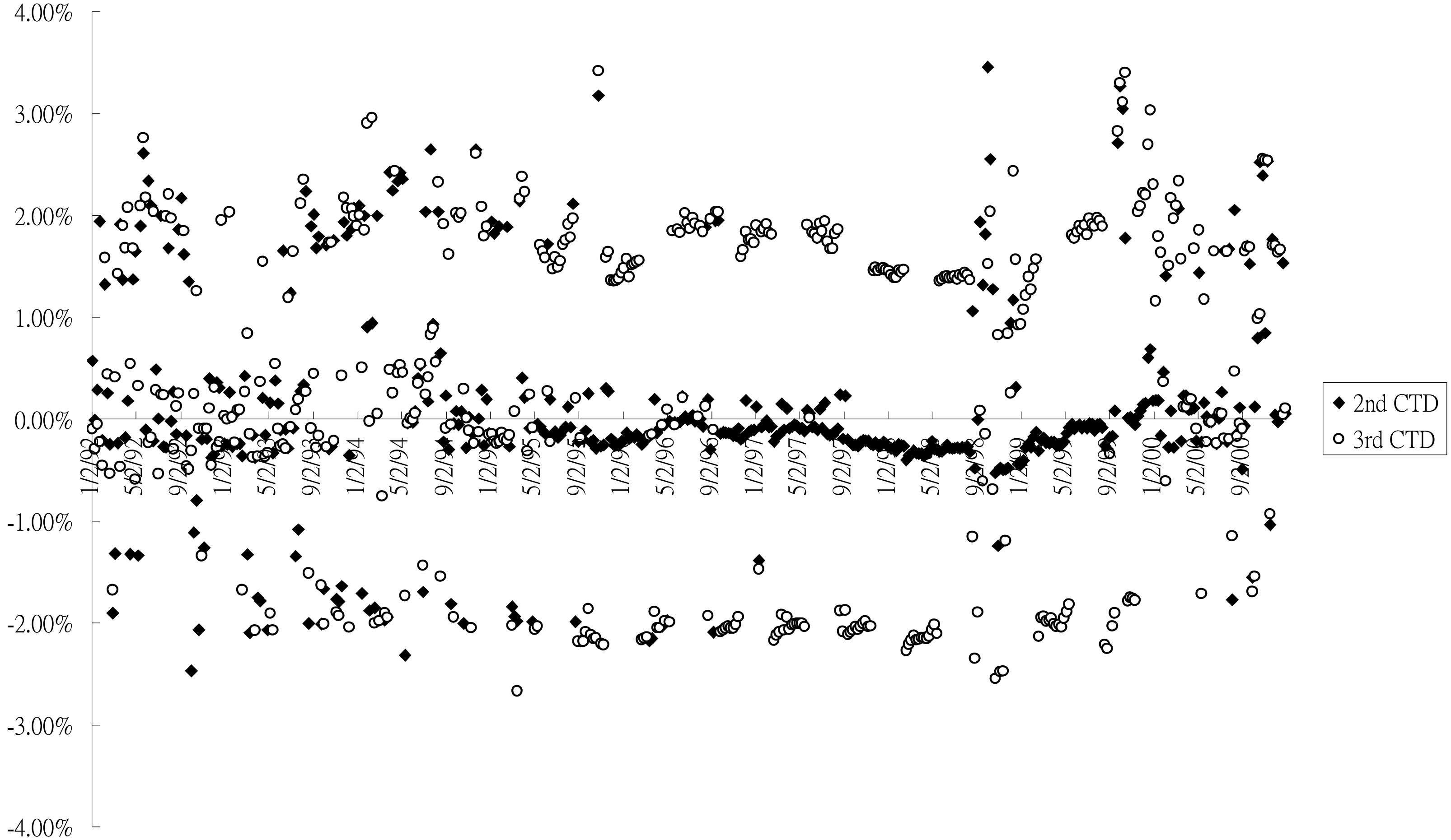


Figure 2: Fitting Performance Of The Second And Third Cheapest-to-deliver Bonds

Contracts 3/92 ~ 12/00



Note: The pricing error is measured as percentage error of the market price: model price ÷ market price − 1. The average percentage errors are 10 basis points and 26 basis points for the 2nd CTD and 3rd CTD respectively. The root mean square errors are 1.04% and 1.61% respectively.

Figure 3: Weekly Time Series the Difference between the Actual Futures Prices and the End-of-month Model Prices

Contracts 3/92 ~ 12/00