# ANALYTICAL BOUNDS FOR TREASURY BOND FUTURES PRICES

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**ABSTRACT**

The pricing of the delivery options, timing options particularly, in Treasury bond futures are prohibitively expensive. A recursive use of the lattice model is unavoidable for valuing such options, as Boyle (1989) demonstrates. As a result, this paper derives an upper bound and a lower bound for Treasury bond futures prices. We first show that the popular preference-free, closed form cost of carry model is an upper bound for the Treasury bond futures price. Then, we derive analytical lower bounds for the futures price under one and two-factor Cox-Ingersoll-Ross models of the term structure. These bounds are then tested empirically with weekly futures prices for the period from January 1987 till December 2000.

# I. INTRODUCTION

The delivery options in Treasury bond futures are generally known as the quality option and three timing options. The *quality* option gives the short the right to deliver any eligible bond (no less than 15 years to maturity or first call) and various timing options give the short the flexibility of making the delivery decision any time in the delivery month. The *end-of-month timing* option refers to the deliveries occurring at the last 7 business days in the delivery month when the futures market is closed to trading. For the remaining about 15 business days of the delivery month, the *wild card timing* option refers to the period from 2:00 p.m. to 8:00 p.m. (Chicago time) every day when the futures market is closed but the bond market is open while the *accrued interest timing* option refers to the period from 7:20 a.m. to 2:00 p.m. when both futures and its underlying bond markets are open.

Delivery options in T bond futures are difficult to price. A recursive use of the lattice model is unavoidable for valuing such options, as Boyle (1989) demonstrates, in that the futures price is effectively a forward price. Furthermore, as we shall demonstrate later, the wild card timing option is actually a compound forward price – one on top of the other, which cannot be priced precisely without a multi-recursive system. As a result, an accurate valuation of these delivery options is very expensive. The goal of this study is therefore to derive fast bounds for the T bond futures price. These bounds can be computed quickly and provide a crude conservative estimate for the T bond futures price.

An early discussion of the valuation of the quality option appears in Cox, Ingersoll, and Ross (1981) in which they state that their valuation can be applied to futures with the quality option when the single spot bond price is replaced with the minimum from the deliverable set. Hemler (1988) uses Margrabe's (1978) exchange option formula to price the quality option but the pricing formula becomes intractable as the number of deliverable bonds increases. Carr (1988) was the first to use factor models to price the quality option and Carr and Chen (1996) extend the Carr model to include a second factor. Ritchken and Sankarasubramanian (1992) use the Heath-Jarrow-Morton (1992) framework to find the quality option value. Livingston (1987) analyzes the quality option on the forward contract.

Timing options in general have no closed form solutions and are therefore studied with lattice methods. Kane and Marcus (1986a) lay out a general framework for analyzing the wild card option. In their analysis, discounting is not considered in the wild card period. Broadie and Sundaresan (1987) develop a lattice model to value the end-of-month option. Their focus is strictly on the futures price in the end-of-month period. Boyle (1989) uses a two-period model to show that the timing option could have a significant impact. His analysis assumes constant interest rates and does not directly apply to T bond futures.

Empiricists in general agree that the quality option has a non-trivial value.[1] However, unlike the evidence for the quality option, the evidence for the timing option is not so clear. This is because most studies do not distinguish between the quality option value and the value from the other timing options, let alone values among various timing options.[2]

The Treasury bond futures contract is one of the most liquid and widely traded interest rate derivative contract worldwide. The bid-ask spread is tight and the volume is large. Usually this is the market that practitioners use to calibrate the models they use to price other less liquid contracts. Hence, a pricing model that prices accurately both the quality and timing options must be derived in order to do such a task. However, as we shall demonstrate later in the paper, such a model is too expensive to be implemented since it involves a recursive search for the futures price at the beginning of the delivery month. In order to have a rough feel for the cost of computation of directly modeling the quality and timing options, we use a similar two factor Cox-Ingersoll-Ross model to the one we use in the paper to compute 6 futures prices. With a Dell Dimension 2400 with a Intel Celeron processor 2.4 GHz CPU, on average, it needs 9719.52 seconds (or 2.7 hours) per calculation under 102 steps. Clearly such high cost of computation is too expensive to be used in any realistic fashion.

In this paper, we derive several results regarding the lower and upper bounds for the futures price. First, we derive both bounds in a model-free format. We prove that the model-free upper bond is the cost of carry model, which is closed-form. The lower bond is in the format of an expectation. Since the bounds are model free, violating the bounds implies arbitrage profits. Secondly, with the two-factor Cox-Ingersoll-Ross model, we derive an analytical solution to the futures price with the quality option, which serves as a tighter upper bound for the Treasury bond futures price. Lastly, we derive an analytical lower bound for the Treasury bond futures price under the Cox-Ingersoll-Ross model. We then provide empirical results to show that these bounds are reasonably tight – about 2 ~ 3% up and below the futures price.[3]

The paper is organized as follows. The next section studies the quality option. We first study the quality option under continuous marking to market, or MTM (i.e. both futures and bond markets are open all the time). Then we show that the futures price with the quality

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[1] See, for example, Carr and Chen (1996), Kilcollin (1982), Benninga and Smirloc (1985), Kane and Marcus (1986b), and Hedge (1990).

[2] See, for example, Arak and Goodman (1987), Hedge (1988), Gay and Manaster (1986).

[3] These bounds are not to be violated, or arbitrage profits should take place. As it will become clear (in Section IV), in the case of the upper bound that is model free, a simple trading strategy can be formed to arbitrage against the violation (under perfect markets). In the case of the model-dependent lower bound, arbitrage profits exist only if the assumed model is correct.

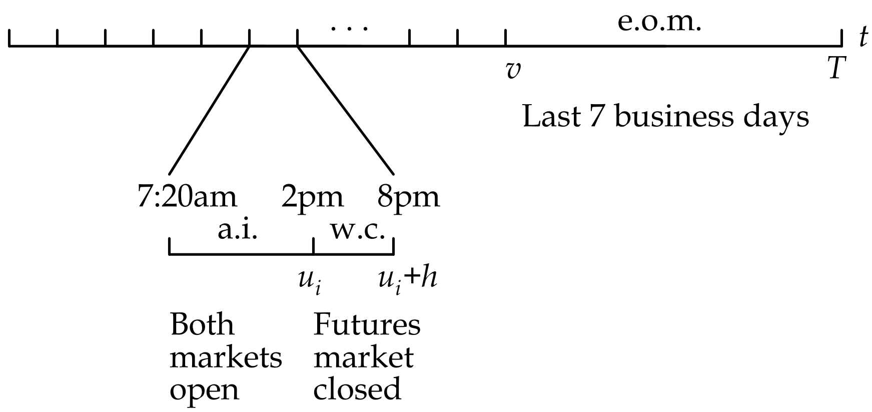
option is effectively a forward price when the futures market is closed but the bond market is open. Section III provides the theoretical analysis and derives lower and upper bounds for the futures price. We derive lower bounds for the futures price under both the quality option and the timing options. We then show that the preference-free cost of carry formula is an upper bound for the futures price. Section IV derives analytical formulas for the lower bound of the futures price (note that the cost of carry formula is model-free) under one and two-factor Cox-Ingersoll-Ross models. Section V contains an empirical study where a two-factor equilibrium term structure model is estimated under the Chen and Scott (1993) technique. Finally, the paper is concluded in Section VI.

## II THE QUALITY OPTION AND THE FUTURES PRICE

The delivery option that has the most economic value is the quality option that gives the short of the futures contract the right to choose the cheapest bond to deliver at the delivery date. Other delivery options that are embedded in T bond futures are known as the three timing options. The short can choose any time in the delivery month to make a delivery. The short can make a delivery even when the futures market is closed. At the end of the delivery month, for 7 business days, the futures market is closed but the short can still make a delivery. This is understood as the end-of-month timing option. For the remaining about 15 business days in the delivery month, the short can deliver either between 7:20 a.m. and 2:00 p.m. (Chicago time) when both the futures market and the underlying bond market are open or after 2:00 p.m. when the futures market is closed.[4] The former timing option is called accrued interest timing option and the latter timing option is also known as the daily wild card play. The following picture explains graphically various timing options.

[4] T bond market is an over the counter market that has no official closing time, even though market practice adopts 3:00 p.m. Eastern time as a symbolic closing time. The futures market allows the short up to 8:00 p.m. Eastern time to make the delivery announcement, and hence theoretically there is a 5-hour window for the wild card.

**DELIVERY MONTH**




The period of the last 7 business days is the end-of-month period. Throughout the paper we use $v$ for the starting time and $T$ for the ending time of this period. For the rest of the delivery month, there are two sections of each day, the accrued interest period and the wild card period. For a regular futures trading day $i$ between 7:20 a.m. and 2 p.m. Chicago time, both bond and futures markets are open simultaneously. The futures market closes at 2 p.m. but there is no official closing time for the bond market (while conventionally 3 p.m. Eastern time is marked as a symbolic closing time for the bond market.) Since the short has till 8 p.m. to make the delivery decision, the wild card period is defined over 2 p.m. ( $u_i$ ) to 8 p.m. ( $u_i + h$ ).

The notation and symbols used in the paper are also summarized as follows:

$\Phi(t) =$ "quoted" futures price with all delivery options

$\Phi^*(t) =$ futures price with the quality option and continuous marking to market

$\Phi^{**}(t) =$ futures price with the quality option at the absence of conti nuous MTM

$\overline{\Phi(t)} =$ upper bound

$\underline{\Phi(t)} =$ lower bound

$\Phi_i(t) =$ futures price of the $i$th quoted bond price

$\Psi_i(t) =$ forward price of the $i$th quoted bond price

$a_i(t) =$ accrued interest of the $i$th bond

$P(t,T) =$ discount bond price at time $t$ of \$1 at time $T$

$Q_i(t) =$ "quoted" coupon bond price of the $i$th bond

$q_i =$ conversion factor of the $i$th bond

$\delta(t,T) =$ random discount factor between $t$ and $T$

Note that under a specific model for the term structure (e.g. Vasicek or Cox-Ingersoll-Ross), the futures price of a specific bond can be priced in analytical form (see Section IV). Before we start our analysis, we need Jamshidian's separation theorem (1987) and his definition of the forward measure.[5]

*Theorem 1 (Forward Measure)*

*Let* $P(t,T)$ *be the price of a pure discount bond delivering* \$1 *at some future date and it follows the dynamics as:*

$$\frac{dP(t,T)}{P(t,T)} = r(t)dt + b(t,T)dW^{Q}(t)$$

*where* $r$ *is the instantaneous risk-free rate,* $b$ *is maturity dependent bond volatility, and* $dW^{Q}(t)$ *is the standard Wiener process defined under the risk-neutral space. Then the forward measure is defined as:*

$$\frac{dP(t,T)}{P(t,T)} = \left(r(t) - b(t,T)^2\right)dt + b(t,T)dW^{F(T)}(t)$$

*where* $dW^{F(T)}(t) = dW^{Q}(t) + b(t,T)dt$ *. Under this forward measure, all expected values taken will be forward prices, that is:*

$$\begin{aligned} E_t^Q[\delta(t,T)X(T)] &= E_t^Q[\delta(t,T)]E_t^{F(T)}[X(T)] \\ &= P(t,T)E_t^{F(T)}[X(T)] \end{aligned}$$

*where* $\delta(t,T) = \exp\left(-\int_t^T r(u)du\right)$ *and* $E_t^{F(T)}[X(T)]$ *computes the forward price of X.*

A simple proof of this theorem is given in an appendix although the original proof is available in Jamshidian (1987).

### *A. The Quality Option with Continuous Marking to Market*

In the absence of all timing options, the quality option gives the short the right to deliver the cheapest bond only at maturity, $T$, and the short receives the following payoff:

(1) $\max\left\{q_i\Phi(T) - Q_i(T)\right\}$

[5] Also see Hull (2003).

Note that the accrued interests of both bond and futures contracts are equal and canceled. Since the delivery value of (1) has to be identically 0 for all states, we can solve for the futures price at maturity as:

$$\Phi(T) = \min\left\{\frac{Q_i(T)}{q_i}\right\} \tag{2}$$

and today's futures price is merely a risk-neutral expectation of this payoff:

$$\begin{aligned}
\Phi^*(t) &= E_t^Q\left[\min\left\{\frac{Q_i(T)}{q_i}\right\}\right] \\
&= \frac{E_t^Q[Q_1(T)]}{q_1} - E_t^Q\left[\max\left\{\frac{Q_1(T)}{q_1} - \frac{Q_i(T)}{q_i}\right\}\right] \\
&= \frac{\Phi_1(t)}{q_1} - E_t^Q\left[\max\left\{\frac{Q_1(T)}{q_1} - \frac{Q_i(T)}{q_i}\right\}\right]
\end{aligned} \tag{3}$$

Note $\Phi_1(t) = E_t^Q[Q_1(T)]$ is the futures price of the first bond with no option and $\Phi^*(t)$ is the futures price of the cheapest bond at maturity. This result has been shown previously by Carr (1988) and other authors. This equation says that the futures contract with the quality option is equivalent to a futures contract without the quality option (only bond 1 is eligible for delivery) with an exchange option held by the short. With a specific term structure model, equation (3) becomes an analytical solution.[6]

### *B. The Quality Option with no Marking to Market When the Futures Market Is Closed*

Equation (3) is correct only if marking to market is applied continuously throughout the life of the futures contract. Unfortunately, in the last 7 business days of the delivery month, the futures market is not open and the futures contract is not marked to market. The futures price used for settlement in this period is the last settlement price at the beginning of the 7-day period. Since the futures price is already determined, the actual payoff at the last delivery day, $T$, is not necessarily 0. The short can actually gain or lose. To avoid arbitrage, the futures price at the

[6] For example, the closed form solution under the one-factor Cox-Ingersoll-Ross model can be found in Carr (1988).

beginning of the 7-day period should be set so that the expected present value of payoffs at maturity is 0. Under this circumstance, the futures price at the beginning of the 7-day period is a *forward price*, not a futures price. Formally, label the futures price as $\Phi^{**}(v)$ to represent the futures price at the beginning of the end-of-month period, $v$, should be so set that:

$$(4) \qquad E_v^Q[\delta(v,T)\max\{\Phi^{**}(v)q_i - Q_i(T)\}] = 0$$

where $\delta$ is the stochastic discount factor assumed to be strictly less than 1. Using Theorem 1, we can then rewrite (4) as:

$$(5) \qquad E_v^{F(T)}[\max\{\Phi^{**}(v)q_i - Q_i(T)\}] = 0$$

which can be expanded as follows:

$$\begin{aligned}
&0 = E_v^{F(T)}[\max\{\Phi^{**}(v)q_i - Q_i(T)\}] \\
(6) \qquad &0 = E_v^{F(T)}[\Phi^{**}(v)q_1 - Q_1(T) + \max\{\Phi^{**}(v)(q_i - q_1) - (Q_i(T) - Q_1(T)), 0\}] \\
&0 = \Phi^{**}(v)q_1 - \Psi_1(v) + E_v^{F(T)}[\max\{Q_1(T) - Q_i(T) - \Phi^{**}(v)(q_1 - q_i), 0\}]
\end{aligned}$$

and the futures price at time $v$ can be written as:

$$(7) \qquad \Phi^{**}(v) = \frac{\Psi_1(v)}{q_1} - \frac{1}{q_1} E_v^{F(T)}\left[\max\left\{Q_1(T) - Q_i(T) - K_i^{**}\right\}\right]$$

where $K_i^{**} = (q_1 - q_i)\Phi^{**}(v)$. Note that $\Psi_1(v) = E_v^{F(T)}[Q_1(T)]$ is the forward price of the first bond. The interpretation of this result is similar to that of (3), except that the risk neutral measure is replaced by the forward measure defined in Theorem 1 and the futures price becomes the forward price. However, unlike (3), the futures price at time $v$ has no easy solution, because it appears on both sides of the equation. This futures price has to be solved recursively using a numerical method. In a lattice framework suggested by Boyle (1989), we first choose an initial value for the futures price at time $v$, calculate payoffs at various states at maturity $T$, and then work backwards along the lattice. We adjust the futures price until the discounted payoff computed from the lattice is 0. Once the futures price at time $v$ is set, we can then travel back along the lattice and use the risk neutral probabilities till the end of the last wild card period, $u_n + h$. Then the similar procedure for the end-of-month period is repeated for the last wild card period to arrive at the futures price at the beginning of the wild card period $u_n$. Again, the risk neutral expectation is taken at $u_{n-1} + h$ and a recursive search is to compute the futures price at $u_{n-1}$. The process is repeated until the delivery month is over. Since the futures price becomes a forward price which cannot be obtained without a recursive search. The search for the "forward price" takes place at every node at all the times (i.e., $u_1$, $u_2$, $\cdots$, $u_n$, $v$). As a result, to compute the futures price with the quality option is prohibitively expensive.

With the presence of the end-of-month timing option, the futures price computed by (7) is an overestimate because the short has additional flexibility of choosing the best timing. If the short is allowed to deliver at any time in this 7-day period, then we need to compare the expected present value of future payoffs with the current delivery value. Higher current delivery value will trigger early deliveries. This is very similar to the American option pricing methodology where the intrinsic value is compared by the expected present value of future payoffs.

## III THE TIMING OPTIONS AND FUTURES PRICE BOUNDS

In the previous section, we see that under the end-of-month and a series of wild card periods, even the quality option alone is very complex to compute, let alone those timing options. In this section, we derive upper and lower bounds for these options in a general framework and analytical formulas are derived in the next section when a specific term structure model is chosen.

### *A. The Accrued Interest Timing Option*

The accrued interest timing option refers to the flexibility for the short to deliver the cheapest bond any time in the delivery month when both futures and spot markets are open. This is everyday from 7:20 a.m. to 2:00 p.m. (Chicago time) from the first day of the delivery month to right before the end-of-month period. Since the futures market is open, the futures contract is marked to market and deliveries can take place any time. As a result, the futures price can never be greater than the cheapest-to-deliver bond price. If the futures price were greater than the cheapest bond price, then deliveries would take place instantly. The short will sell the futures, buy the cheapest bond, make the delivery, and earn an arbitrage profit. Formally, for $t < v$,

(8) $$\Phi(t) > \min\left\{\frac{Q_i(t)}{q_i}\right\} \Leftrightarrow \max\left\{\Phi(t)q_i - Q_i(t)\right\} > 0$$

Therefore, the futures price in the period where both markets are open must be less than the cheapest-to-deliver bond price to avoid arbitrage. On the other hand, if the futures price is lower, one can long futures and short spot but the delivery will not occur because the short position of the futures contract will lose money if he makes a delivery. Consequently, the delivery will be postponed and there is no arbitrage profit to be made. If the futures price is always less than the cheapest-to-deliver bond price (adjusted by its conversion factor), the delivery payoff now is negative as opposed to 0 at the end. As a result, the short will never deliver until the last day. Consequently, the accrued interest timing option has no value. We restate this result in the following proposition.

*Proposition 1*

*The accrued interest timing option without the wild card and end-of-month options has no value.*[7]

□

The existence of the other timing options will lower the current futures price, further reducing the incentive for the short to deliver early. We state this result in the following Corollary.

*Corollary 1-1*

*The accrued interest timing option with the wild card and end-of-month options has no value.*

□

While the accrued interest timing option is worthless, the timing options at the end-of- month and the wild card periods are not. When the futures market is closed, there is no marking to market in the futures market and the futures contract becomes a forward contract. Boyle (1989) has demonstrated that in a case of forward contracts timing options will have value. We shall extend Boyle's analysis to stochastic interest rates so that we can evaluate T bond futures timing options.

### *B. The End-of-Month Timing Option*

Without the end-of-month timing option, we know that the futures price should be set according to (7). With the end-of-month timing option, deliveries can occur any time in the end-of-month period as long as the current delivery payoff is more than the present value of the expected payoff.

When both quality and timing options exist, the short makes a rational delivery decision when the immediate delivery value is higher than the expected discounted value should delivery

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[7] The name "accrued interest" comes in because in the delivery month, the bond price increases due to accrued interests. Here, $Q$ is a traded price that included accrued interests.

takes place later. This is like the early exercise of an American option. There is no closed form solution to compute American option prices. Precisely as Boyle (1989) has observed, the pricing of quality and timing options would need a lattice model.

To avoid arbitrage, today's futures price needs to be set so that the expected discounted payoff is nil. As a result, if we can identify a function that is always greater than both the delivery payoff and the discounted present value, this function is guaranteed to have a positive present value at time $v$. This is in spirit similar to the application in Chen and Yeh (2002). The trick is to identify a function that is always greater than the delivery value and the continuation value (ft: Continuation value is the value if it is not optimal to exercise (i.e. delivery). In the binomial model, the continuation value is the value at the node that reflects all possible exercises.)

We guess the function of the following, for $v < t < T$ :

(9)
$$E_t^Q\left[\max\left\{\frac{1}{\delta(t,T)}\Phi(v)q_i - Q_i(T)\right\}\right] > E_t^Q\left[\max\left\{\Phi(v)q_i - \delta(t,T)Q_i(T)\right\}\right]$$
$$> \max\left\{\Phi(v)q_i - Q_i(t)\right\}$$

where $\delta$ is the stochastic discount factor which is assumed to be strictly less than 1. This value is greater than the present value of the delivery payoff at any time $t \in [v,T]$. Equation (9) states that the upper bound is always greater than the exercise value of the futures contract. The last line is obtained as follows. Note that the martingale result states that: $E_t^Q[\delta(t,T)(Q_i(T)+a_i(T))] = Q_i(t)+a_i(t)$ , in other words, discounted market price of a bond should equal its current value, assuming there is no coupon in between $t$ and $T$.[8] Since the accrued interest is linear but discounting is not (i.e., $P(t,T)a_i(T) > a_i(t)$ ) it follows that $E_t^Q[\delta(t,T)Q_i(T)] < Q_i(t)$ but the difference is small.

Equation (9) shows that the proposed function is greater than the delivery value at any time. We can also show that the function has a higher value at an earlier time than at a later time. That is:

[8] If there is a coupon in between $t$ and $T$, we simply subtract the coupon value from the expected value.

$$
\begin{aligned}
& E_t^Q\left[\max\left\{\frac{1}{\delta(t,T)}\Phi(v)q_i - Q_i(T)\right\}\right] \\
(10) \qquad & > E_t^Q\left[\max\left\{\frac{1}{\delta(t+\Delta t,T)}\Phi(v)q_i - Q_i(T)\right\}\right] \\
& > E_t^Q\left[\delta(t,t+\Delta t)E_{t+\Delta t}^Q\left(\max\left\{\frac{1}{\delta(t+\Delta t,T)}\Phi(v)q_i - Q_i(T)\right\}\right)\right]
\end{aligned}
$$

It is seen that the proposed function is always greater than the delivery value and the discounted continuation value, it must be the case that it is an upper bound for the end-of-month period timing option value. Hence, at time $v$, the payoff should be positive:

$$
(11) \qquad E_v^Q\left[\max\left\{\frac{1}{\delta(v,T)}\Phi(v)q_i - Q_i(T)\right\}\right] > 0
$$

which can be expanded as follows:

$$
(12) \qquad
\begin{aligned}
& E_v^Q\left[\frac{1}{\delta(v,T)}\Phi(v)q_1 - Q_1(T) + \max\left\{\frac{1}{\delta(v,T)}\Phi(v)(q_i - q_1) - (Q_i(T) - Q_1(T))\right\}\right] > 0 \\
& E_v^Q\left[\frac{1}{\delta(v,T)}\right]\Phi(v)q_1 - \Phi_1(v) + E_v^Q\left[\max\left\{Q_1(T) - Q_i(T) - \frac{1}{\delta(v,T)}\Phi(v)(q_1 - q_i)\right\}\right] > 0
\end{aligned}
$$

This implies that the futures price should be bounded from below as follows:

$$
(13) \qquad
\begin{aligned}
\Phi(v) & > \frac{\Phi_1(v)\Delta(v,T)}{q_1} - \frac{\Delta(v,T)}{q_1}E_v^Q\left[\max\left\{Q_1(T) - Q_i(T) - \frac{1}{\delta(v,T)}K_i, 0\right\}\right] \\
& > \frac{\Phi_1(v)\Delta(v,T)}{q_1} - \frac{\Delta(v,T)}{q_1}E_v^Q\left[\max\left\{Q_1(T) - Q_i(T) - K_i, 0\right\}\right]
\end{aligned}
$$

where

$K_i = (q_1 - q_i)\Phi(v)$ and
$\Delta(v,T) = \frac{1}{E_v^Q[1/\delta(v,T)]}$

Note that the second inequality holds because $\delta$ is strictly less than 1. Therefore, the right hand side of the above equation is a lower bound. The lower bound for any time $t$, $\underline{\Phi(t)}$, is the risk neutral expectation of the above lower bound at time $v$:

$$\begin{aligned}\underline{\Phi(t)} &= E_t^Q\left\{\frac{\Phi_1(v)\Delta(v,T)}{q_1} - \frac{\Delta(v,T)}{q_1}E_v^Q\left[\max\left\{Q_1(T)-Q_i(T)-K_i,0\right\}\right]\right\} \\ &= \frac{\Phi_1(t)\Delta(v,T)}{q_1} - \frac{\Delta(v,T)}{q_1}E_t^Q\left[\max\left\{Q_1(T)-Q_i(T)-K_i,0\right\}\right]\end{aligned} \tag{14}$$

Note that $K_i$ is a function of $\Phi(v)$ which cannot be solved without a recursive search procedure, to arrive at an analytical lower bound, we replace this value with a closed form futures price $\Phi^*(v)$. We state this result in a following proposition.

*Proposition 2*

*The futures price under only the end-of-month timing option is bounded from below by the following risk neutral expectation:*

$$\frac{\Phi_1(t)\Delta(v,T)}{q_1} - \frac{\Delta(v,T)}{q_1}E_t^Q\left[\max\left\{Q_1(T)-Q_i(T)-K_i^*,0\right\}\right] \tag{15}$$

*where* $K_i^* = (q_1 - q_i)\Phi^*(v)$ *and* $\Phi^*(v)$ *is the futures price with only the quality option defined in equation* (3).

□

It is interesting to note that the end-of-month option has value even if there exists no quality option. When there is no quality option but the timing option is allowed, the delivery may occur early. The short always compares the delivery payoff $\Phi(v)q - Q(t)$ where $v < t < T$ with the expected present value of the delivery payoff at maturity. We can show that:

$$E_t^Q\left[\delta(t,T)(\Phi(v)q - Q(T))\right] > P(t,T)\Phi(v)q - Q(t) < \Phi(v)q - Q(t) \tag{16}$$

Since the direction of the inequality can go either way, it is likely that early deliveries can take place. This demonstrates that the timing option does have value even in the absence of the quality option. The difference between the first two terms in (16) is $P(t,T)a(T) - a(t)$ where $a$ is the accrued interest and the difference of the last two terms is $(1 - P(t,T))\Phi(v)$. As a result, whether or not deliveries will occur early depends upon which effect is larger. This result should not be confused with the result from Boyle (1989) where the timing option is defined differently.

### *C. The Wild Card Timing Option*

In addition to the end-of-month period where the futures market is closed but the bond market is open, there is a 6-hour period every day for about 15 days where the futures market is also closed. This is called the daily wild card timing option. The wild card option is different from the end-of-month option in that the futures market will reopen after each wild card period and the futures contract will be marked to market. If bond prices drop in the wild card period, given that the futures price is fixed, the short can benefit from delivering a cheaper bond. However, the short can equally benefit from the marking to market when the futures market reopens. As a result, the incentive for the short to deliver in the wild card period is minimal. Delivery can take place in a wild card period only when the payoff from immediate delivery exceeds the expected present value of marking to market on the next day.

We now proceed to derive the bound of the wild card option. For each daily wild card period, $(u, u+h)$, we define the following function as the upper bound of the delivery payoff (for $u < t < u+h$):

$$E_t^Q[\max\{\Phi(u)q_i - \delta(t, u+h)Q_i(u+h)\}] \tag{17}$$

This is an upper bound of the payoff because it is greater than (i) the payoff from immediate delivery:

$$\begin{aligned} E_t^Q[\max\{\Phi(u)q_i - \delta(t, u+h)Q_i(u+h)\}] &\geq \max\{\Phi(u)q_i - E_t^Q[\delta(t, u+h)Q_i(u+h)]\} \\ &> \max\{\Phi(u)q_i - Q_i(t)]\} \end{aligned} \tag{18}$$

where the second line is obtained by the fact that $E_t^Q[\delta(t,T)Q_i(T)] < Q_i(t)$ proved earlier and (ii) the discounted expected payoff from delivering at the end of the wild card period:

$$E_t^Q[\max\{\Phi(u)q_i - \delta(t, u+h)Q_i(u+h)] > E_t^Q[\delta(t, u+h)\max\{[\Phi(u)q_i - Q_i(u+h)]\}] \tag{19}$$

Hence, (19) is indeed an upper bound for the wild card option value, which is greater than 0:

$$
\begin{aligned}
&E_t^Q[\max\{\Phi(u)q_i - \delta(t,u+h)Q_i(u+h)\}] > 0 \\
&E_t^Q\left[\max\left\{\Phi(u) - \delta(t,u+h)\frac{Q_i(u+h)}{q_i}\right\}\right] > 0 \\
(20) \qquad &\Phi(u) - E_t^Q\left[\delta(t,u+h)\min\left\{\frac{Q_i(u+h)}{q_i}\right\}\right] > 0 \\
&\Phi(u) > P(t,u+h)E_t^{F(u+h)}\left[\min\left\{\frac{Q_i(u+h)}{q_i}\right\}\right]
\end{aligned}
$$

Note that $\min\left\{\frac{Q_i(u+h)}{q_i}\right\} \geq \Phi(u+h)$ when both markets are open from Section IIIA.[9] Therefore, $\Phi(u) > P(t,u+h)E_t^{F(u+h)}[\Phi(u+h)]$. This is no surprise because the end-of-month option will reduce the futures price prior to time $v$, which in turn will reduce the futures price at time $u+h$. Hence, it is

*Proposition 3*

*Given the futures price next morning (i.e., $\Phi(u_i+h)$) when the futures market re-opens at day $i+1$ (assuming continuous marking to market), The futures price prior to each wild card period (i.e., $\Phi(u_i)$) is bounded from below by:*

$$(21) \qquad P(u_i, u_i+h)E_t^{F(u_i+h)}[\Phi(u_i+h)]$$

*where $u_i$ is the beginning of a wild card period depicted on page 3 and $u_i+h$ is end of the wild card period (which is assumed to be the same as the time when the futures market re-opens next morning).*

□

***D. Putting It All Together for the Lower Bound***

So far, we have derived the lower bound for the futures price of the end-of-month period, $\Phi(v)$, and each of the wild card period, $\Phi(u)$, where $u$ represents the beginning time of any wild card period. The futures price of any given time, is a recursive calculation of (21). The easiest way to understand the calculation is to picture a univariate lattice model. The lower bound for the futures price at time $v$ is calculated by (15). We shall label it $\underline{\Phi(v)}$ for the lower bound at time $v$. Then, the regular risk neutral expectation is taken until the end of the last wild card period,

---

[9] Note that in the second line of (17) where $q_i$ is divided through is due to the fact that there exists a bond $i$ such that $\max\{\Phi(u)q_i - \delta(t,u+h)Q_i(u+h)\} > 0$ in all states.

$u_n + h$ where $u_n$ represents the beginning of the $n$-th (last) wild card period, is reached. The correct futures price, $\Phi(u_n + h)$, at this moment is unknown since it requires a repeated recursive process described in Section III. But we can replace it with the lower bound $\underline{\Phi(u_n + h)} = E^Q_{u_n+h}[\underline{\Phi(v)}]$. Then, we apply (21) to compute the lower bound at time $u_n$ to get $\underline{\Phi(u_n)} = P(u_n, u_n + h)E^{F(u_n+h)}_{u_n}[\underline{\Phi(u_n + h)}]$. Repeat this process through all the wild card periods, $u_{n-1}, u_{n-2}, \cdots, u_1$ to get $\underline{\Phi(u_1)} = P(u_1, u_1 + h)E^{F(u_1+h)}_{u_1}[\underline{\Phi(u_1 + h)}]$. Then the regular risk neutral expectation is taken to the current time: $\underline{\Phi(t)} = E^Q_t[\underline{\Phi(u_1)}]$. Repeated substitutions yield the following general result for the lower bound at the current time $t < u_1$,

$$
\begin{aligned}
\underline{\Phi(t)} &= E^Q_t[\underline{\Phi(u_1)}] \\
&= E^Q_t[\delta(u_1, u_1 + h)E^Q_{u_1+h}[\underline{\Phi(u_2)}]] \\
&= E^Q_t[\delta(u_1, u_1 + h)E^Q_{u_2}[\delta(u_2, u_2 + h)[\underline{\Phi(u_2 + h)}]] \\
&= \cdots \\
&= E^Q_t\left[\prod\nolimits_{j=1}^{n} \delta(u_j, u_j + h)\underline{\Phi(v)}\right] \\
&= E^Q_t\left[\prod\nolimits_{j=1}^{n} \delta(u_j, u_j + h)\left(\frac{\Phi_1(v)\Delta(v,T)}{q_1} - \frac{\Delta(v,T)}{q_1}\max_i \left\{Q_1(T) - Q_i(T) - K_i^*\right\}\right)\right]
\end{aligned}
\tag{22}
$$

The second line of the above equation is obtained by substituting the lower bound for $\underline{\Phi(u_1)}$ (i.e., $\underline{\Phi(u_1)} = E^Q_{u_1}[\delta(u_1, u_1 + h)[\underline{\Phi(u_1 + h)}])$ and the law of iterative expectations under the risk neutral measure. We summarize in a proposition:

*Proposition 4*

*The futures price is bounded from below by the following risk neutral expectation*

$$
\underline{\Phi(t)} = E^Q_t\left[\prod\nolimits_{j=1}^{n} \delta(u_j, u_j + h)\left(\frac{\Phi_1(v)\Delta(v,T)}{q_1} - \frac{\Delta(v,T)}{q_1}\max_i \left\{Q_1(T) - Q_i(T) - K_i^*\right\}\right)\right]
\tag{23}
$$

□

***E. The Cost of Carry Model – the Upper Bound***

After deriving the lower bound of the futures price, in the next proposition, we show that the cost of carry model provides an upper bound for the futures price. The well-known cost of carry formula is the following:

$$(24) \qquad \Phi_*(t) = \frac{\frac{Q_*(t)+a_*(t)}{P(t,T)} - a_*(T)}{q_*}$$

where $Q_*$, $q_*$, and $a_*$ are quoted price, conversion factor, and accrued interest of the cheapest bond at time $t$. Rearranging terms to get:

$$(25) \qquad \begin{aligned} \Phi_*(t) &= \frac{\frac{Q_*(t)+a_*(t)}{P(t,T)} - a_*(T)}{q_*} \\ &= E_t^{F(T)}\left[\frac{Q_*(T)}{q_*}\right] \\ &= E_t^{F(T)}\left[\min\left\{\frac{Q_i(T)}{q_i}\right\}\right] \\ &> E_t^{Q}\left[\min\left\{\frac{Q_i(T)}{q_i}\right\}\right] \end{aligned}$$

As we can see, the cost of carry model is equal to a forward expectation of the payoff. The futures price without the timing options is a risk-neutral expectation of the payoff (see (3)). The last inequality is obtained due to the following:

$$(26) \qquad \operatorname{cov}\left[\delta(t,T), \min\left\{\frac{Q_i(T)}{q_i}\right\}\right] > 0$$

This is easy to see because when $r$ increases (decreases), both discount factor, $\delta$, and all quoted bond prices, $Q_i$'s, decrease (increase), and the sign of the covariance is therefore positive. Note that the futures price without timing options is already an upper bound, the cost of carry model used by practitioners is a more conservative upper bound of the futures price. We state the result in the following proposition.

*Proposition 5*

*The futures price is bounded from above by the cost of carry model.*

$$(27) \qquad \overline{\Phi(t)} = \Phi_*(t)$$

□

It is generally believed that the futures price with the quality option (equation (3)) is the upper bound of the futures price, since it ignores the timing options. Indeed, if (3) can be evaluated accurately, it is a much tighter lower bound than the cost of carry model shown above. However, note that the cost of carry model is a “model free” result while (3) relies upon a specific term structure model. As a result, if the term structure model is not correctly specified, (3), may not serve the role of upper bound well. As we shall see in the empirical section, under a two-factor Cox-Ingersoll-Ross model, equation (3) does not always provide an upper bound. On the other hand, the violation of the cost of carry upper bound implies arbitrage opportunities.

## IV ANALYTICAL BOUNDS FOR EXPLICIT TERM STRUCTURE MODELS

In this section, we use the one- and two-factor Cox-Ingersoll-Ross (1985) models to demonstrate how one can calculate the upper bounds of the delivery options and the lower bound of the futures price analytically. Quoted coupon bond price should be equal to:

(28) $$Q(t) = \sum_{j=1}^{m} P(t, T_j) c_j - a(t) ,$$

Define additional notation $\Phi(t, T_i, T_j) = E_t^Q[P(T_i, T_j)]$ to be the futures price of a pure discount bond delivered at time $T_i$, and $\Psi(t, T_i, T_j) = P(t, T_i) / P(t, T_j)$ to be the forward price of a pure discount bond. These general results are independent of model assumption and of the number of factors.

### *A. Single-Factor Model*

For the sake of easy exposition and no loss of generality, we shall derive analytical lower bound for the futures price at time $v$ (beginning of end-of-month period). The lower bound at an arbitrary time $t$ can be derived similarly. Assume $Q_1 > Q_i$ for $i \neq 1$. We follow Carr (1988) that in a single factor model the whole distribution of $r$ can be partitioned into $n$ disjoint segments, denoted by $\Omega_i \equiv [r^*_{k(i)-1}, r^*_{k(i)})$ where $r^*_0 = 0$ and $r^*_n = \infty$, each of which represents a segment where $Q_i$ maximizes the payoff function: $\max\{Q_1 - Q_i - K^*_i, \}$. The analytic result of the expected value (taken at time $v$) of (15) is then derived as follows:

$$
\begin{aligned}
W_v &= E_v^Q\Big[\max\Big\{Q_1 - Q_i - K_i^*,\, 0\Big\}\Big] \\
&= \sum\nolimits_{i=2}^{n} \int\limits_{\Omega_i} \Big[\sum\nolimits_{j=1}^{b(1)} c_1 P(T,T_{1j}) - \sum\nolimits_{j=1}^{b(i)} c_i P(T,T_{ij}) - K_i^*\Big]\varphi(r)dr \qquad (29)\\
&= \sum\nolimits_{j=1}^{m_1} c_1 \int\limits_{\Sigma_{i=2}^{n}\Omega_i} P(T,T_{1j})\varphi(r)dr - \sum\nolimits_{i=2}^{n} \sum\nolimits_{j=1}^{b(i)} c_i \int\limits_{\Omega_i} P(T,T_{ij})\varphi(r)dr - K_i^*
\end{aligned}
$$

where $b(i)$ is the last coupon payment time for bond $i$, $K_i^* = (q_1 - q_i)\Phi^*(v)$, and $\Phi^*(v)$ is the futures price under continuous marking to market (defined by (3)). Note that in each region $\Omega_i$, bond $i$ maximizes the payoff $\max\{Q_1 - Q_i - K_i, 0\}$ and $\varphi(r)$ is the risk-neutral density of the interest rate.

Without the consideration of any wild card, the lower bound for the futures price at any arbitrary time $t$ is a risk neutral expectation of (29):

(30)

$$
\begin{aligned}
\underline{\Phi(t)} &= E_t^Q\left[\prod\nolimits_{j=1}^{n} \delta(u_j, u_j + h)\left(\frac{\Phi_1(v)\Delta(v,T)}{q_1} - \frac{\Delta(v,T)}{q_1}\max_i\Big\{Q_1(T) - Q_i(T) - K_i^*\Big\}\right)\right] \\
&= \int_{-\infty}^{\infty} dr(u_1)\int_{-\infty}^{\infty} dr(u_2)\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dr(u_n)\prod\nolimits_{j=1}^{n} \delta(u_j, u_j + h)\int_{-\infty}^{\infty} dr(v)\frac{\Phi_1(v)\Delta(v,T)}{q_1} \\
&\quad - \frac{\Delta(v,T)}{q_1}\int_{-\infty}^{\infty} dr(T)\max_i\Big\{Q_1(T) - Q_i(T) - K_i^*\Big\}\varphi(r(u_1),\cdots,r(T)) \\
&= \int_{-\infty}^{\infty} dr(u_1)\int_{-\infty}^{\infty} dr(u_2)\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dr(u_n)\prod\nolimits_{j=1}^{n} \delta(u_j, u_j + h)\int_{-\infty}^{\infty} dr(v)\frac{\Phi_1(v)\Delta(v,T)}{q_1} \\
&\quad - \frac{\Delta(v,T)}{q_1}\sum\nolimits_{i=2}^{n} \int_{\Omega_i} \Big[\sum\nolimits_{j=1}^{b(1)} c_1 P(T,T_{1j}) - \sum\nolimits_{j=1}^{b(i)} c_i P(T,T_{ij}) - K_i^*\Big]\varphi(r(u_1),\cdots,r(T)) \\
&= \int_{-\infty}^{\infty} dr(u_1)\cdots\int_{-\infty}^{\infty} dr(u_n)\prod\nolimits_{j=1}^{n} \delta(u_j, u_j + h)\int_{-\infty}^{\infty} dr(v)\frac{\Phi_1(v)\Delta(v,T)}{q_1} - \frac{\Delta(v,T)}{q_1}W_v\varphi(r(u_1),\cdots,r(v))
\end{aligned}
$$

where $W_v$ is defined in (29).

In the case of CIR, the interest rate process follows the square root process:

$$
dr = (\alpha\mu - (\alpha + \varsigma)r)dt + \sigma\sqrt{r}dW^Q \qquad (31)
$$

where $\alpha$ is the reverting speed, $\mu$ is the reverting level, $\sigma$ is the volatility parameter, and $\varsigma$ is the market price of risk which is constant under log utility. The futures price with only the quality option is in Carr (1988) as:

$$(32) \qquad \Phi^*(t) = \sum_{i=1}^{n}\sum_{j=1}^{b(i)} \frac{C_{ij}}{q_i}\Phi(t,T,T_{ij})\sum_{k=1}^{m} I_{ik}[\chi_j^2(r_k^*) - \chi_j^2(r_{k-1}^*)]$$

where

$$\Phi(t,u,v) = C(t,u,v)e^{-r(t)D(t,u,v)},$$

$$C(t,u,v) = A(u,v)\left(\frac{\eta(t,u)}{\eta(t,u)+B(u,v)}\right)^{2\alpha\mu/\sigma^2}$$

$$D(t,u,v) = \frac{B(u,v)\eta(t,u)e^{-(\alpha+\varsigma)(u-t)}}{\eta(t,u)+B(u,v)}$$

$$A(u,v) = \left(\frac{2\gamma e^{(\alpha+\varsigma+\gamma)(v-u)/2}}{(\alpha+\varsigma+\gamma)\left(e^{\gamma(v-u)}-1\right)+2\gamma}\right)^{\frac{2\alpha\mu}{\sigma^2}}$$

$$B(u,v) = \frac{2\left(e^{\gamma(v-u)}-1\right)}{(\alpha+\varsigma+\gamma)\left(e^{\gamma(v-u)}-1\right)+2\gamma}$$

$$\eta(t,u) = \frac{2(\alpha+\varsigma)}{\sigma^2\left(1-e^{-(\alpha+\varsigma)(u-t)}\right)}$$

$$\chi_j^2(r^*) = \chi^2[2\eta(t,u)r^*; \frac{4\alpha\mu}{\sigma^2}, 2\eta(t,u)re^{-(\alpha+\varsigma)(T_{ij}-t)}]$$

Note that $I_{ik}$ is the indicator function equal to 1 for the $i$-th bond and between the critical values of $r_{k-1}^*$ and $r_k^*$ and $\chi^2(x,y,z)$ is a non-central chi-square probability function with limit $x$, degrees of freedom $y$, and degrees of non-centrality $z$.

Under the CIR model, equation (29) becomes:

$$(33) \qquad \begin{aligned} W_v^{\text{CIR}} &= E_v^Q\left[\max\left\{Q_1 - Q_i - K_i^*,\, 0\right\}\right] \\ &= \left\{\sum_{i=1}^{n-1}\left[\sum_{j=1}^{m} c_1\Phi(v,T,T_{1j})\chi^2(\overline{r}_1) - \sum_{j=1}^{n} c_i\Phi(v,T,T_{ij})\chi^2(\overline{r}_i) - K_i^*\chi^2(\overline{r}_1)\right]\right\} \end{aligned}$$

and the lower bound under the CIR model of the term structure is still equation (30) but with $W_v^{\text{CIR}}$ replacing $W_v$.

Carr and Chen (1996) show that one-factor term structure models are incapable of pricing delivery options because all bonds are perfectly correlated and hence crossover strike rates are rarely identified. Hence, for this study, we adopt a two-factor Cox-Ingeroll-Ross model for the empirical work. We derive the analytical solution to the lower bound.

### *B. Two-Factor Model*

We use the two-factor model of the following kind:[10]

(34) $$r = y_1 + y_2$$

where each factor follows a square root process as in (31):

(35) $$dy_i = (\alpha_i \mu_i - (\alpha_i + \varsigma_i) y_i) dt + \sigma_i \sqrt{y_i} dW_i^Q$$

where $i = 1,2$ and $dW_1^Q dW_2^Q = 0$. Under this framework, the two-factor model works the same way as the one-factor models. The difference is that the univariate integrals in the one factor models are replaced with two dimensional integrals.

(36) $$\begin{aligned} \underline{\Phi(t)} = & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dr(u_1)\cdots \int \int_{-\infty}^{\infty} dr(u_n) \prod_{j=1}^{n} \delta(u_j, u_j + h) \\ & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dr(v) \frac{\Phi_1(v)\Delta(v,T)}{q_1} - \frac{\Delta(v,T)}{q_1} W_v^{\mathrm{CIR}(2)} \hat{\varphi}(r(u_1),\cdots,r(v)) \end{aligned}$$

where

$$\begin{aligned} W_v^{\mathrm{CIR}(2)} = & \left\{ \sum_{i=1}^{n-1} \left[ \sum_{j=1}^{m_1} c_1 \Phi(v,T,T_{1j}) \iint_{\Omega_i} \hat{\varphi}(y_1,y_2) dy_1 dy_2 \right.\right. \\ & \left.\left. - \sum_{j=1}^{m_i} c_i \Phi(v,T,T_{ij}) \iint_{\Omega_i} \hat{\varphi}(y_1,y_2) dy_1 dy_2 - K_i^* \iint_{\Sigma_{i=1}^{n-1}\Omega_i} \hat{\varphi}(y_1,y_2) dy_1 dy_2 \right] \right\} \end{aligned}$$

The bivariate integrals may become quadruple integrals as we move backwards in time. The lattice approach proposed by Longstaff and Schwartz (1992) can be efficiently implemented to calculate the result. Since the lower bound requires only risk neutral expectations, it can be computed without recursive loops and be extremely fast.

## V EMPIRICAL STUDY

In this section, we empirically examine the magnitude of each bound using a two-factor CIR model. We provide evidence for two non-overlapping periods: 1987 ~ 1991 and 1992 ~ 2000. In

[10] This two-factor model is adopted by a number of authors. See Chen and Scott (1993), Turnbull and Milne (1991), Langetieg (1980), Hull and White (1990).

each period, we perform both in-sample and out-of-sample tests. The results of two periods are similar, implying that the model is robust. Furthermore, for both periods, out-of-sample performance is pleasantly satisfactory.

### *A. Term Structure Model Estimation*

In estimating the two-factor CIR term structure model, we use weekly (Friday) four Treasury interest rate series: the 3-month and 6-month Treasury-bills and the 5-year and 30-year Constant Maturity Treasury (CMT) interest rates to estimate the parameters for the two-factor CIR model. The weekly data is from January 4, 1991 to December 29, 1998, which contains 416 observations in total. Data source is from the Aremos USFIN databank. The estimation procedure is identical to that described in Chen and Scott (1993). In addition to our estimates, as a robustness comparison, we also use the results from Chen and Scott (1993) who use a weekly data set from 1980 to 1988 and the estimates from both estimations are reported in Table 1. We can see that the estimates do not change much from one period to another, while the new estimates do show slightly lower reverting level and slower mean reversion. The first factor remains strong mean reversion while the second remains to be close to a random walk.

The term structure estimation must also estimate factor values. In Chen and Scott (1993), factor values are computed by fitting the long and short rates of the yield curve. For our purposes (that we need to price the cheapest-to-deliver bond correctly),[11] the factor values are solved for by matching the short rate and the cheapest-to-deliver bond price. In reality, the delivery options are priced off the cheapest-to-delivery bond and a series of exchange options to the next cheapest, the third cheapest, and so on. By calibrating the term structure model of Chen-Scott to the cheapest-to-delivery bond, we shall provide the most accurate valuation of the delivery options using the two-factor CIR model. It is generally understood that the two-factor CIR model does not fit the yield curve well.[12] In order to mitigate the concern of Jagannathan, Kaplin and Sun (2003), we must examine how good our term structure fit is for the set of deliverable bonds. We are not particularly concerned with the whole yield curve fit because the majority of the risk of the delivery options resides in the set of deliverable bonds. Furthermore, as a practical concern, we present the fitting performance of the three most relevant bonds – the cheapest, second cheapest, and third cheapest. The probability of other bonds become the cheapest is small and the impact of other deliverable bonds is believed to be negligible.

---

[11] T bond futures prices are affected by all bonds underlying the yield curve, and yet doubtlessly the cheapest-to-deliver bond has the most influence.

[12] See, for example, Chen and Scott (1993) and Jagannathan, Kaplin and Sun (2003).

Theoretically, the cheapest bond at any point in time should be fitted perfectly by tweaking the second factor, since there is one equation and one unknown. However, there is no solution to the second factor at the following dates when we try to fit the cheapest bond: 980903, 980910, 980917, 980924, 981001, 981015, 981029, 981203, 981210, and 981217. Figure 1 plots the yield curves for a sub-period (January 2, 1998 ~ December 28, 2000) from our CMT dataset. It can be seen that the above dates where the second factor fails to coincide (CTD bond fails to fit) with the period when the yield curve is steeply sloped and the short rates are small. This is a problem already described in Chen and Scott (1993). Chen and Scott recommend a three-factor model to improve the fit. However, due to the reality that this problem is only present for 10 out of 722 cases (252 observations in the first sub-period: 1987 ~ 1991 and 470 observations in the second sub-period: 1992 ~ 2000)[13] and the complexity of estimating a three-factor model, we decide to stay with the two-factor model.[14] Or alternatively, we can allow the first factor to be flexible until we are able to fit the CTD bond. But in order to maintain consistency, we allow the CTD bond to be not perfectly fitted for those 10 dates.[15] As it will be clear later, the ill-fitted CTD bonds for those 10 dates will hurt the tightness of the bounds. The following summary illustrates the cheapest bond that fails to be fitted and the difference between the market price and the model price.

| date | coupon | maturity | market price | model price | % diff |
|---:|---:|---:|---:|---:|---:|
| 980903 | 11.250 | 150215 | 164.6250 | 159.2477 | 3.38% |
| 980910 | 11.250 | 150215 | 167.9063 | 158.6218 | 5.85% |
| 980917 | 11.250 | 150215 | 167.2500 | 163.3068 | 2.41% |
| 980924 | 11.250 | 150215 | 168.3438 | 163.2337 | 3.13% |
| 981001 | 11.250 | 150215 | 171.7188 | 163.7861 | 4.84% |
| 981015 | 11.250 | 150215 | 169.3438 | 165.2365 | 2.49% |
| 981029 | 11.250 | 150215 | 168.2813 | 164.6084 | 2.23% |
| 981203 | 11.250 | 150215 | 168.7813 | 162.1672 | 4.08% |
| 981210 | 11.250 | 150215 | 169.3438 | 161.4943 | 4.86% |
| 981217 | 11.250 | 150215 | 167.9688 | 161.1123 | 4.26% |

[13] All 10 cases are in the second sub-period: 1992 ~ 2000.

[14] Chen and Scott (1993) argue that the three-factor model does not necessarily dominate the two-factor model in that the three-factor model, although fits better the term structure, generates extra volatility. See Chen and Scott for details.

[15] The result of the alternative fitting is available upon request.

Note that other than these 10 dates, the CTD bond is fitted perfectly. In order to mitigate the criticism of Jagannathan, Kaplin and Sun (2003), we must also examine the fitting performance of the second cheapest and the third cheapest. Figure 2 presents the fitting performance of the two-factor model (with the 3-month short rate and the CTD bond perfectly fitted). The percentage fitting error (theoretical price ÷ market price − 1) is plotted. The second CTD bonds are fitted very well. The average percentage error (APE) is 30 basis points in the first period (1987 ~ 1991) and 10 basis points in the second sub-period (1992 ~ 2000). The root mean square errors (RMSE) for both periods are 1.07% and 1.04% respectively. The numbers may seem to suggest that the second sample period provides a better fit, but by eyeballing the graphs we can see most of the time in the first sample period the second CTD is well fitted and only half of time in the second period is well fitted.

The fitting performance of the third CTD bonds presents a very different profile. In the first sample period (1987 ~ 1991), the third CTD bonds are fitted as well as the second CTD bonds but substantially poorly in the second sub-period (1992 ~ 2000). As opposed to 30 basis points APE and 1.07% RMSE for the second CTD bond, the APE and the RMSE for the third CTD bond are 14 basis points and 1.2% in 1987 ~ 1991. However, in the period of 1992 ~ 2000, the APE and RMSE grow from 10 basis points and 1.04% respectively for the second CTD bond to 26 basis points and 1.61% respectively for the third CTD bond. The worse fit of the third CTD bond and the 10 cases of unsuccessful fit of the CTD in the second sub-period might explain the slightly worse bound performance (show later) of the second period.

### *B. Futures Data*

Daily futures prices are obtained from the Chicago Board of Trade (CBOT) between January 1987 and December 2000. The summary statistics are given in Table 2. Note that the drop in the futures price in for March 2000 contract is due to the change of the discount rate in the conversion factor (from 8% to 6%). But for our study, the futures prices are collected weekly (Thursday) for two different (non-overlapping) periods. One is from January 8, 1987 through October 31, 1991 (252 observations) and the other is from November 7, 1991 through November 2, 2000 (470 observations). The first period, which covers the quarterly contracts of March 87 through December 1991, uses the Chen-Scott estimates and the second period, which covers contracts of March 1992 to December 2000 uses the new estimates. We select weekly futures prices that have 6 weeks to 4½ months to maturity from the CBOT daily price data set.

The cost of carry model requires the knowledge of all deliverable bonds at the trade date. We collect all deliverable bonds from the Wall Street Journal for all the trade dates. We use the average of the bid and ask for the bond price. We also use the three-month T bill rates for the cost of carry model. There are about 26 bonds for any given trade date. Conversion factors are computed by the CBOT formula.[16]

***C. Results***

We assume no gap between the close of the bond market for any given day and the open of the futures market in the next morning. As a result, in order to correctly date all the timing periods in the lattice, we have to count the number of trading days. As been pointed out previously, there are about 22 trading days in a month. The last 7 days attribute to the end-of-month period and each of the remaining 15 days has about 6 hours for the day period where both bond and futures markets are open and another about 6 hours for the night period where only bond market is open. In order to accurately calculate various timing option values, the time to maturity in this study is not measured by calendar days but by business days.[17] Accurate day count is necessary because we need to calculate expectations at various times.

We first rank all deliverable bonds by their conversion factors. We then choose the bond with the largest conversion factor as our primary bond to deliver and calculate its futures price using the two-factor version of the Cox-Ingersoll-Ross model (1981). The quality option represents the option for the short to exchange a cheaper bond for this bond at delivery. Various timing options give the short additional flexibility of choosing the best timing.

To calculate the upper bound value for the end-of-month option for any given time prior to $v$, we first need to calculate (36) and then use (14). As noted earlier, the wild card value can be ignored if the lower bound of the futures price at the beginning time of the end-of-month period, $v$, is already low enough. That is, if we use the lower bound for the end-of-month option to substitute for $\Phi(v)$, the loss of the wild card value is translated into the end-of-month option. In other words, we can efficiently incorporate the wild card value into the lower bound for the end-of-month option. If the wild card value is eliminated completely by this substitution, then the lower bound for the end-of-month option becomes a lower bound for both end-of-month and wild card options. As we shall see this is indeed the case for the periods we examine.

Finally, the cost of carry model of (24) is computed to compare with the futures price with only the quality option, i.e. (3), and the actual futures price.

The empirical examination of the upper and lower bounds is presented in two periods where the term structure model is separately estimated. The first period contains the futures

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[16] Hull (2003) has an excellent demonstration of such a computation.

[17] That is, we do the business day count between trade day and the last day of the delivery month and assume 252 trading days for a given year.

contracts from March 1987 to December 1991. The empirical results in this period use the parameter estimates of Chen and Scott (1993), which use data from January 1980 to December 1988 to estimate the term structure. Hence, contracts from March 1987 to December 1988 are considered in-sample and contracts from March 1989 to December 1991 are considered out-of-sample. The second period contains the futures contracts from March 1992 to December 2000. We re-estimate the parameters with the Treasury data from January 1991 till December 1998 to estimate the term structure and perform also in-sample and out-of-sample tests. We re-estimate the parameters because we observe that interest rates are significantly lower in the later period.

The first part of Table 3 presents results in averages for the 20 contracts (8703 through 9112) studied in the paper. The first 3 columns of Table 3 present actual futures prices, lower bound futures prices using (36) which considers only the end-of-month option, and upper bound futures prices using which is the cost-of-carry model. The average of the whole period is given at the bottom of the table. The cost of carry model is on average 2% higher than the actual futures price while the lower bound is 2% lower than the actual futures price. Weekly prices of these three series are plotted in Figure 1. Since the futures price with only the quality option should be a tighter upper bound, we report this result using (3) in column 4. It is seen that the futures price with only the quality option not only provides a tighter upper bound, it also approximates the actual futures price amazingly well. For all 20 contracts together, the average futures price with the quality option is 92.909 which is less than 50 basis points higher than the average actual futures price. This result supports Carr and Chen (1996) in which the value of the quality option should explain most of the total delivery option value. It also supports the evidence that the cost-of-carry model is insufficient to explain the total delivery option value.

Since the true futures price contains all embedded options, the total value of timing options can be implied by subtracting the actual futures price from the futures price with the quality option, i.e., subtracting column 1 from column 4.[18] The results are reported in column 5. As we have argued, this value is quite small. Nonetheless, an average of 70 basis points is not a negligible quantity.

The end-of-month option bound values are given in column 6. This value includes both the quality option and the timing option values. It is difficult to separate these two values because there is no consistent way to measure the quality option.[19] It is seen in Figure 3 that the

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[18] The futures price with the quality option sometimes is less than the actual futures price. In this case, the timing option value is recorded as 0.

[19] Carr and Chen (1996) measure the quality option value by looking at the difference between column 3 and column 4 in the Table. The quality option value, on the other hand, can be defined

lower bound for the futures price provided by this upper bound is conservative enough to include all daily wild card values. And the bound is as tight as the cost of carry model, about 2% on average lower than the actual futures price.

We also estimate the two-factor Cox-Ingersoll-Ross term structure model for a more recent dataset (weekly, from January 4, 1991 through December 29, 1998). The in-sample test is for the contracts from March 1991 to December 1998 and the out-of-sample test is for the contracts from March 1999 to December 2000. We see somewhat different and yet very interesting results. Similar to the first half of Table 3, the second half of the table presents the results from the second period in a parallel fashion. First, we note that, even theoretically so, the futures price with the quality option on longer is an upper bound for the actual futures price. This is a clear indication of the poor fitting result in the term structure model, as mentioned the previously in the sub-section Term Structure Model Estimation. Since the futures price with the quality option can be an upper bound only when the term structure model is correct, a poor fit of the term structure model certainly affects its performance. On average, we find that the model-dependent futures price with the quality option falls below the actual futures price by 84 basis points. In short, the two-factor Carr-Chen futures pricing model (1996) with the quality option performs poorly in the second period, which includes in-sample (for 1992 ~ 1998 contracts) and out-of-sample (for 1999 ~ 2000 contracts). The plot of the weekly actual futures prices and theoretical futures prices with the quality option is provided in Figure 4.

Interestingly, the model-free upper bound, the cost-of-carry model, performs equally well as in the first period. It remain roughly 2% above the actual price, a very robust result.

The very surprising result is the model-dependent lower bound. With the same term structure model, the lower bound on average remains within about 2% below the actual futures price. Furthermore, from Table 3 (second part), throughout all contracts, the lower bound constantly falls below the actual futures price. The lower bound performance, week by week, can be seen in the second part of Figure 3. This observation raises an interesting issue. When we use an ill-fitted term structure model to estimate the value of a contract, the performance of the estimate relies extremely on the performance of the underlying model. However, when we estimate a range of values for the contract, the accuracy of the underlying model becomes less sensitive. In reality, no trader is seeking "the price," since model assumptions are always inconsistent with reality. However, robust models (models that are robust to parameter changes) are useful in that they provide useful implications traders can use to gain insights. What we have learned from this empirical test precisely enhances this point.

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as the difference between the futures price without the quality option and the futures price with the quality option. Then, there is more than one measure for the quality option.

### *D. Discussions*

The importance of the bounds is clear if one realizes that it is nearly impossible to compute the delivery options accurately. Yet it is almost equally important to recognize that violating such bounds implies arbitrage opportunities. This is particularly interesting for the upper bound because the upper bound of the futures price – the cost of carry, is model-free. Our results show that 164 out of 722 (or 22.71%) weeks that the futures price exceeded its upper bound. The magnitude of violation is on average 28.6 basis points (annualized) or half a basis point a week should such violation occur. In such times, investors can sell futures and buy the CTD bond and then dynamically switch to the new CTD bond if necessary. Such a strategy, as suggested by the forward measure, should yield an arbitrage profit of half a basis point each time. This profit must overweigh the transaction costs to be profitable.

The violation of the lower bound only generates arbitrage profits when the adopted model is true. Our results show that there are 32 out of 722 cases (or 4.4%) where the lower bound is violated. Note that the fewer violations of the lower bound than upper bounds are mainly due to a more conservative estimate of the lower bound (i.e., we compound upper bound values of the embedded delivery options). The results suggest that there is a more efficient lower bound to be discovered. We defer that for future research.

From Figure 3, we note that there are periods where bounds are tight and others where bounds are loose. To examine any potential systematic biases, we run regressions of the "tightness" of the bound against a number of possible factors that affect the futures price. For consistency, we run the following regressions for the period of January 2, 1992 till November 2, 2000, total of 462 weekly observations.

$$\Phi_t - \underline{\Phi}_t = a_0 + a_1(\mathrm{CTD}_t - \mathrm{SCTD}_t) + a_2(3\mathrm{MTB}_t - 3\mathrm{MTB}_{t-1}) + a_3(30\mathrm{YTB}_t - 30\mathrm{YTB}_{t-1}) + a_4(\mathrm{CF}_t) + e_t$$

$$\overline{\Phi}_t - \Phi_t = b_0 + b_1(\mathrm{CTD}_t - \mathrm{SCTD}_t) + b_2(3\mathrm{MTB}_t - 3\mathrm{MTB}_{t-1}) + b_3(30\mathrm{YTB}_t - 30\mathrm{YTB}_{t-1}) + b_4(\mathrm{CF}_t) + u_t$$

where

$\Phi$ = market futures price

$\overline{\Phi}$ = upper bound (COC)

$\underline{\Phi}$ = lower bound

CTD = cheapest to deliver bond

SCTD = second cheapest to deliver bond

3MTB = three-month T bill rate

30YTB = 30-year T bond rate

CF = conversion factor of the cheapest bond

and the results are reported in Table 4. It is interesting to note that the lower bound performance is more sensitive to the fitting of the second cheapest bond and the upper bound performance is more sensitive to the long rate. This result is not surprising because the lower bound is a model-driven result while the upper bound is model-free and hence relies on the long rate.

Finally, we can argue that the timing options are more valuable in the first period than in the second period. Note that the timing options are negatively related with the interest rates. Lower interest rates in the second period reduce the value of the timing options.

## VI CONCLUSION

In this paper, we derive lower and upper bound formulas for the Treasury bond futures price. The lower bound of the futures price is obtained by integrating all upper bounds for the delivery options. The cost of carry model is found to be an upper bound of the futures price. These bounds are model free and can be used with any choice of the term structure model. Analytical results are obtained when a two-factor Cox-Ingersoll-Ross model is used. They provide investors with efficient range of how much futures prices can move. In two sample periods of 1987~1991 and 1992~2000, the cost of carry model is found to be about 2% above the actual futures price and the lower bound is found to be about 2% below.

It is generally believed that a tighter upper bound is the futures price with the quality option (equation (3)), since it ignores the timing options. However, this is true only if the chosen term structure model correctly specifies the markets. We show empirically that the futures price with the quality option approximates the actual futures price well in the first period but not in the second period. Nevertheless, the same model-dependent lower bound performs robustly in both first and second periods.

As opposed to recursively using the lattice model to iteratively obtain an accurate estimate of the futures price, which is prohibitively expensive, as Boyle (1989) demonstrates, the bounds provide in the paper can be computed quickly and accurately. Thus, these bounds can provide traders with a useful guideline of the true futures price.

**APPENDIX**

From Theorem 1, we have:

$$\text{(A1)} \qquad \begin{aligned} E_t^Q\left[\delta(t,T)X(T)\right] &= E_t^Q[\delta(t,T)]E_t^{F(T)}[X(T)] \\ &= P(t,T)E_t^{F(T)}[X(T)] \end{aligned}$$

where $\delta$ is strictly less than 1. Due to the risk neutral pricing result we have, the LHS must equal $X(t)$, and hence:

$$\text{(A2)} \qquad X(t) = \frac{E_t^{F(T)}[X(T)]}{P(t,T)}$$

Note that the forward measure is maturity dependent. Clearly, the Radon-Nikodym Derivative (RND) is:

$$\text{(A3)} \qquad \eta(t,T) = \frac{\delta(t,T)}{P(t,T)}$$

Since the measure is $T$-dependent, so should be the RND (usually, RND is just $\eta(t)$.) Let the interest rate process be:

$$\text{(A4)} \qquad dr(t) = \hat{\mu}(r,t)dt + \sigma(r,t)dW^Q(t)$$

Applying Ito's lemma,

$$0 = \ln P(T,T) = \ln P(t,T) + \int_t^T \frac{1}{P(u,T)}\left[P_u(u,T)du + P_r(u,T)dr + \frac{1}{2}P_{rr}(u,T)(dr)^2\right]d\hat{W}(u)$$

$$-\int_t^T \frac{1}{2}\left[\frac{\sigma(r,u)P_r(u,T)}{P(u,T)}\right]^2 du$$

(A5) $$= \ln P(t,T) + \int_t^T \frac{1}{P(u,T)}\left[P_u(u,T)du + P_r(u,T)\hat{\mu}(r,u) + \frac{1}{2}P_{rr}(u,T)\sigma(r,u)^2\right]du$$

$$+\int_t^T \frac{1}{P(u,T)}P_r(u,T)\sigma(r,u)d\hat{W}(u) - \int_t^T \frac{1}{2}\left[\frac{\sigma(r,u)P_r(u,T)}{P(u,T)}\right]^2 du$$

$$= \ln P(t,T) + \int_t^T r(u)du + \int_t^T \frac{1}{P(u,T)}P_r(u,T)\sigma(r,u)d\hat{W}(u) - \int_t^T \frac{1}{2}\left[\frac{\sigma(r,u)P_r(u,T)}{P(u,T)}\right]^2 du$$

Letting:

(A6) $$\theta(t,T) = -\frac{\sigma(r,t)P_r(t,T)}{P(t,T)}$$

and moving the first two terms to the left:

(A7) $$-\int_t^T r(u)du - \ln P(t,T) = \int_t^T -\theta(u,T)d\hat{W}(u) - \int_t^T \frac{1}{2}\theta(u,T)^2 du$$

$$\frac{\delta(t,T)}{P(t,T)} = \eta(t,T) = \exp\left(\int_t^T -\theta(u,T)d\hat{W}(u) - \int_t^T \frac{1}{2}\theta(u,T)^2 du\right)$$

This implies the Girsanov transformation of the following:

(A8) $$W^{F(T)}(t) = W^Q(t) + \int_t^T \theta(u)dt$$

$$= W^Q(t) - \int_t^T \sigma(r,u)\frac{P_r(u,T)}{P(u,T)}du$$

The interest rate process under the forward measure henceforth becomes:

(A9) $$dr(t) = \left[\hat{\mu}(r,t) + \sigma(r,t)^2 \frac{P_r(t,T)}{P(t,T)}\right]dt + \sigma(r,t)dW^{F(T)}(t)$$

Note that the forward measure is quite general. It does not depend on any specific assumption on the interest rate process.

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Table 1: Parameter Estimates of the Two-factor Cox-Ingersoll-Ross Model

| | Chen-Scott Estimation | | | | | New Estimation | | | |
|---|---|---|---|---|---|---|---|---|---|
| | factor 1 | std.err. | factor 2 | std.err. | | factor 1 | std.err. | factor 2 | std.err. |
| $\alpha$ | 1.834100 | 0.222800 | 0.005212 | 0.115600 | $\alpha$ | 0.879967 | 0.001014 | 0.004423 | 0.000014 |
| $\mu$ | 0.051480 | 0.005321 | 0.030830 | 0.683300 | $\mu$ | 0.043822 | 0.000009 | 0.029555 | 0.000097 |
| $\sigma$ | 0.154300 | 0.005529 | 0.066890 | 0.002110 | $\sigma$ | 0.097855 | 0.001429 | 0.095974 | 0.000018 |
| $\varsigma$ | -0.125300 | 0.180600 | -0.066500 | 0.115400 | $\varsigma$ | -0.146140 | 0.000151 | -0.178846 | 0.000361 |
| | likelihood function = 7750.82 | | | | | likelihood function = 11722.81 | | | |
| | # of obs. 470 | | | | | # of obs. 416 | | | |

Note:

Chen-Scott estimates are taken from Exhibit 2, Panel B on page 21 of Chen and Scott (1993) who take Thursday weekly prices of 13-week, 26-week, 5-year, and longest maturity Treasuries. The period of study is January 1980 to December 1988. The new estimates use Friday weekly T-Bill rates of 3 months and 6 months and CMT rates of 5 years, and 30 years. The period of study is January 1991 to December 1998. The new estimates are estimated with RATS where the number of usable observations in the estimation is 387.

Table 2: Summary Statistics of Daily Futures Prices

| Contract Month | N | Mean | Std. Dev | Min | Max |
|---|---|---|---|---|---|
| | | | | | |
| All maturities | 3537 | 103.69 | 11.4274 | 77.78 | 134.66 |
| | | | | | |
| 8703 | 21 | 100.62 | 0.6833 | 99.47 | 101.59 |
| 8706 | 63 | 97.23 | 3.2328 | 88.56 | 101.38 |
| 8709 | 64 | 90.72 | 1.5606 | 86.84 | 93.19 |
| 8712 | 65 | 84.59 | 3.3369 | 77.78 | 90.09 |
| | | | | | |
| 8803 | 63 | 88.23 | 2.147 | 83.72 | 93.91 |
| 8806 | 64 | 91.19 | 1.9662 | 97.34 | 94.16 |
| 8809 | 64 | 86.68 | 1.2191 | 84.44 | 89.56 |
| 8812 | 65 | 87.53 | 2.1673 | 83.94 | 91.41 |
| | | | | | |
| 8903 | 63 | 89.09 | 1.1461 | 86.97 | 91.44 |
| 8906 | 64 | 88.73 | 1.0879 | 86.5 | 91.28 |
| 8909 | 64 | 95.23 | 3.1389 | 88.34 | 100.38 |
| 8912 | 65 | 97.42 | 1.1772 | 95.25 | 99.84 |
| | | | | | |
| 9003 | 63 | 98.29 | 1.9 | 93.22 | 100.28 |
| 9006 | 64 | 92.26 | 1.6765 | 88.59 | 94.72 |
| 9009 | 64 | 93.15 | 1.1783 | 89.78 | 95.19 |
| 9012 | 65 | 89.6 | 1.2881 | 87.16 | 93.09 |
| | | | | | |
| 9103 | 63 | 94.91 | 1.6509 | 91.09 | 97.56 |
| 9106 | 64 | 95.95 | 1.1031 | 93.44 | 97.94 |
| 9109 | 63 | 93.83 | 0.925 | 92.28 | 95.94 |
| 9112 | 64 | 98.17 | 1.4522 | 95.25 | 100.41 |
| | | | | | |
| 9203 | 62 | 101.25 | 2.106 | 97.78 | 105.25 |
| 9206 | 62 | 98.94 | 0.8243 | 97.28 | 100.31 |
| 9209 | 64 | 100.79 | 2.0015 | 97.31 | 105.16 |
| 9212 | 64 | 104.46 | 1.1772 | 102.31 | 106.91 |
| | | | | | |
| 9303 | 61 | 103.9 | 1.7221 | 100.28 | 107.22 |
| 9306 | 64 | 109.79 | 1.9499 | 105.69 | 112.66 |
| 9309 | 64 | 112.3 | 2.3068 | 108.44 | 115.97 |
| 9312 | 64 | 118 | 2.83 | 102.63 | 121.94 |

Note:

Daily futures prices are taken with maturity between 6 weeks and 4½ months for each contract. Such a selection enjoys high liquidity and rare overlapping between contracts.

Table 2 Continued

| Contract Month | N | Mean | Std. Dev | Min | Max |
|---|---|---|---|---|---|
| | | | | | |
| 9403 | 62 | 110.33 | 0.9456 | 113.34 | 117.44 |
| 9406 | 64 | 108.69 | 3.4739 | 103.25 | 115.34 |
| 9409 | 64 | 103.02 | 1.2617 | 100.31 | 105.44 |
| 9412 | 64 | 100.08 | 1.9172 | 97.06 | 103.81 |
| | | | | | |
| 9503 | 60 | 98.64 | 1.5588 | 95.44 | 101.47 |
| 9506 | 64 | 103.57 | 1.4609 | 100.5 | 106.31 |
| 9509 | 64 | 112.34 | 2.2275 | 106.97 | 115.75 |
| 9512 | 61 | 113.51 | 2.6347 | 108.69 | 117.44 |
| | | | | | |
| 9603 | 63 | 119.33 | 1.4683 | 116.75 | 121.56 |
| 9606 | 65 | 112.94 | 3.5877 | 106.75 | 120.22 |
| 9609 | 62 | 108.1 | 1.0822 | 105.88 | 111.84 |
| 9612 | 62 | 109.72 | 1.7013 | 106.41 | 113 |
| | | | | | |
| 9703 | 59 | 113.06 | 1.9931 | 109.78 | 120.06 |
| 9706 | 61 | 109.45 | 1.9678 | 106.63 | 113.44 |
| 9709 | 62 | 111.83 | 0.3393 | 108.31 | 116.75 |
| 9712 | 62 | 114.62 | 1.7645 | 112.06 | 118.47 |
| | | | | | |
| 9803 | 59 | 120.13 | 1.8508 | 117.03 | 123.72 |
| 9806 | 61 | 120.56 | 0.8548 | 118.66 | 122.44 |
| 9809 | 63 | 122.1 | 1.3907 | 118.88 | 124.16 |
| 9812 | 62 | 127.7 | 2.8354 | 122.97 | 134.66 |
| | | | | | |
| 9903 | 58 | 127.75 | 1.4533 | 124.72 | 130.63 |
| 9906 | 64 | 121.89 | 1.4661 | 119.47 | 126.19 |
| 9909 | 63 | 116.3 | 1.4175 | 113.63 | 119.38 |
| 9912 | 62 | 113.38 | 1.2839 | 110.84 | 116.16 |
| | | | | | |
| 0003 | 60 | 92.38 | 1.9398 | 89.22 | 95.66 |
| 0006 | 63 | 95.8 | 1.7785 | 92.47 | 99.34 |
| 0009 | 63 | 96.46 | 1.8667 | 92.66 | 99.38 |
| 0012 | 62 | 99.54 | 0.8819 | 97.63 | 101.22 |

Note:

Daily futures prices are taken with maturity between 6 weeks and 4½ months for each contract. Such a selection enjoys high liquidity and rare overlapping between contracts.

Table 3: Empirical Performance of Upper and Lower Bounds

March 1987 ~ December 1991

| contract month | # of obs. | (1) | (2) | (3) | (4) | (5) | (6) |
|---|---|---|---|---|---|---|---|
| 8703 | 4 | 100.703 | 99.486 | 100.585 | 100.990 | 0.562 | 1.505 |
| 8706 | 13 | 97.875 | 96.136 | 100.132 | 98.228 | 0.740 | 2.092 |
| 8709 | 13 | 90.719 | 89.204 | 93.853 | 91.235 | 0.528 | 2.031 |
| 8712 | 13 | 84.546 | 83.357 | 86.323 | 85.244 | 0.867 | 1.887 |
| 8803 | 13 | 88.269 | 87.545 | 89.407 | 89.753 | 1.483 | 2.208 |
| 8806 | 13 | 91.267 | 89.973 | 95.513 | 91.945 | 0.712 | 1.971 |
| 8809 | 13 | 86.628 | 84.728 | 87.053 | 86.684 | 0.999 | 1.956 |
| 8812 | 13 | 87.269 | 85.947 | 87.530 | 88.314 | 1.045 | 2.367 |
| 8903 | 13 | 89.123 | 87.205 | 89.912 | 89.700 | 0.972 | 2.496 |
| 8906 | 13 | 88.712 | 86.533 | 92.655 | 89.024 | 0.631 | 2.491 |
| 8909 | 14 | 94.980 | 92.763 | 96.251 | 95.357 | 0.401 | 2.594 |
| 8912 | 13 | 97.471 | 95.300 | 100.400 | 97.563 | 0.103 | 2.264 |
| 9003 | 12 | 98.430 | 96.106 | 100.170 | 98.598 | 0.170 | 2.491 |
| 9006 | 14 | 92.252 | 90.549 | 95.804 | 92.971 | 0.733 | 2.421 |
| 9009 | 13 | 93.238 | 91.168 | 94.006 | 93.470 | 0.570 | 2.302 |
| 9012 | 13 | 89.572 | 88.004 | 92.559 | 90.149 | 0.772 | 2.146 |
| 9103 | 13 | 95.195 | 93.275 | 95.982 | 95.278 | 0.377 | 2.003 |
| 9106 | 13 | 96.003 | 94.693 | 97.561 | 96.547 | 0.555 | 1.855 |
| 9109 | 13 | 93.902 | 92.723 | 94.322 | 94.579 | 0.898 | 1.856 |
| 9112 | 13 | 98.152 | 96.792 | 100.638 | 98.395 | 0.581 | 1.603 |
| | | | | | | | |
| all maturities | 252 | 92.414 | 90.758 | 94.306 | 92.909 | 0.691 | 2.151 |

Note:

(1) is actual futures price

(2) is lower bound (equation (36))

(3) is cost of carry price, also upper bound (equation (24)

(4) is the futures price with the quality option (equation (3))

(5) is average of (4) - (1), a measure of the market value of the timing options

(6) is average of (1) - (2), a measure of bound tightness

The theoretical values are computed using the Chen-Scott estimates (left panel of Table 1)

March 1992 ~ December 2000

| contract month | # of obs. | (1) | (2) | (3) | (4) | (5) | (6) |
|---|---|---|---|---|---|---|---|
| 9203 | 13 | 101.216 | 99.800 | 102.820 | 103.402 | 2.349 | 1.392 |
| 9206 | 13 | 98.875 | 95.497 | 101.650 | 100.550 | 1.944 | 3.315 |
| 9209 | 13 | 100.930 | 98.520 | 101.607 | 101.803 | 1.116 | 2.366 |
| 9212 | 13 | 104.577 | 100.955 | 106.022 | 104.363 | -0.029 | 3.563 |
| 9303 | 13 | 103.926 | 101.612 | 107.472 | 104.020 | 0.274 | 2.267 |
| 9306 | 13 | 110.099 | 107.530 | 110.817 | 109.325 | -0.732 | 2.531 |
| 9309 | 13 | 112.274 | 109.783 | 113.701 | 110.169 | -2.086 | 2.463 |
| 9312 | 13 | 118.125 | 115.264 | 120.051 | 114.977 | -3.226 | 2.843 |
| 9403 | 13 | 115.250 | 114.138 | 117.530 | 113.434 | -1.877 | 1.096 |
| 9406 | 13 | 108.777 | 107.669 | 110.239 | 108.880 | 0.131 | 1.090 |
| 9409 | 13 | 103.132 | 100.401 | 104.535 | 104.280 | 1.324 | 2.712 |
| 9412 | 13 | 100.277 | 97.763 | 100.997 | 102.553 | 2.505 | 2.473 |
| 9503 | 13 | 98.438 | 94.237 | 100.829 | 101.028 | 2.832 | 4.153 |
| 9506 | 13 | 103.394 | 100.743 | 104.408 | 105.689 | 2.424 | 2.616 |
| 9509 | 14 | 112.212 | 108.228 | 113.331 | 112.049 | -0.187 | 3.958 |
| 9512 | 12 | 113.485 | 112.215 | 114.642 | 114.273 | 0.746 | 1.240 |
| 9603 | 14 | 119.299 | 115.368 | 123.136 | 117.049 | -2.362 | 3.905 |
| 9606 | 13 | 112.681 | 111.661 | 113.476 | 113.569 | 0.858 | 0.993 |
| 9609 | 13 | 108.375 | 104.510 | 108.601 | 108.917 | 0.574 | 3.840 |
| 9612 | 13 | 109.630 | 108.310 | 111.131 | 111.098 | 1.473 | 1.294 |
| 9703 | 13 | 112.834 | 109.111 | 116.659 | 112.241 | -0.634 | 3.700 |
| 9706 | 13 | 109.301 | 108.105 | 110.123 | 110.984 | 1.679 | 1.172 |
| 9709 | 13 | 111.875 | 107.909 | 114.585 | 111.327 | -0.578 | 3.944 |
| 9712 | 13 | 114.690 | 113.795 | 115.964 | 115.040 | 0.270 | 0.871 |
| 9803 | 13 | 120.329 | 116.394 | 124.512 | 117.533 | -2.939 | 3.911 |
| 9806 | 13 | 120.625 | 120.158 | 121.157 | 119.546 | -1.241 | 0.443 |
| 9809 | 13 | 122.120 | 118.071 | 123.111 | 118.598 | -3.681 | 4.026 |
| 9812 | 13 | 127.772 | 124.326 | 129.021 | 122.219 | -5.497 | 3.054 |
| 9903 | 13 | 127.916 | 121.848 | 131.740 | 120.926 | -6.878 | 5.584 |
| 9906 | 13 | 121.709 | 121.580 | 122.289 | 120.135 | -1.737 | 0.104 |
| 9909 | 13 | 116.298 | 112.651 | 116.150 | 114.429 | -1.973 | 3.629 |
| 9912 | 13 | 113.397 | 111.580 | 114.369 | 113.170 | -0.305 | 1.797 |
| 0003 | 13 | 92.378 | 89.690 | 93.742 | 91.195 | -2.603 | 3.746 |
| 0006 | 13 | 95.856 | 93.482 | 97.851 | 93.488 | -3.020 | 2.848 |
| 0009 | 14 | 96.574 | 95.796 | 97.963 | 96.060 | -2.724 | 2.662 |
| 0012 | 13 | 99.606 | 97.489 | 101.067 | 98.023 | -3.324 | 3.729 |
| | | | | | | | |
| all maturities | 470 | 109.940 | 107.378 | 111.584 | 109.326 | -0.8427 | 2.651 |

Note:

(1) is actual futures price

(2) is lower bound (equation (36))

(3) is cost of carry price, also upper bound (equation (24)

(4) is the futures price with the quality option (equation (3))

(5) is average of (4) - (1), a measure of the market value of the timing options

(6) is average of (1) - (2), a measure of bound tightness

The theoretical values are computed using the new estimates (right panel of Table 1)

Table 4: Regression Results

| | Lower Bound | | | Upper Bound | | |
|---|---|---|---|---|---|---|
| | coefficient | std.err. | t | coefficient | std.err. | t |
| Intercept | 2.620156 | 0.080332 | 32.61647 | 1.62892 | 0.097364 | 16.73019 |
| $CTD_t - SCTD_t$ | 0.615754 | 0.240687 | 2.558313 | 0.043733 | 0.291717 | 0.149915 |
| $3MTB_t - 3MTB_{t-1}$ | -0.85906 | 1.056233 | -0.81332 | -1.35587 | 1.280171 | -1.05913 |
| $30YTB_t - 30YTB_{t-1}$ | -1.55869 | 0.919818 | -1.69456 | -3.24591 | 1.114835 | -2.91156 |
| $CF_t$ | -0.79337 | 0.970453 | -0.81753 | -0.42436 | 1.176205 | -0.36079 |
| Adjusted $R^2$ | 2.28% | | | 1.87% | | |
| # of obs. | 462 | | | 462 | | |

Note: Regression period is from January 2, 1992 till November 2, 2000, total of 462 weekly observations. Regression equations are:

$$\Phi_t - \underline{\Phi}_t = a_0 + a_1(\mathrm{CTD}_t - \mathrm{SCTD}_t) + a_2(3\mathrm{MTB}_t - 3\mathrm{MTB}_{t-1}) + a_3(30\mathrm{YTB}_t - 30\mathrm{YTB}_{t-1}) + a_4(\mathrm{CF}_t) + e_t$$

$$\overline{\Phi}_t - \Phi_t = b_0 + b_1(\mathrm{CTD}_t - \mathrm{SCTD}_t) + b_2(3\mathrm{MTB}_t - 3\mathrm{MTB}_{t-1}) + b_3(30\mathrm{YTB}_t - 30\mathrm{YTB}_{t-1}) + b_4(\mathrm{CF}_t) + u_t$$

where

$\Phi$ = market futures price

$\overline{\Phi}$ = upper bound (COC)

$\underline{\Phi}$ = lower bound

CTD = cheapest to deliver bond

SCTD = second cheapest to deliver bond

3MTB = three-month T bill rate

30YTB = 30-year T bond rate

CF = conversion factor of the cheapest bond

Figure 1: Yield Curves for the Selected Period

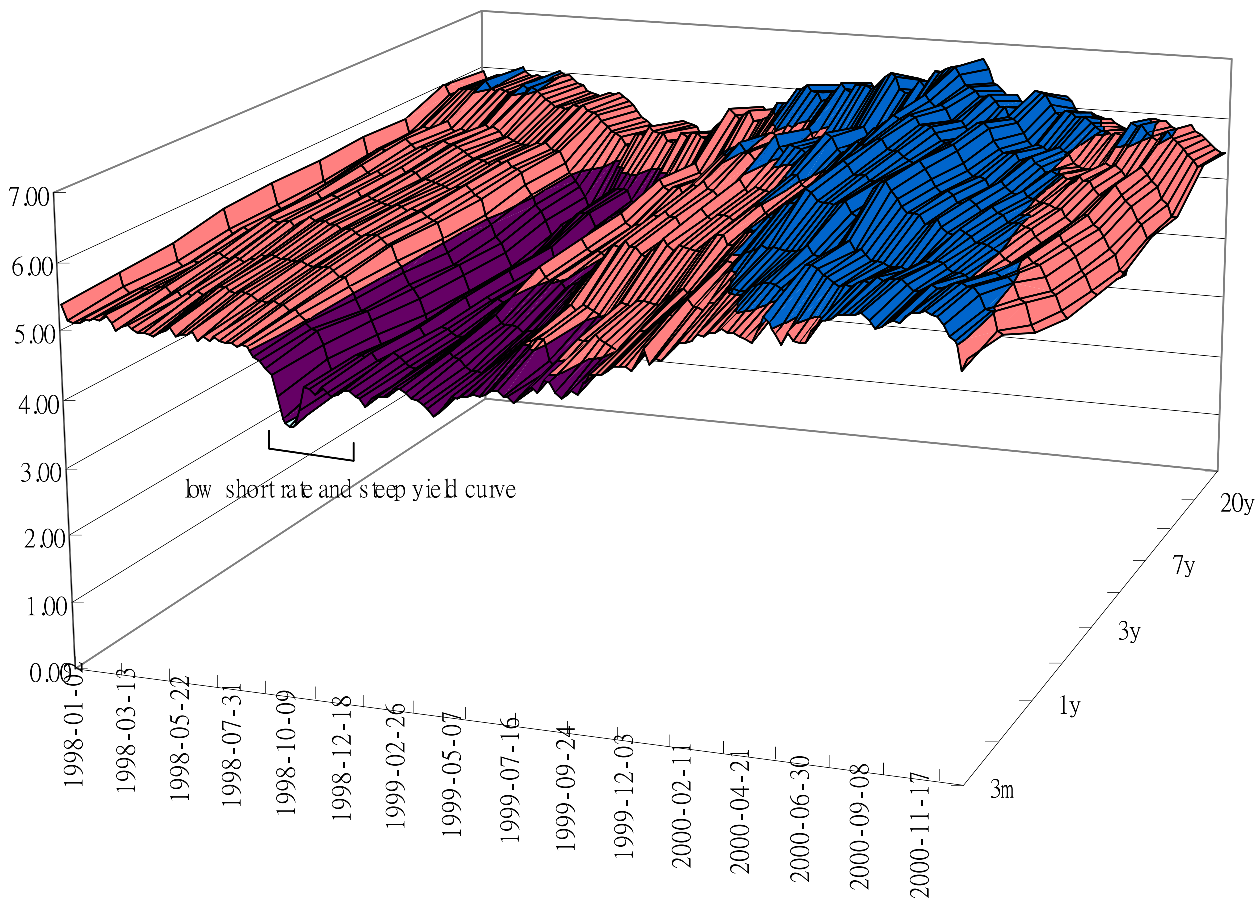
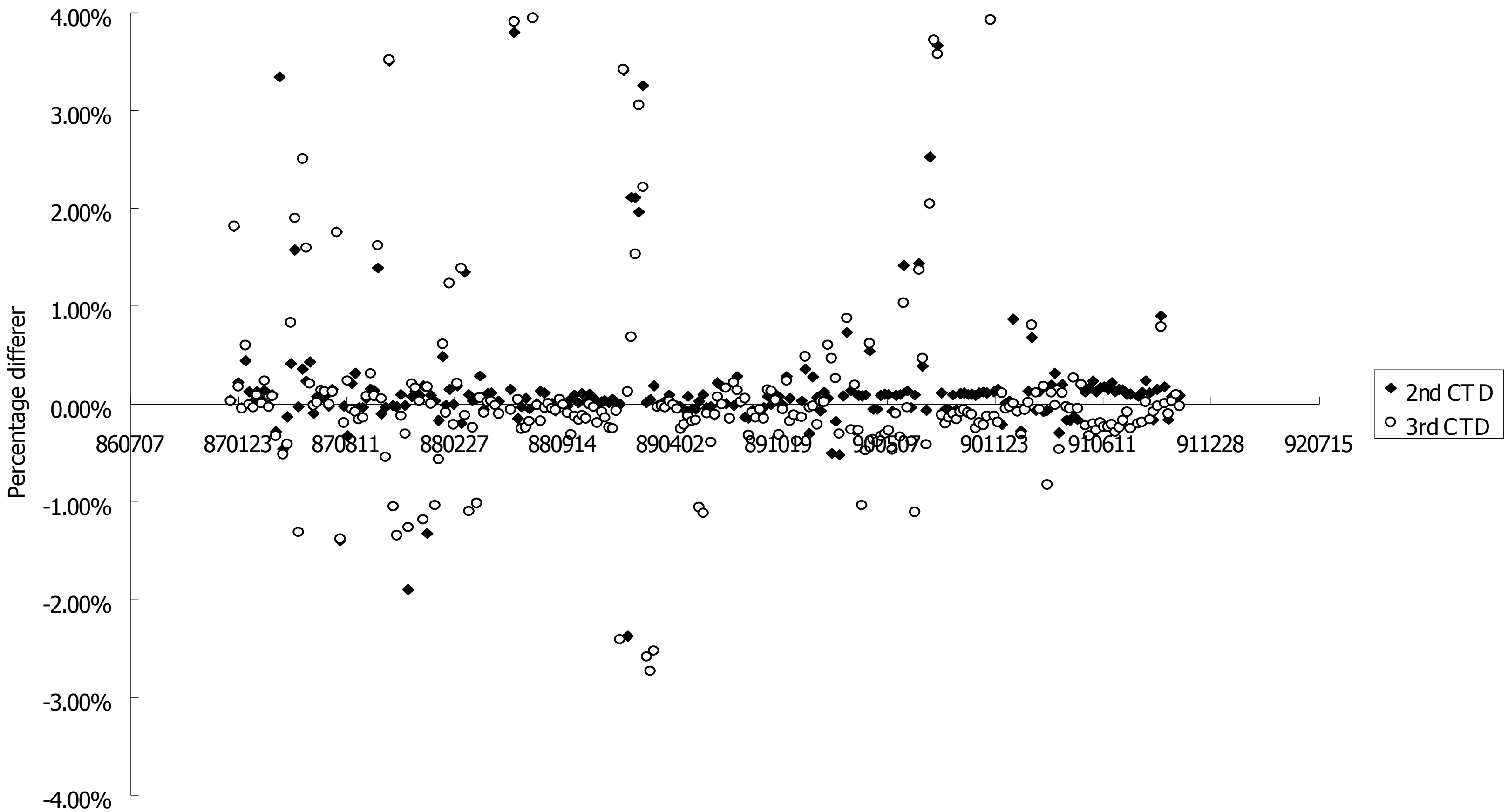


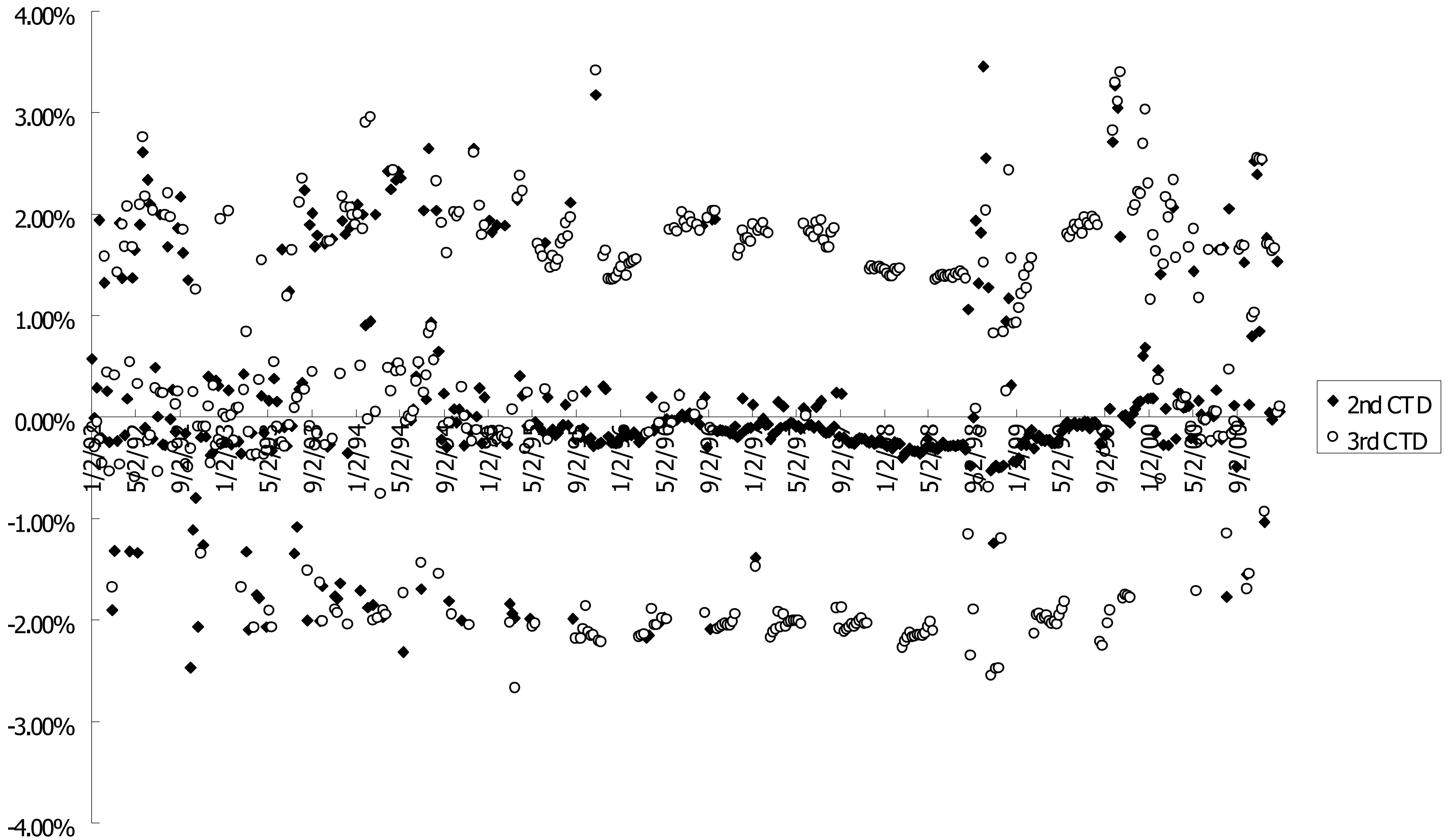
Figure 2: Fitting Performance Of The Second And Third Cheapest-to-deliver Bonds

Contracts 3/87 ~ 12/91




Note: The pricing error is measured as percentage error of the market price: model price ÷ market price − 1. The average percentage errors are 30 basis points and 14 basis points for the 2nd CTD and 3rd CTD respectively. The root mean square errors are 1.07% and 1.20% respectively.

Contracts 3/92 ~ 12/00




Note: The pricing error is measured as percentage error of the market price: model price ÷ market price – 1. The average percentage errors are 10 basis points and 26 basis points for the 2nd CTD and 3rd CTD respectively. The root mean square errors are 1.04% and 1.61% respectively.

Figure 3: Weekly Time Series Plot of Actual Futures Prices (Actual), Their Upper (COC) and Lower (LBB) Bounds

Contracts 3/87 ~ 12/91

Contracts 3/92 ~ 12/00

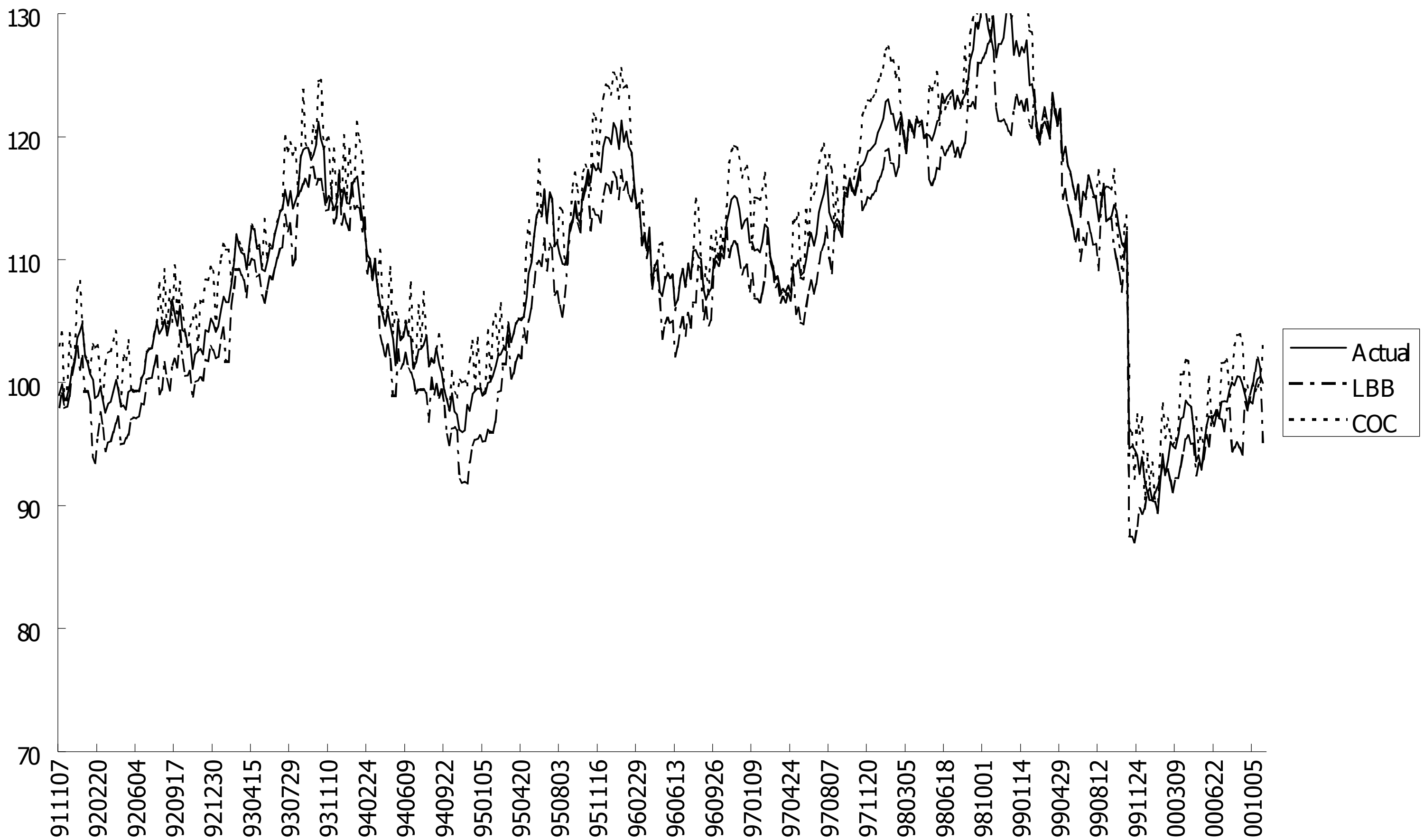
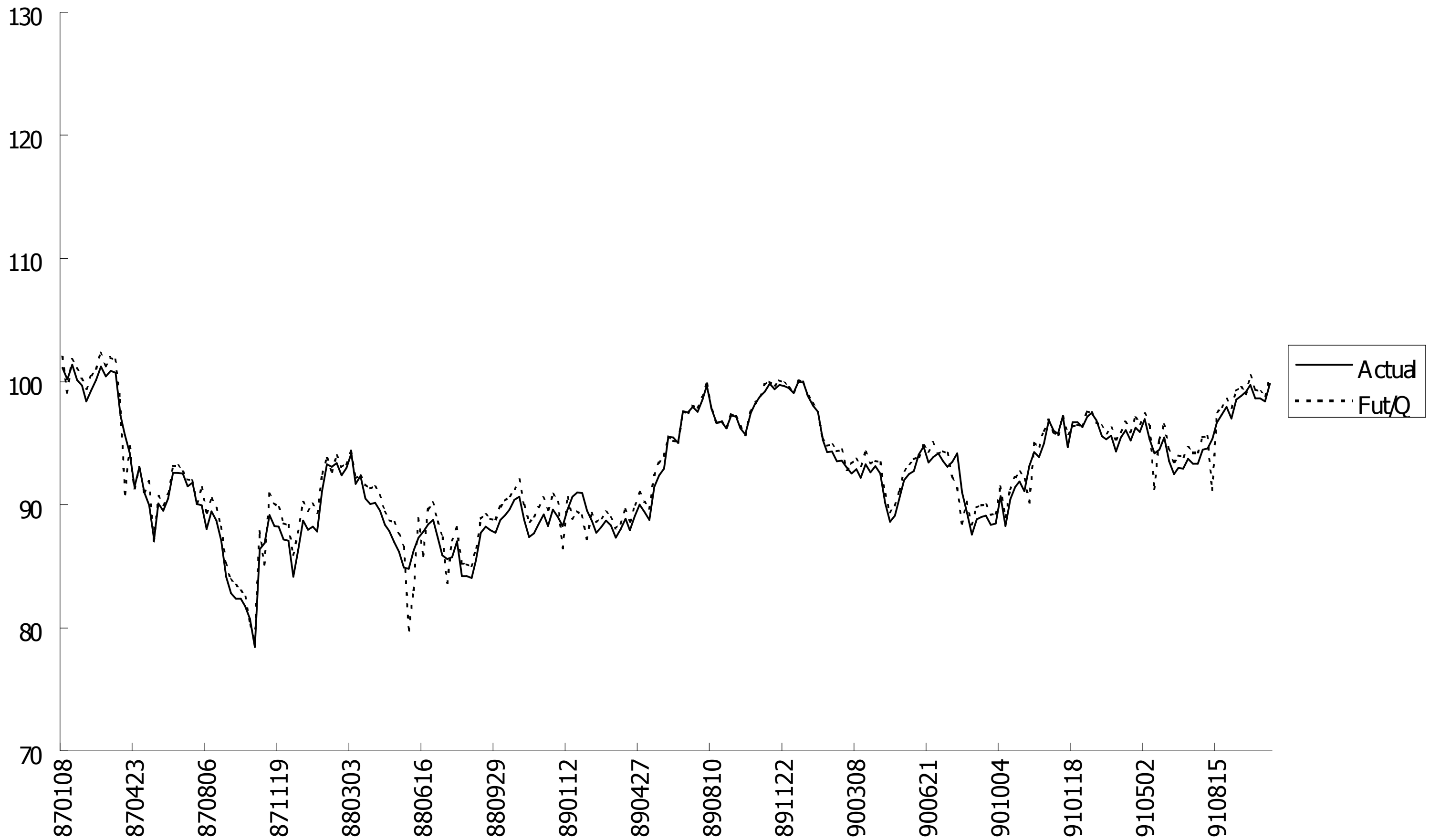


Figure 4: Weekly Time Series Plot of Actual Futures Prices (Actual), and the Theoretical Futures Prices with the Quality Option (Fut/Q)

Contracts 3/87 ~ 12/91



Contracts 3/92 ~ 12/00