Implied Volatility, Volatility Smile/Skew/Smirk, and Risk-Neutral Density (RND)

Historical Volatility versus Implied Volatility

In our VaR calculations, we use historical volatility. Historical volatility has three major problems:

- period dependent
- frequency dependent
- backward looking

Hence, to have an accurate VaR, we seek a measure of the volatility that does not suffer from the above problems. The answer is the implied volatility from options (or derivatives).

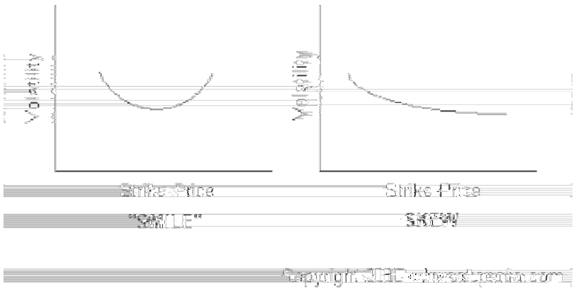
Implied vol is forward looking. Hence it provides an accurate picture of the future of the underlying asset.

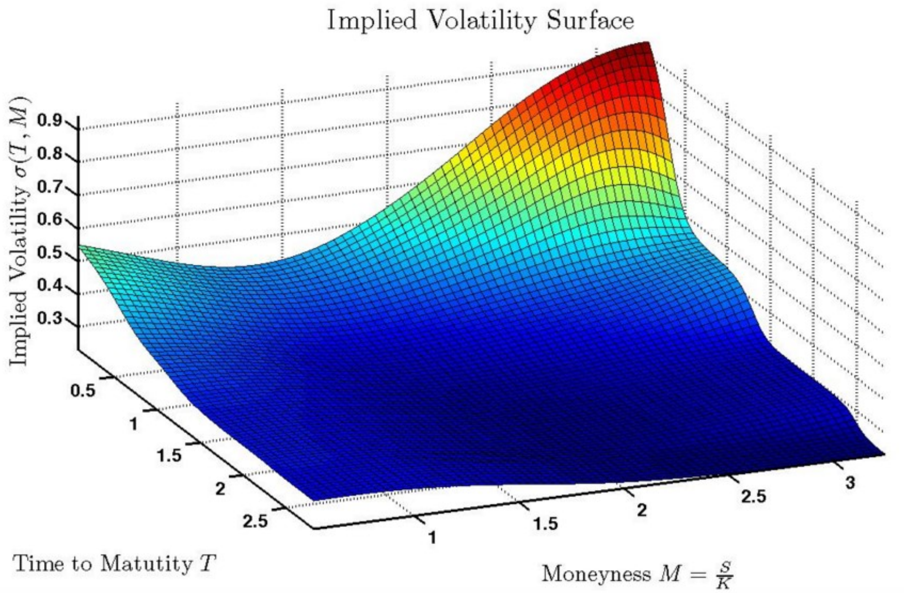
Volatility Smile and Volatility Term Structure

The seminal Black-Scholes model is too limited to be used in reality. It assumes constant volatility which is not true in reality. In particular, ever since the VIX index was introduced in 1993, it has demonstrated a very volatile pattern, especially when the market is in turmoil, and hence gained its name as the "fear index". If the volatility is stochastic, then clearly the Black-Scholes model does not work properly. The Black-Scholes model also assumes normally distributed returns (or log normally distributed prices)

The Black-Scholes model also suffers from its assumption of normality (i.e. stock returns are normally distributed). Normal distributions have thin tails. As a result, the distribution gives very little probability to very high and very low stock prices (relative to today's price) in the future, leading to smaller (smaller than the market) in- and out-the-money option prices. Hence, when using the same volatility to compute option prices, we find that the options that are deep in and out of the money have too low values and near the money too high values. As a result, the "correct" volatility values for deep in and out of money options are much higher than those of the near the money options. This is known as the volatility smile (when the smile is asymmetric, it is called skew or smirk).

Across strikes, we have volatility smile (or skew or smirk). Across maturities, we have volatility term structure. Together, we have volatility surface.





The Black-Scholes model has two major problems:

- normal distribution
- constant volatility

As a result, the model cannot price all options (with different maturities and different strikes) in the market. Amazingly, as it turns out, this is not the reason to abandon the Black-Scholes. If anything, people embrace the Black-Scholes even more. HOW? By using the implied volatility by the Black-Scholes model. People find that the Black-Scholes model, although incorrect, is remarkably robust. If one is willing to accept the smile, one can still use the Black-Scholes model to evaluate options or predict future asset prices.

More Realistic Model

Does the vol smile or vol term structure disappear when we use a fat-tailed distribution or stochastic vol?

To answer this question, we must first ask ourselves where do fat tails come from? How vol is changing over time? Many believe that fat tails come from "over hedging" or "fear".

- When we look at the smile, we find that to disappear for long dated options.
- When we look at the term structure, we find that it is steeply downward sloping.

This tells us that people are willing to pay high vol for short-dated, deep out of money options (deep in for puts is deep out for calls and vice versa). First, these options are cheap. So overpaying in vols looks bad but not so bad in premiums.

So how to build a model for this? Jumps add to short term vol. Stochastic vol adds to long term vol.

Model-free Model – RND

Take the Black-Scholes model as an example:

$$C = SN(d_1) - e^{-r(T-t)}KN(d_2)$$

$$\frac{\partial C}{\partial K} = e^{-r(T-t)}N(d_2)$$

$$e^{r(T-t)}\frac{\partial C}{\partial K} = N(d_2) = \Pr[S > K]$$

This is the cum probability of in the money ($\Pr[S > K] = F(K)$). Hence,

$$\frac{\partial F(K)}{\partial K} = f(K)$$

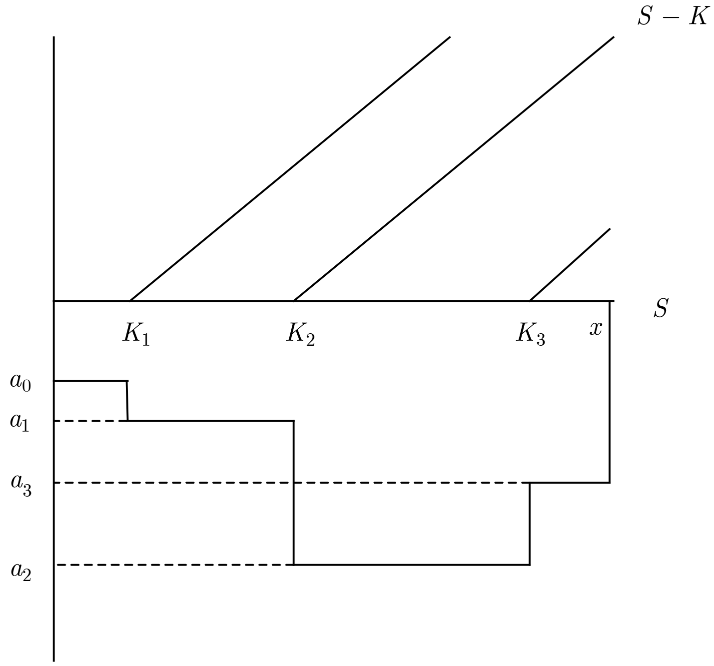
is the probability density function.

Carr-Madan Theorem

They argue that if there are infinite number of European options across strikes and maturities, then all exotic options (e.g. American) can be priced via forming a portfolio of relevant European options.

Of course, in reality there are only a handful of European options. Hence we cannot price American options using European options. A lattice must be used. However, it would be nice if there exists an interpolation scheme allows the pricing American options using the European options.

This is the model-free model – using European options only (and some interpolation) to price American options without a parametric model.



This density function can exactly re-price all options for that maturity.

(1) $$C_k = \int_{K_k}^{x} (S-K_k)\phi(S)dS = \frac{1}{2}\sum_{i=k}^{n} a_i(K_{i+1}^2 - K_i^2) - K_k\sum_{i=k}^{n} a_i(K_{i+1}-K_i)$$

(2) $$x = K_n + \frac{1}{a_n}\left[1 - \sum_{i=1}^{n} a_{i-1}(K_i - K_{i-1})\right]$$

where $K_0 = 0$.

These can be solved in closed-form:

(3) $$a_n = \frac{2C_n}{(x-K_n)^2}$$

and

(4) $$a_k = \frac{2}{(K_{k+1}-K_k)^2}\left[C_k - \frac{1}{2}\sum_{i=k+1}^{n} a_i\left[(K_{i+1}^2 - K_i^2) - 2K_k(K_{i+1}-K_i)\right]\right]$$

with $C_0 = S_0$ and $K_0 = 0$. By iterating *x*, we can solve for all the a 's. Note that there is no guarantee that a 's are positive. For that to happen, $\Delta_k > \sum_{i=k+1}^{n} a_i(K_{i+1}-K_i)$. In other words, the delta of option *k* must be greater than the probability of the area right-side of k. This is equivalent to $N(d_1) > N(d_2)$ in the Black-Scholes case. Otherwise, there is an opportunity for arbitrage.

An Example

Strike Prices						
3M	1.163347	1.214108	1.260902	1.302704	1.342671	1.382695
1Y	1.049059	1.16682	1.272601	1.368343	1.467118	1.579103

Call Prices						
3M	0.099672	0.056007	0.024819	0.008891	0.002866	
1Y	0.222911	0.122904	0.054187	0.019453	0.006356	
Densities						
3M	0.007565	5.160465	2.68231	10.65532	0.37813	3.577975
1Y	0.02557	2.104793	1.431359	4.410113	0.386236	1.013703

Mini HW:
Use any stock option.