Chapter 13

Reduced-Form Models

13.1 Introduction

There are two general approaches in modeling default – the structural approach and the reduced-from approach. The reduced-form models for default, like any other reduced-form models, take market information as given. Analogously, the structural models, like many others, are built on economic fundamentals.

While details vary, the basic principle of the reduced-form models is that defaults occur according to a Poisson process. In other words, a default event is a Poisson jump event. The representative reduced-form models for default are the Jarrow-Turnbull and Duffie-Singleton models. When a Poisson jump event happens, a firm is in default. Once a firm is in default, it is assumed that it will not become live again. As a result, a default here represent complete bankruptcy. Assets of the company must be liquidated. The usual notion of default such as Chapter 11 (bankruptcy protection) is not a default event by these models.

13.2 Survival Probability

We compute survival probabilities when we model default. The survival probability between now and some future time T for defaults is:

$$Q(t,T) = e^{-\lambda(T-t)} \tag{13.1}$$

where λ is the "intensity" parameter of the Poisson process. This intensity parameter intuitively represents the likelihood of default. When the recovery is 0, then this value is almost identical (exactly identical in continuous time) to the "forward" probability of default.

As we can see, there is extreme similarity between the result of survival probability and risk-free discount, which is $P(t,T) = e^{-r(T-t)}$. In fact, we shall show that they can be combined linearly if the recovery of a risky bond is 0. Due to this similarity, we shall proceed without proof (which is difficult in many situations) to borrow what we have known for the risk-free rate and use it for the intensity. In the insurance literature, the intensity parameter is called the hazard rate.

If the intensity parameter (hazard rate) is non-constant, then we can express the survival probability as:

$$Q(t,T) = \exp\left(-\int_t^T \lambda_u du\right) \tag{13.2}$$

Furthermore, if the hazard rate is random, then we simply compute the risk neutral expectation:

$$Q(t,T) = \hat{\mathbb{E}}_t\left[\exp\left(-\int_t^T \lambda_u du\right)\right] \tag{13.3}$$

Finally, if the interest rate is random, then we must use the forward measure:

$$Q(t,T) = \tilde{\mathbb{E}}_t^{(T)}\left[\exp\left(-\int_t^T \lambda_u du\right)\right] \tag{13.4}$$

Equation 13.2 represent the survival probability till a certain future time (T). As a result, we can have a whole "curve" of survival probabilities, known as the survival probability curve.

Taking the derivative with respect to an arbitrary future time T of equation 13.2, we get:

$$\lambda_T = -\frac{d \ln Q(t,T)}{dT} \tag{13.5}$$

Again, we remind the readers of the similarity between default probabilities and forward rates in the risk-free world.[1]

[1] The risk-free forward rate is $f_{t,T} = -d \ln P(t,T)/dT$.

13.3 Zero Recovery Risky Bond

A zero recovery risky bond has the same analogy to the risk-free bond as the hazard rate to the risk-free rate. First we shall look at the risky discount factor.

13.3.1 Risky Discount Factor

A risky discount factor, similar to the risk-free discount factor, discounts \$1 paid in the future. Assuming the same notation for the risk-free discount factor, $P(t,T)$, that represents the present value of \$1 paid in time T, we can denote the survival probability as $Q(t,T)$. A risky discount factor is the present value of \$1 paid in T only if default does not occur. As a result, the present value of \$1 paid in time T is $P(t,T)Q(t,T)$.

13.3.2 Zero Recovery Risky Bond

A zero recovery risky coupon bond pays a periodic coupon c till maturity T_n. If the bond has no recovery, then its price must be:

$$B(t,\underline{T}) = \sum\nolimits_{j=1}^{n} cP(t,T_j)Q(t,T_j) + P(t,T_n)Q(t,T_n) \tag{13.6}$$

where $\underline{T} =< T_1 \cdots T_n >$.

13.4 Positive Recovery Risky Bond

With recovery, the valuation of the risky bond becomes much more complex. Deciding a recovery value of a defaulted bond is a complex process. When a firm defaults, its assets are under a liquidation process and when it ends, bond holders know what they can recover. This process can sometimes take multiple years to finish. For some bond holders who do not wish to wait, they can sell their bonds to the marketplace (distressed bond market) to gain cash earlier. This is similar to Account Receivable factorization.

When such a market exists, then investors estimate a fair present value for the ultimate recovery. This then represents the fair market value of recovery of the bond. When such a market does not exist, then the recovery value must be estimated. It is common for practitioners to use a historical average. Rating agencies provide historical averages for various categories of bonds. For example, senior unsecured bonds recover on average 35% to 45% and junior unsecured bonds recover on average 15% to 25%.

In the literature, there are two major approaches to model positive recovery. Jarrow and Turnbull (1995) assume recovery to be a fixed percentage of the face value of the bond and Duffie and Singleton (1997) assume the recovery to be a fixed percentage of the market value of the bond immediately prior to default. As we shall show later, the Jarrow-Turnbull model is particularly useful in building the credit curve (i.e. bootstrapping) and the Duffie-Singleton model is useful in integrating with the term structure models.

13.4.1 Recovery of Face Value – The Jarrow-Turnbull Model

When the recovery rate is a fixed amount, we can modify the pricing formula of (13.6) as follows:

$$\begin{aligned} B(t,\underline{T}) &= \sum_{j=1}^{n} cP(t,T_j)Q(t,T_j) + P(t,T_n)Q(t,T_n) \\ &\quad + R\sum_{j=1}^{n} P(t,T_j)[Q(t,T_{j-1}) - Q(t,T_j)] \end{aligned} \tag{13.7}$$

where c is coupon (or cash flow), $P(t,T)$ is risk-free discount factor between now and time T, $Q(t,T)$ is survival probability between now and time T, and R is the recovery rate that is assumed constant. The last term is added due to recovery. Note that $Q(t,T_{j-1}) - Q(t,T_j)$ is the default probability between T_{j-1} and T_j. Note that in continuous time, this is $-dQ(t,T)$ which is equal to $\pi(t,T)dT$. As a result, the above formula can be written in continuous time as:

$$\begin{aligned} B(t,\underline{T}) &= \sum_{j=1}^{n} cP(t,T_j)Q(t,T_j) + P(t,T_n)Q(t,T_n) \\ &\quad + R\int_t^{T_n} P(t,u)[-dQ(t,u)] \end{aligned} \tag{13.8}$$

The above equation is not a closed-form solution as it requires integration over the default probability measure. One particularly easy way to keep the closed-form solution is to assume the recovery to be received at a fixed time (and not upon default). Then the above equation can be simplified assuming the recovery is received at T_n:

$$B(t,\underline{T}) = \sum_{j=1}^{n} cP(t,T_j)Q(t,T_j) + P(t,T_n)Q(t,T_n) + RP(t,T_n)[1 - Q(t,T_n)] \tag{13.9}$$

The last term $1 - Q(t, T_n)$ is the cumulative default probability. This is the Jarrow-Turnbull model.

We shall demonstrate numerically how the Jarrow-Turnbull model is used in practice, which is known as "bootstrapping" or "curve cooking". The model has become the industry standard in retrieving survival probability information from market quotes (such as credit default swaps, or CDS). To do that the model needs to be slightly adjusted. We shall discuss this in a separate section later.

13.4.2 Recovery of Market Value – The Duffie-Singleton Model

Another easy way to arrive at a closed-form solution is to let the recovery be proportional to the otherwise undefaulted value. That is, upon default (at default time u), the recovery value is $R_t = \delta Z(t, T)$ where $Z(t, T)$ is the price of a zero coupon risky bond as if it has not defaulted.

Under the Poisson process, for a very small time interval Δt, we can write the bond equation as:

$$Z(t, T) = \frac{Z(t + \Delta t, T)\delta\lambda\Delta t + Z(t + \Delta t, T)(1 - \lambda\Delta t)}{1 + r\Delta t} \tag{13.10}$$

which then can be simplified to, assuming n periods between now t and maturity T:

$$\begin{aligned} Z(t, T) &= \frac{Z(t + \Delta t, T)(1 - \lambda\Delta t(1 - \delta))}{1 + r\Delta t} \\ &= Z(T, T) \left[\frac{1 - \lambda\Delta t(1 - \delta)}{1 + r\Delta t} \right]^n \\ &\sim Z(T, T) \left[\frac{e^{-\lambda\Delta t(1-\delta)}}{e^{r\Delta t}} \right]^n \\ &\sim Z(T, T) e^{-(r+s)(T-t)} \end{aligned} \tag{13.11}$$

where $s = \lambda(1 - \delta)$ can be viewed as a spread over the risk-free rate. This is the Duffie-Singleton model. $Z(T, T)$ is the terminal value of the bond which is usually the face value.

A nice feature of the Duffie-Singleton model is that a coupon bond can then a portfolio of such zeros, as each coupon is treated as a zero bond it recovers market value. That is:

$$B(t, \underline{T}) = \sum\nolimits_{j=1}^{n} cZ(t, T_j) + Z(t, T_n) \tag{13.12}$$

This model is practically appealing in that it reflects the usual industry practice that credit risk is reflected in spreads. This model is also convenient to be combined with existing term structure models. It simply adds a second state variable.

The drawback of the model is that the recovery parameter and the intensity parameter always are inseparable. This adds to difficulty in calibration this model to the market.

Both the Jarrow-Turnbull and the Duffie-Singleton models assume defaults to be unexpected like Poisson events. Different from the Jarrow-Turnbull model that assumes fixed amount recovery (or known as recovery of face value), the Duffie-Singleton model assumes the recovery to be proportional to the market value of the debt (known as recovery of market value) immediately prior to default.

The Jarrow-Turnbull model is suitable for bootstrapping and the Duffie-Singleton model is convenient to combine with term structure models. Following the similar analysis for equations (13.2) $\sim$ (13.4), we can write (13.11) as:

$$Z(t, T) = \hat{\mathbb{E}}_t \left[\exp \left(- \int_t^T (r(u) + s(u)) du \right) \right] \tag{13.13}$$

which allows us to directly model "spread" as another state variable. Note that $s(t) = \lambda(t)(1 - \delta)$ according to (13.11) and hence the spread process is similar to the intensity process. If the intensity and the risk-free rate are independent, then the Duffie-Singleton model of (13.12) is similar to (13.6) but with positive recovery.

We can easily conduct a CIR model with the Duffie-Singleton approach. We can have the following joint square-root process:

$$\begin{aligned} dr(t) &= \hat{\alpha}_r(\hat{\mu}_r - r(t))dt + \sigma_r \sqrt{r(t)} d\hat{W}_r(t) \\ ds(t) &= \hat{\alpha}_s(\hat{\mu}_s - s(t))dt + \sigma_s \sqrt{s(t)} d\hat{W}_s(t) \end{aligned} \tag{13.14}$$

where $d\hat{W}_r(t) d\hat{W}_s(t) = 0$. Then (13.13) has a closed-form solution as each expectation in the following equation is a CIR solution.

$$Z(t, T) = \hat{\mathbb{E}}_t \left[\exp \left(- \int_t^T r(u) du \right) \right] \hat{\mathbb{E}}_t \left[\exp \left(- \int_t^T s(u) du \right) \right] \tag{13.15}$$

The following example is to demonstrate how the Duffie-Singleton model can be easily combined with any interest rate model. In the following a simple binomial model for the risk-free rate is given. The probabilities of the up and down branches are $\frac{1}{2}$ and $\frac{1}{2}$.

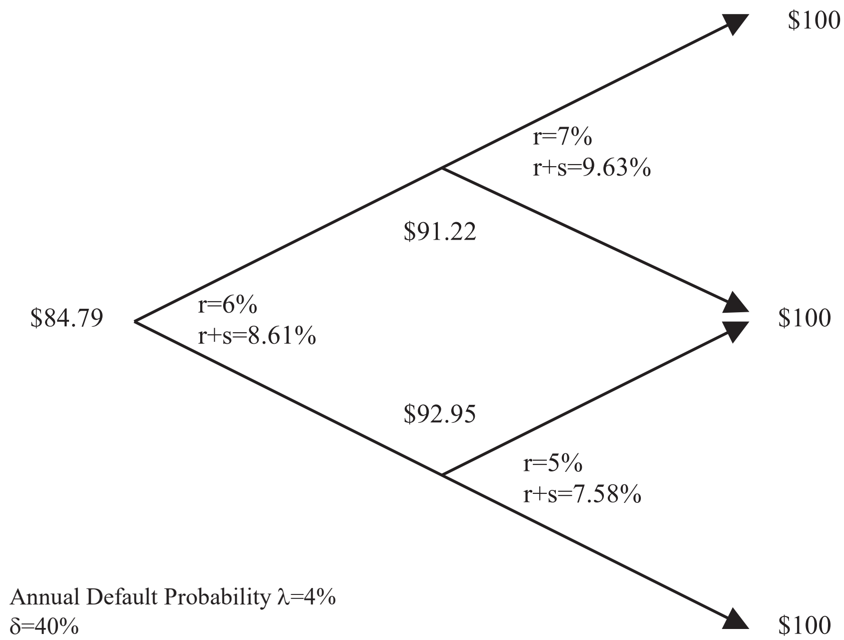

Figure 13.1: Duffie-Singleton Model

From the input information provided, we can compute the bond prices in the diagram, as follows:

$$91.215 = \frac{1}{1.07}[(1-4\%) \times 100 + 4\% \times 40]$$
$$92.95 = \frac{1}{1.05}[(1-4\%) \times 100 + 4\% \times 40]$$
$$84.79 = \frac{1}{1.06}\left[(1-4\%) \times \frac{91.215 + 92.95}{2} + 4\% \times \frac{91.215 + 92.95}{2} \times 0.4\right]$$

The spreads of these bonds are computed as follows:

$$\frac{100}{91.215} - 1 - 7\% = 9.63\% - 7\% = 2.63\%$$
$$\frac{100}{92.95} - 1 - 5\% = 7.58\% - 5\% = 2.58\%$$
$$\frac{92.0825}{84.79} - 1 - 6\% = 8.61\% - 6\% = 2.61\%$$

where $92.0825 = \frac{1}{2}(91.215 + 92.95)$. Note that these spreads are not (even though close to) the continuous spread in the Duffie-Singleton model, which is computed as follows:

$$
\begin{aligned}
&\frac{1}{\text{Default Prob} \times \text{Recovery} + \text{Survival Prob}} - 1\\
&= \frac{1}{4\% \times 0.4 + 96\%} - 1\\
&= 2.46\%
\end{aligned}
$$

13.5 Credit Default Swap

A Credit Default Swap, or CDS, is a bilateral contract which allows an investor to buy protection against the risk of default of a specified reference credit. The fee may be paid up front, but more often is paid in a "swapped" form as a regular, accruing cashflow. A CDS is a negotiated contract and there are a number of important features that need to be agreed between the counterparties and clearly defined in the contract documentation.

First and foremost is the definition of the credit event itself. This is obviously closely linked to the choice of the reference credit and will include such events as bankruptcy, insolvency, receivership, restructuring of debt and a material change in the credit spread. This last materiality clause ensures that the triggering event has indeed affected the price of the reference asset. It is generally defined in spread terms since a fall in the price of the reference asset could also be due to an increase in the level of interest rates.

Many CDS contracts define the triggering of a credit event using a reference asset. However, in many cases the importance of the reference asset is secondary as the credit event may also be defined with respect to a class of debt issued by a reference entity. In this case the importance of the reference asset arises solely from its use in the determination of the recovery price used to calculate the payment following the credit event.

The contract must specify what happens if the credit event occurs. Typically, the protection buyer will usually agree to do one of the following:

- Deliver the defaulted security to the protection seller in return for Par in cash. Note that the contract usually specifies a basket of securities which are ranked *pari passu* which may be delivered in place of the reference asset. In effect the protection seller is long a "cheapest to deliver" option.

- Receive Par minus the default price of the reference asset settled in cash. The price of the defaulted asset is typically determined via a dealer poll conducted within a few weeks to months of the credit event, the purpose of the delay being to let the recovery price stabilize.

Some CDS have a different payoff from the standard Par minus recovery price. The main alternative is to have a fixed pre-determined amount which is paid out immediately after the credit event. This is known as a binary default swap. In other cases, where the reference asset is trading at a significant premium or discount to Par, the payoff may be tailored to be the difference between the initial price of the reference asset and the recovery price.

The protection buyer automatically stops paying the premium once the credit event has occurred, and this property has to be factored into the cost of the swap payments. It has the benefit of enabling both parties to close out their positions soon after the credit event and so eliminates the ongoing administrative costs which would otherwise occur.

A CDS can be viewed as a form of insurance with one important advantage – efficiency. Provided the credit event in the default swap documentation is defined clearly, the payment due from the triggering of the credit event will be made quickly. Contrast this with the potentially long and drawn out process of investigation and negotiation which may occur with more traditional insurance.

However it is possible to get a very good idea of the price of the CDS using a simple "static replication" argument. This involves recognizing that buying a CDS on a risky par floating rate asset which only defaults on coupon dates is exactly equivalent to going long a default-free floating rate note and short a risky floating rate note of the same credit quality. If no default occurs, the holder of the position makes a net payment equal to the asset swap spread of the asset on each coupon date until maturity. This spread represents the credit quality of the risky floater at issuance. If default does occur, and we assume that it can only occur on coupon payment dates, the position can be closed out by buying back the defaulted asset in return for the recovery rate and selling the par floater. The net value of the position is equal to the payoff from the default swap. The following table summarizes.

CDS vs. Floater			
Event	Riskless FRN	Risky FRN	CDS
At inception	Pay par	Pay par	0
No default	Receive LIBOR	Receive LIBOR + spread	Pay spread
Upon default	Receive par	Receive recovery	+ par − recovery
Maturity	Receive par	Receive par	0

From the above table, it is clear that the spread of a CDS must equal to the spread of the equivalent risky FRN to avoid arbitrage.

13.6 Restructuring Definitions by ISDA

CDS contracts provide default protection. When a default occurs, CDS buyers stop paying the premium (spread), deliver the defaulted bond (cheapest if possible), and collect full face value as payment. However, default is hard to define. It is extremely rare for a company to file bankruptcy. What is usually happening is that losses happen over the years and reduce the asset value of the company, to a point where the company is at the brink of bankruptcy. Then the management of the company will start looking for alternatives to save the firm. One popular alternative to save the firm is to ask debt holders to change their debt contracts to the company – known as debt restructuring. Debt restructuring often means that debt holders convert parts of their debts into equity and participate in the operation of the firm. To protect their own interests, debt holders, especially large ones, will be willing to agree to debt restructuring.

Hence debt restructuring is commonly regarded as a form of default. However, each restructuring can be very different. Some restructurings are major and equivalent to defaults. But some could be minor as precautionary actions to avoid further deterioration of the firm, which are not equivalent to default.

To regulate if a CDS is triggered, ISDA (International Swaps and Derivatives Association) defines various restructuring standards:

- Full restructuring (FR), based on the ISDA 1999 Definition
- Modified restructuring (MR), based on the ISDA 2001 Supplement Definition
- Modified-modified restructuring (MMR), based on the ISDA 2003 Definition,
- No restructuring (NR).
- The definitions are as follows:

  FR Any bond of maturity up to 30 years

  MR $T \leqslant \bar{T} < T + 30$ months

  MMR Allow additional 30 months for the restructured bond.

  For other obligations, same as MR.

CDS contracts traded in different regions follow different ISDA conventions.

13.7 Why Has the CDS Market Developed So Rapidly?

CDS is the most popular credit derivatives contract and has grown rapidly in late 90's and early 00's. The following is a direct quote of Rene Stulz's article on CDS (2010):[2]

Back in the mid-1990s, one of the first credit default swaps provided protection on Exxon by the European Bank for Reconstruction and Development to JP Morgan (Tett, 2009). It took months to negotiate. By 1998, the total size of the credit default swap market was a relatively small $180 billion (Acharya, Engle, Figlewski, Lynch, and Subrahmanyam, 2009). The credit default swap market has grown enormously since then, although there is no definite measure of how much. Based on survey data from the Bank for International Settlements (BIS) at http://www.bis.org/statistics/derstats.htm, the total notional amount of the credit default swap market was $6 trillion in 2004, $57 trillion by June 2008, and $41 trillion by the end of 2008. Credit-default swap contracts that insure default risk of a single firm are called single-name contracts; in contrast, contracts that provide protection against the default of many firms are called multi-name contracts.[3]

In addition to the efficiency in hedging and transferring credit risk, the potential benefits of CDS include:

- A short positioning vehicle that does not require an initial cash outlay
- Access to maturity exposures not available in the cash market
- Access to credit risk not available in the cash market due to a limited supply of the underlying bonds
- The ability to effectively "exit" credit positions in periods of low liquidity
- Off-balance sheet instruments which offer flexibility in terms of leverage
- To provide important anonymity when shorting an underlying credit

[2] "Credit Default Swaps and the Credit Crisis," Journal of Economic Perspectives, Volume 24, Number 1, Winter 2010, pp. 73-92.

[3] Stulz noted that DTCC statics are a lot smaller.

13.8 Relationship between Default Probabilities and CDS Spreads – Use of the Jarrow-Turnbull Model

There is a simple formula (using the Jarrow-Turnbull model) that relates the CDS spread, the risk-free rate, default/survival probabilities, and the fixed recovery rate. Due to the swap nature, CDS, similar to IRS (interest rate swap), has two legs – floating and fixed. The floating leg of a CDS contract is called the protection leg as it pays only if default occurs. The fixed leg of a CDS contract is called the premium leg because the fixed payments (i.e. spreads) are like insurance premiums. As in a standard swap contract, at inception, the values of the two legs must equal each other. This is how CDS spreads are calculated. Recently affected by the crisis, CDS premiums have been split into an upfront and a spread (which is the index, such as CDX, trading convention). As we shall see later, this extra calculation does not add any complexity to the model. We simply deduct the upfront amount from the protection value of the CDS. For now, we shall proceed with no upfront.

Using the formulation given earlier, the protection and premium values of a CDS are as follows:

$$\begin{aligned} V_{\text{prot}}(t,T) &= (1-R)\sum_{i=1}^{n} P(t,T_i)[Q(t,T_{i-1}) - Q(t,T_i)] \\ V_{\text{prem}}(t,T) &= s(t,T)\sum_{i=1}^{n} P(t,T_i)Q(t,T_i) \end{aligned} \tag{13.16}$$

where $T_n = T$. As a result, the spread (known as par spread) can be computed as:

$$s(t,T) = \frac{(1-R)\sum_{i=1}^{n} P(t,T_i)[Q(t,T_{i-1}) - Q(t,T_i)]}{\sum_{i=1}^{n} P(t,T_i)Q(t,T_i)} \tag{13.17}$$

Note that (13.17) (for CDS) and (13.7) (for bond) are very similar. The numerator of (13.17) is similar to the recovery value in (the second line of) (13.7) and the denominator is similar to the (first line of (13.7) coupon value. This should not be surprising as CDS is a natural hedge to the bond. In other words, buy a bond and a CDS is equivalent to buying a default-free bond. If we add the protection value in (13.16) to the coupon bond value in (13.7), recovery disappears and the bond as a result becomes default-free.

Note that (13.17) assumes no accrued interest if default occurs in between coupons. In reality there are accrued interests on both legs and they may not be equal. If default is assumed to happen on cashflow days only, then there is no accrued interest. Note that if there is an upfront, we simply deduct it from the protection value, V_{prot}.

13.9 Back-of-the-envelope Formula

In a one-period model where default is a Bernulli event, as the following picture demonstrates,

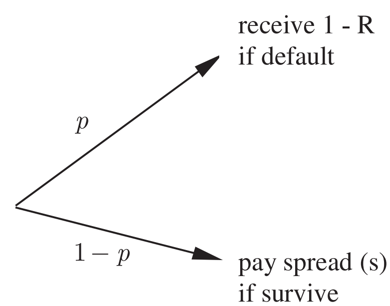

Figure 13.2: One-period Default Diagram

We know that for the CDS to have 0 value it must be true that:

$$p(1-R) = (1-p)s \tag{13.18}$$

Note that risk-free discount cancel from both sides. Hence, we arrive at the famous back-of-the-envelope formula for the default probability (by ignoring the term $p \times s$ which is small):

$$p = \frac{s}{1-R} \tag{13.19}$$

This formula, while simple, provides a powerful intuition of spreads and default probabilities. If the recovery is 0, then spread is (forward) default probability. This is not only true in (13.19) but also true in continuous time. Spreads are not equal to (i.e. smaller) forward default probabilities in that they are compensated by recovery. Note that CDS buyers acquire default protection by paying spreads as an insurance premium. If recovery is high, the protection value is low, and so should be the spread. In an extreme case where the recovery is 100%, the spread should be 0, which is suggested by (13.19).

13.10 Bootstrapping (Curve Cooking)

We need credit curves to price credit derivatives. Credit curves are obtained from liquid "cash" products such as CDS or corporate bonds. Due to the liquidity concern, CDS is a better choice for curve cooking.

The basic bootstrapping idea of constructing a risky curve is the same as the risk-free curve. We use a pricing formula to back out the parameter(s). In the traditional fixed income world (Treasuries and IRS), we back out spot and forward rates from the market prices of bonds and swap rates. Here, we back out survival probabilities from a series of CDS contracts. As in the world of traditional fixed income, we need a term structure of CDS spreads in order to back out the entire survival probability curve. A popular smoothing technique in LIBOR curves is piece-wise flat.

The CDS market has been standardized over the years to have the following on-the-run maturities: 1, 2, 3, 5, 7, and 10 years to maturities. As in the IRS market, these contracts are "on-the-run" which are issued periodically. In the early years, only 5-year CDS contracts were issued. A few years ago, the market started to trade 10-year CDS contracts. The other maturities have gradually been introduced to the market but their liquidity is still a concern. Assume for now that we observe market prices of these CDS spreads.

13.11 Poisson Assumption

From (13.2) $\sim$ (13.4), we know that if we assume piece-wise flat intensity values, then the survival probability can be approximated as follows:

$$Q(t, T_n) = \exp\left\{-\sum_{i=1}^{n} \lambda_i (T_i - T_{i-1})\right\} \tag{13.20}$$

where $t = T_0$.

13.12 Simple Demonstration (annual frequency)

To make matters simple, we assume CDS spreads are paid annually. There are 6 CDS spreads observed in the market (1, 2, 3, 5, 7, 10). Take Disney as an example. On 12/23/2005, we observe the following spreads (in basis points):

CDS quotes	
term	sprd
1	9
2	13
3	20
5	33
7	47
10	61

Continue to assume fixed 0.4 recovery ratio under MR for Disney. The following binomial chart presents possible cash flows. Let's assume 5% risk-free rate.

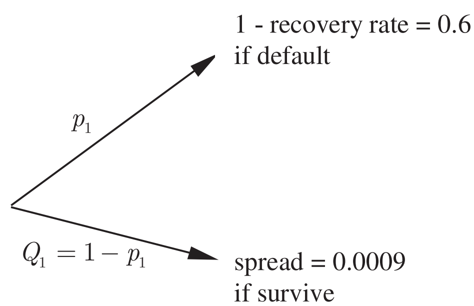

Figure 13.3: First period

CDS is a swap contract so there is no cash changed hands on day 1. Hence, it must be the case that the expected payment (9 basis points) equals the expected compensation (60%). In a single period, since both payment and compensation are discounted, the risk-free does not matter. Note that the survival probability for one year is $Q_1 = 1 - p_1$ (which is also equal to $e^{-\lambda_1}$ if we assume the Poisson process for defaults). Hence, using 5% interest rate, we have:

$$\frac{0.6 \times (1 - Q_1) - 0.0009 \times Q_1}{1.05} = 0$$

which is solved as:

$$0.6 \times (1 - Q_1) = 0.0009 \times Q_1$$

and $Q_1 = 0.9985$. λ_1 can be solved for as $-\ln Q_1$ to be 0.001499, or 14.99 basis points. Hence, the survival and default probabilities are 99.85% and 0.15% respectively.

Now we proceed to bootstrap the second period.

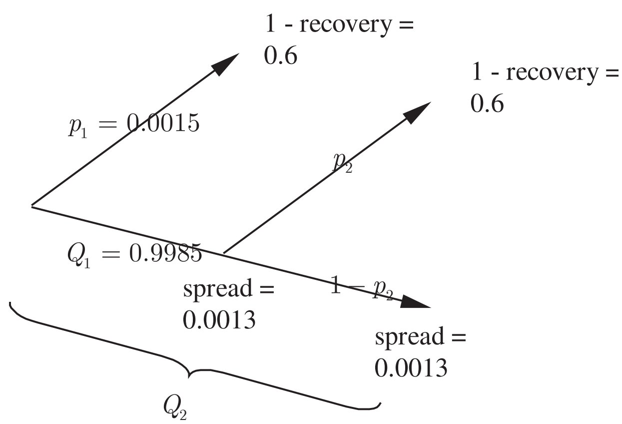

Figure 13.4: Two-period Default Diagram

There are three scenarios. Either Disney defaults in period 1, or default in period 2, or survive in period 2 (note that to survive till period 2, Disney must first survive period 1). We know the first default probability, which is $p_1 = 0.0015$. But we do not know the other two probabilities. Using the same Poisson algorithm, we can compute the second year present value as:

$$\frac{0.6 \times p_2 - 0.0013 \times (1 - p_2)}{1.05}$$

Note that this quantity itself is not 0; but combining it with the first year cash flows is:

$$\frac{0.6 \times 0.0015 + 0.9985 \left(-0.0013 + \frac{0.6 \times p_2 - 0.0013 \times (1-p_2)}{1.05} \right)}{1.05} = 0$$

Solve for p_2 to get 28.62 basis points. Under the Poisson assumption, $e^{-\lambda_2} = 1 - p_2$ and as a result, $\lambda_2 = -\ln(1 - p_2) = 0.002865$ or 28.65 basis points. The conditional survival probability is $1 - p_2 = 99.7138\%$. The unconditional survival probability, Q_2, equals $(1-p_1)(1-p_2)$ which is 99.5645%. The unconditional default probability is $1 - 99.5645\% = 0.4355\%$.

Similar process applies to all periods as the following figure depicts. Due to the limitation of the space in the table, $Q(t, v)$ is replaced with $Q(\tau)$ where $\tau = v-t$.

The results are given as follows. The first three columns are taken from the

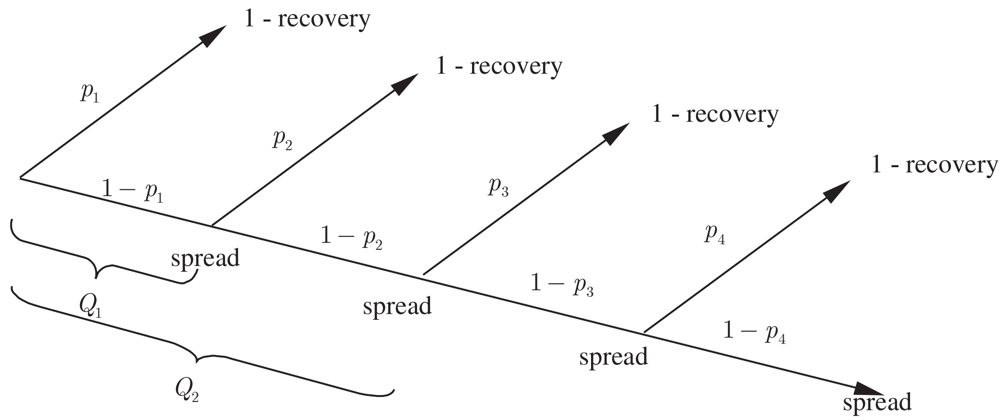

Figure 13.5: Multi-period Default Diagram

market. Columns A and B are the same spread input given earlier. Column C is the risk-free discount factors that are computed using 5% flat in the example.

CDS Bootstrapping					
A	B	C	D	E	F
Term τ	Market Spread	Risk-free $P(\tau)$	Fwd. $\lambda(\tau)$	Surv.Pr. $Q(\tau)$	Def.Pr. $-dQ(\tau)$
1	0.0009	0.9512	0.0015	0.9985	0.0015
2	0.0013	0.9048	0.0029	0.9956	0.0029
3	0.0020	0.8607	0.0059	0.9898	0.0058
4		0.8187	0.0092	0.9808	0.0091
5	0.0033	0.7788	0.0092	0.9718	0.0090
6		0.7408	0.0150	0.9573	0.0144
7	0.0047	0.7047	0.0150	0.9431	0.0142
8		0.6703	0.0176	0.9267	0.0164
9		0.6376	0.0176	0.9106	0.0161
10	0.0061	0.6065	0.0176	0.8948	0.0158

CDS Bootstrapping (cont'ed)					
G	H	I	J	K	L
$P(\tau)Q(\tau)$	$P(\tau)\times$ $[-dQ(\tau)]$	Prem. Leg	Prot. Leg	Model Spread	Cond. Def.Pr.
0.9498	0.0014	0.9498	0.0009	0.0009	0.0015
0.9009	0.0026	1.8507	0.0024	0.0013	0.0029
0.8520	0.0050	2.7027	0.0054	0.0020	0.0058
0.8030	0.0074	3.5056	0.0099	0.0028	0.0092
0.7568	0.0070	4.2625	0.0141	0.0033	0.0092
0.7092	0.0107	4.9717	0.0205	0.0041	0.0148
0.6646	0.0100	5.6363	0.0265	0.0047	0.0148
0.6212	0.0110	6.2575	0.0331	0.0053	0.0174
0.5806	0.0103	6.8381	0.0393	0.0057	0.0174
0.5427	0.0096	7.3808	0.0450	0.0061	0.0174

Column E presents the survival probabilities that are computed sequentially as in equation (13.20):

$$\begin{aligned} Q(t,T_n) &= \exp\left\{-\sum\nolimits_{i=1}^{n} \lambda_i (T_i - T_{i-1})\right\} \\ &= Q(t,T_{n-1}) \exp\left\{-\lambda_n (T_n - T_{n-1})\right\} \end{aligned} \tag{13.21}$$

For example, $Q(0,2) = 0.9956 = 0.9985 \times e^{-0.0029\times(2-1)}$. Column F is the *unconditional* default probabilities which is the differences are two consecutive survival probabilities. For example, $0.0029 = 0.9985 - 0.9956$. Column G is known as the risky discount factor (introduecd earlier), or \$1 present value with default risk. These values are needed in order to compute the default swap spread, i.e. the denominator of (13.17). Similarly, column H provides the values for the numerator of (13.17). Columns I and J are accumulations of columns G and H respectively. Column K is the division of column J by column I, which is the spread of CDS. The values of this column must match the market quotes in column B. In fact, we solve for column D so that values in column K are identical to values in column B. Finally, column L contains conditional default probabilities, each of which equals $1-Q(t,T_j)/Q(t,T_{j-1})$. We note that the conditional default probabilities are close to the intensity values (λ), as they should, in that they are exactly equal in continuous time.

The table presented here provides a nice algorithm for further automate the calculations for more complex situations in reality, which we shall demonstrate later. Once all the λ_j values are "bootstrapped" out, we can then compute any survival probability of any future time, as follows:

$$Q(t, v) = Q(t, T_{n-1}) \exp\{-\lambda_n (v - T_{n-1})\} \tag{13.22}$$

where $T_{n-1} < v < T_n$. For example, the survival probability of 6.25 years is $0.9769 \times e^{-0.0098\times(6.5-5)} = 0.9627$.

13.13 In Reality (quarterly frequency)

In the above example, we assume spreads are paid annually. As a result, the calculation is quite simple. In reality, this is not the case. Spreads are paid by the swap market convention which is quarterly. In this case, default can occur at any quarter. We then need to alter the one period calculation shown above.

Note that within a year (for the first few spreads), all per-quarter default probabilities are equal. This is because we have only one spread (e.g. 0.0009 in year 1) to cover four quarters. The basic formula is still the same. Mainly we solve the following equation for p (note that at each period, the interest rate is 1.25%):

$$\begin{aligned} x_i &= x_0 + 0.6p_1 + (1 - p_1)x_{i-1} \\ x_n &= x_0 \end{aligned} \tag{13.23}$$

where n represents the number of periods that shares the probability. For the first year, $n = 4$ and $x_0 = 0.000225$. We then solve for $p_1 = 0.000375$ or 3.75 basis points and $\lambda_1 = 0.0015$. The full expansion of this equation is shown in the Appendix.

Note that while this equation is solvable by hand if proper re-arrangement of terms is performed, it is much faster if we set up the equation and use the Solver in Excel. This equation can be set up recursively as the discounting and expected values are nested. We can set up an Excel sheet to compute all the results. As a demonstration, we provide the results up the 3 years.

The layout of the table is the same as before. The frequency of the CDS premium payments is now quarterly (see column A). Column B is still market CDS spreads that are available every four quarters. Column C is quarterly risk-free discount factors (at 5%). Columns E $\sim$ H are computed similarly to the previous section, only with quarterly frequency.

Columns, I, J, and K are computed similarly to the previous section but only every year. Note that column K is column J $\div$ column I $\times$ 4 in order to annualize to match annual CDS market quotes in column B.

Readers should complete the table for the full 10 years.

CDS Bootstrapping					
A	B	C	D	E	F
Term	Market Spread	Risk-free $P(\tau)$	Fwd. $\lambda(\tau)$	Surv.Pr. $Q(\tau)$	Fwd.Def.Pr. $-dQ(\tau)$
0.25		0.9876	0.0015	0.9996	0.000375
0.5		0.9753	0.0015	0.9993	0.000375
0.75		0.9632	0.0015	0.9989	0.000375
1	0.0009	0.9512	0.0015	0.9985	0.000374
1.25		0.9394	0.002868	0.9978	0.000716
1.5		0.9277	0.002868	0.9971	0.000715
1.75		0.9162	0.002868	0.9964	0.000715
2	0.0013	0.9048	0.002868	0.9956	0.000714
2.25		0.8936	0.00586	0.9942	0.001457
2.5		0.8825	0.00586	0.9927	0.001455
2.75		0.8715	0.00586	0.9913	0.001453
3	0.002	0.8607	0.00586	0.9898	0.001451

CDS Bootstrapping					
G	H	I	J	K	L
$P(\tau)Q(t)$	$P(\tau)[-dQ(\tau)]$	premium leg	protection leg	computed spread	cond. surv.prob.
0.9872	0.0004				
0.9746	0.0004				0.9996
0.9621	0.0004				0.9996
0.9498	0.0004	3.8737	0.0009	0.0009	0.9996
0.9373	0.0007				0.9993
0.925	0.0007				0.9993
0.9129	0.0007				0.9993
0.9009	0.0006	7.5498	0.0025	0.0013	0.9993
0.8884	0.0013				0.9985
0.8761	0.0013				0.9985
0.8639	0.0013				0.9985
0.8519	0.0012	11.0302	0.0055	0.002	0.9985