# Bond (Fixed Income) Primer

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## Treasuries

On-the-run Treasuries are those that get auctioned by the Treasury department. Roughly there are 11 of them: 1M, 3M, 6M, 12M (1Y), 3Y, 5Y, 7Y, 10Y, 15Y, 20Y, 30Y. A bond stays on the run until its replacement. For example, a 30-year bond can still be on the run after 4 months of its auction (i.e. 29 years and 8 months left). When a new 30-year bond is issued, then the old one becomes off the run. Due to this "roll-down" feature, the Treasury department computes CMT (constant maturity Treasury) rates. These are inter(extra)polated rates. CMT rates are rolling so we can find, e.g., the 30-year T rate every day.

### *1. Zero-coupon bonds*

Zero-coupon bonds (commonly called zeros) are discount bonds because they are sold at discounts (due to no interests). For example, a 30-year zero with \$100 notional can be sold at \$55. That is, \$45 interest to be made in a lump sum at maturity in 30 years (however, there is a tax issue – that is taxes must be paid annually so an accrued interest is computed annually and usually it is continuously compounded).

It is customary to use just \$1 notional as the base unit. Then the 30-year zero is priced at 55 cents. This is convenient because we can apply any notional amount on the base unit. A \$25,000 notional zero-coupon bond is priced at \$25,000 × 0.55 = \$13,750.

Among Treasuries, only T bills pay no coupons. Yet T bills are no more than 1 year. Longer term zeros are STRIPS (Separate Trading of Interest and Principal Securities which have coupon strips and principal strips) but they are not liquid (coupon strips are more liquid than principal strips, for obvious reasons).

The price-yield relationship of zeros can be described as follows:

$$y(t,T) = \frac{-1}{T-t}\ln P(t,T) \qquad\qquad P(t,T) = e^{-y(t,T)(T-t)}$$

where t is current date and T is maturity date (in years). This is continuous time.

Discrete time formulas are messy, depending on frequency and if y is adjusted.

$$y(t,T) = \sqrt[T-t]{\frac{1}{P(t,T)}} - 1 \qquad\qquad P(t,T) = \frac{1}{(1+y(t,T))^{(T-t)}}$$

$$y(0,n) = \sqrt[m\times n]{\frac{1}{P(0,n)}}m - 1 \qquad\qquad P(0,n) = \frac{1}{\left(1+\frac{y(0,n)}{m}\right)^{m\times n}}$$

where the second equation is used mainly in cases of integer years – not very realistic.

In reality, both t and T are calendar dates,say 3/5/2021 and 12/31/2050 respectively. Hence T-t is 18903 days but in years it is 18903/365.25=29.82. However, this is a rough approximation. The correct number in years should adopt the "daycount" convention, which is to use `yearfrac` (year fraction) in Excel. There are four popular daycount conventions:

- 30/360
- Act/Act
- Act/360
- Act/365

## 2. Coupon-bearing bonds

T notes and T bonds are coupon bonds. There is no structural difference between the two other than the fact that T notes are no more than 10 years and T bonds are longer than 10 years. Also, T notes are auctioned more frequently than T bonds by the Treasury department.

Price of a coupon bond can be determined as a portfolio of zeros:

$$\Pi(c;t,T_1,\cdots,T_n) = cP(t,T_1) + cP(t,T_2) + \cdots + (1+c)P(t,T_n)$$

$$= c\sum_{i=1}^{n} P(t,T_i) + P(t,T_n)$$

This is because arbitrage profits could be made should it not hold. If the coupon bond price is higher than the portfolio value, then short the coupon bond and buy c units (remember each unit has a face value of \$1) of the zeros (till $T_{n-1}$) and 1+c units of the $T_n$-maturity zero. At each coupon date, you sell the zero to receive \$c to pay for the coupon. As a result, you end up with no obligation whatsoever. Yet you get to keep the arbitrage profit (the difference between the coupon bond value and the value of the portfolio of zeros) on day 1. If opposite is true, then buy the coupon bond and sell zeros. At each coupon date, you receive a coupon of \$c which is exactly the obligation you owe to the short zero. Again, you end up with no obligation whatsoever and you can keep the day 1 arbitrage profit.

Again, it is customary to keep the notional of coupon bond at \$1. Hence, \$c is a percent (like \$0.025).

## 3. IRS

The standard IRS (known as vanilla IRS) is a fixed-floating interest rate swap. A long IRS is to pay fixed and in return receive floating (indexed to LIBOR). Notionals are not exchanged (FXS do though) as they are equal and hence cancelled.

The fixed rate (called the swap rate) is determined at inception of swap. The fixed leg is like a fixed rate bond (when principal added back) and the floating leg is like a floater. Floater has a price of par always.

Not only are zeros useful in pricing coupon bonds, they are also useful in measuring the risk of a swap (most importantly, IRS, FXS and CDS). The fixed leg of a swap. The fixed leg of an IRS is an annuity. Its PV is:

$$V_{\text{fixed}}(t) = w(t)\sum_{i=1}^{n} P(t,T_i)$$

and as a result, it is clear that $\sum_{i=1}^{n} P(t,T_i)$ is an annuity factor (i.e. PV of $1). The P&L of an IRS is:

$$V_{\text{fixed}}(t) - V_{\text{fixed}}(t_0) = (w(t) - w(t_0))\sum_{i=1}^{n} P(t,T_i)$$

If $w(t) - w(t_0)$ is 1 bp, then $\sum_{i=1}^{n} P(t,T_i)$ is the 1 bp risk of an IRS. This is known as DV01 (or PV01) – 1 basis point risk.

In FXS or CDS, an additional adjustment is added. The former needs the exchange rate adjustment and the latter needs the default/survival probability adjustment. But the basic idea is the same. The annuity is the risk measure. And the *sum of discount factors is the annuity*. [Recall how you learned before about the annuity using a flat interest rate. That sounds silly, doesn't it?]

### *4. Forward of zero*

This contract does not exist in reality but it is very useful to help calculate forward prices of other contracts. So we pretend it to exist. Should it exist, the prices must be:

$$\Psi(t,T_i,T_j) = \frac{P(t,T_j)}{P(t,T_i)}$$

due to no arbitrage. This is the forward price of $P(t,T_j)$ (say 3-year zero) settled at time $T_i$ (say 1 year). If the forward price of $P(t,T_j)$ is not so, then one can buy/sell the forward and sell/buy the two zeros to make an arbitrage profit.

The forward rate:

$$\begin{aligned} f(t,T_i,T_j)(T_j - T_i) &= -\ln \Psi(t,T_i,T_j) \\ &= \ln P(t,T_i) - \ln P(t,T_j) \\ &= y(t,T_j)(T_j - t) - y(t,T_i)(T_i - t) \end{aligned}$$

which is the difference of two yields (unannualized). If $T_j \to T_i$, then the forward rate is like a first-order derivative of the yield curve.

### *5. Forward of coupon bond*

$$
\begin{aligned}
\Psi_{\Pi}(c;t,T_1\cdots T_n) &= \frac{\Pi(c;t,T_1\cdots T_n)}{P(t,T_i)} \\
&= \frac{1}{P(t,T_i)}\Big(c\sum\nolimits_{j=1}^{n} P(t,T_j)+P(t,T_n)\Big) \\
&= c\sum\nolimits_{j=1}^{n} \Psi(t,T_i,T_j)+\Psi(t,T_i,T_n)
\end{aligned}
$$

which is a portfolio of forwards on zero.

An example of such contract is FRA (forward rate agreement). Suppose you want to lock in interest rate of a loan for the house you want to buy. You can then engage in an FRA. The forward rate you lock in today will last for the next, say, 30 years and the contract is valid (won't expire) for 3 months (time needed to inspect/sign/close the house). Hence, effectively, FRAs are forwards on coupon bond. Note that FRAs are all short term. Hence, long term forwards are non-existent but we still need them to conduct trading and investment strategies.)

In reality, you do not go out to buy an FRA (you are not qualified as an individual – must be a licensed financial institution). So the mortgage bank to which you apply your mortgage loan will do that on your behalf. These mortgage banks do FRAs in large volumes and distribute them to their customers such as yourself. This is how you can get a locked commitment from your bank.

### *6. Forward of anything*

For any asset, its forward price is:

$$
\Psi(t,T_F) = \frac{S(t)}{P(t,T_F)}
$$

Forward price is considered as the median (as opposed to mean) of future spot prices (at settlement time $T_F$). Which means the probabilities of $S(T_F) \geq \Psi(t,T_F)$ and $S(T_F) \leq \Psi(t,T_F)$ are equal (50%). This is so because otherwise there would be no trade. A forward contract is a pure bet on spot rising/falling.

As a result, forward prices are good benchmark to measure the future. Note that futures prices, on the amazing contrary, are not good benchmarks of the future. [Why?]

## LIBOR (SOFR)

LIBORs are another benchmark interest rates and used very similarly to Treasuries. However, the market of LIBORs is very different from the market of Treasuries.

LIBORs are close-door determined (by a handful of major banks in London) and like T bills, no longer than 1 year. There are short term offer rates (not bid rates), not bonds.

Longer term LIBORs are implied by LIBOR derivatives like Eurodollar futures and IRSes.

Strictly speaking, LIBORs are not risk-free. They are higher than Treasuries due to its default risk (after all, these are private banks) – known as LIBOR-Treasury basis. Yet, due to its popularity in floating rate bonds (mainly corporates who benchmark their coupons to LIBORs), practically they are regarded as risk-free.