Triple Crown Conference

Keynote Speech

Modeling — Why, What, and How

Jonathan Ingersoll

May 8, 2015

Modeling — Why, What, and How

What about when?

Modeling — Why, What, and How

Why do we model?

To help us understand a too-complicated world

What do we model?

Many things. Markets, the behavior of prices, the actions of agents

What do models tell us?

We hope – a portion of the truth

How do we model?

By building a logical, often mathematical, structure
(This is true in many fields, not just Economics and Finance)

Modeling — Why, What, and How

We model by building a mathematical structure, but why should this work?

> *How can it be that mathematics, being after all a product of human thought which is independent of experience, is so admirably appropriate to the objects of reality?*
>
> — Albert Einstein

> *The most incomprehensible thing about the world is that it is comprehensible.* — Albert Einstein

Einstein was talking about Physics. In Physics, Math works wonderfully — perhaps illogically so, but work it does. It works so well that even Math that seems totally illogical works.

Modeling — Why, What, and How

Einstein was talking about Physics. In Physics, Mathematics works wonderfully — perhaps illogically so, but work it does. It works so well that even Math that seems totally illogical works.

Quantum mechanics is based on probabilities which are computed from wave functions — functions that can be positive or negative where probabilities cancel. this is most famously seen in wave-particle duality in the two-slit experiment

Even more amazingly, the result

$$1+2+3+\cdots=-\tfrac{1}{12}$$

is used in string theory[†]

In mathematics, you don't understand things. You just get used to them.

— John von Neumann

[†]This result is based on the analytical continuation of the Reimann zeta function $\zeta(s)\equiv\sum_{n=1}^{\infty}n^{-s}$. The sum is convergent for all real $s > 1$. Its analytical continuation into the complex plane is defined for all complex z, except $z = 1$. Analytical continuations are unique, and $\zeta(-1)=-\tfrac{1}{12}$ so this is a mathematical, if illogical, answer.

Modeling — Why, What, and How

into why Lorentz invariance would be lost in the light-cone approach for the wrong A or D.

First, we assert that the operator ordering constant in the Hamiltonian for a free field comes from summing the zero-point energies of each oscillator mode, $\frac{1}{2}\omega$ for a bosonic field like X^μ. Equivalently, it always works out that the natural operator order is averaged, $\frac{1}{2}\omega(aa^\dagger + a^\dagger a)$, which is the same as $\omega(a^\dagger a + \frac{1}{2})$. In H this would give

$$A = \frac{D-2}{2} \sum_{n=1}^{\infty} n \, , \tag{1.3.31}$$

the factor of $D-2$ coming from the sum over transverse directions. The zero-point sum diverges. It can be evaluated by regulating the theory and then being careful to preserve Lorentz invariance in the renormalization. This leads to the odd result

$$\sum_{n=1}^{\infty} n \to -\frac{1}{12} \; . \tag{1.3.32}$$

To motivate this, insert a smooth cutoff factor

page 22 of *Bosonic String Theory* by Polchinski

Modeling — Why, What, and How

Will or should mathematical structures work as well in economics as they do in physics?

> Perhaps — both math and economics are products of human thought.
> This blunts Einstein's concern

On the other hand, our problems do seem more difficult in a different fashion. Particles in physics obey fairly simple laws at least to first order. People are always looking for loopholes, and they know they're playing the game.

> *While physics and mathematics may tell us how the universe began, they are not much use in predicting human behavior because there are far too many equations to solve. I'm no better than anyone else at understanding what makes people tick, particularly women.* —Stephen Hawking

> *If scientific reasoning were limited to the logical processes of arithmetic, we should not get very far in our understanding of the physical world. One might as well attempt to grasp the game of poker entirely by the use of the mathematics of probability.* —Vannevar Bush

Modeling — Why, What, and How

Consider a typical model: The CAPM

1) Investors have homogeneous beliefs
2) Asset prices have joint normal returns
   etc.

Assumption 1 lets us talk about <u>some</u>thing. Without homogeneous beliefs what expected returns would we be talking about?

Assumption 2 allows us to use mean-variance mathematics simplifying the analysis.

These are strong and clearly incorrect assumptions, but we accept them because they lead to a strong and interesting result.

All investors hold combinations of the risk-free asset and the market portfolio so

$$\mathbb{E}[\tilde{r}_i] = r_f + \beta_i \left(\mathbb{E}[\tilde{r}_{mkt}] - r_f \right)$$

Modeling — Why, What, and How

Compare the CAPM to the APT:

$$\tilde{\mathbf{r}}_n = \mathbf{a}_n + \mathbf{B}_n \tilde{\mathbf{f}} + \tilde{\boldsymbol{\varepsilon}}_n \qquad \text{with } \mathbb{E}[\tilde{\boldsymbol{\varepsilon}}_n]=\mathbf{0}_n \;\; \mathbb{E}[\tilde{\mathbf{f}}]=0 \;\; \mathbb{E}[\tilde{\boldsymbol{\varepsilon}}_n \tilde{\mathbf{f}}'] = \mathbf{0}_{n\times k} \text{ and } \left\| \mathbb{E}[\boldsymbol{\varepsilon}_n \boldsymbol{\varepsilon}_n'] \right\| \leq \omega < \infty \;\forall n$$

along with no "asymptotic arbitrage" this guarantees:

$$\Rightarrow \mathbf{a}_n = r_f \mathbf{1}_n + \mathbf{B}_n \boldsymbol{\lambda} + \mathbf{v}_n \qquad \text{with } \mathbf{v}_n' \mathbf{v}_n \leq V \;\forall n$$

$$i.e., \quad \mathbf{a}_n \approx r_f \mathbf{1}_n + \mathbf{B}_n \boldsymbol{\lambda}$$

This is a much weaker set of assumptions leading to a weaker result.

Which is to be preferred?

Modeling — Why, What, and How

How do we test the CAPM?

$$\text{Does } \overline{r}_i \stackrel{?}{=} r_f + \hat{\beta}_i \left(\overline{r}_{mkt} - r_f \right)$$

Modeling — Why, What, and How

How do we test the CAPM?

$$\text{Does } \overline{r}_i \stackrel{?}{=} r_f + \hat{\beta}_i \left(\overline{r}_{mkt} - r_f \right)$$

But not by asking. “Do you hold an S&P 500 index† fund?”

We are interested in the former rather than the latter because we are often not concerned about how individual agents act, but only the macro consequences of their actions.

If agents deviate from optimal modeled strategies, but do so in random and independent ways, the mistakes they introduce may cancel leaving us with a model that is (approximately) correct.

However if we are interested in behavior, then the CAPM isn’t very good.

† With apologies to Dick Roll. That should read the Wilshire 5000 + real estate + human capital + …

Modeling — Why, What, and How

How do we test the APT?

$$\text{Does } \bar{r}_i \stackrel{?}{=} r_f + \sum_k \hat{\beta}_{ik} \left(\bar{r}_k - r_f \right)$$

for *k* portfolios mimicking the *k* factors

This is where we put in the strong assumption: $\approx \quad \rightarrow \quad =$

Also do people hold well-diversified portfolios?

Now the behavioral prediction does seem to be satisfied approximately and seems to be of interest.

Modeling — Why, What, and How

Do investors behave as models assume at least approximately?

Perhaps, but consider the Merton Model

Does the average investor do this

$$\max \mathbb{E}\left[\int_0^T U(C,t)dt + B(W,T)\right]$$

In fact does s/he even know what it means?

Perhaps — just possibly

Modeling — Why, What, and How

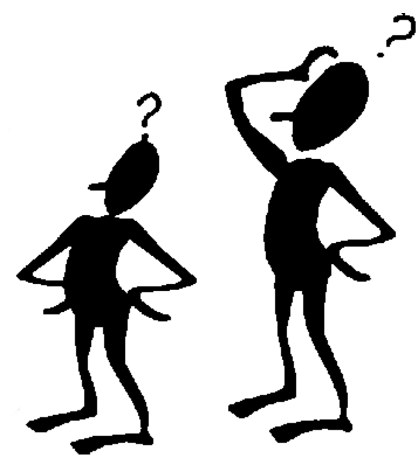
Just possibly, but how about the continuation and solution?

$$
\begin{aligned}
J(W,\mathbf{x},t) &= \max \mathbb{E}\left[\int_t^T U(C_s,s)ds + B(W,T)\ \middle|\ \mathbf{x}_t\right] \\
&= \max\left\{U(C_t,t)dt + \mathbb{E}\left[\int_{t+dt}^T U(C_s,s)ds + B(W,T)\middle|\mathbf{x}_t\right]\right\} \\
&= \max_{C_t,\mathbf{w}_t}\left\{U(C_t,t)dt + J(W+dW,\mathbf{x}+d\mathbf{x},t+dt)\right\} \\
\Rightarrow \quad 0 &= \max_{C_t,\mathbf{w}_t}\left\{U(C_t,t)dt + dJ(W,\mathbf{x},t)\right\}
\end{aligned}
$$

$$
\begin{aligned}
0 = U(C_t,t)dt &+ J_W\mathbb{E}[dW] + \tfrac{1}{2}J_{WW}\operatorname{var}[dW] + J_t dt + \sum J_{Wx_i}\operatorname{cov}[dW,dx_i] \\
&+ \sum J_{x_i}\mathbb{E}[dx_i] + \sum\sum J_{x_i}J_{x_j}\mathbb{E}[dx_i dx_j]
\end{aligned}
$$

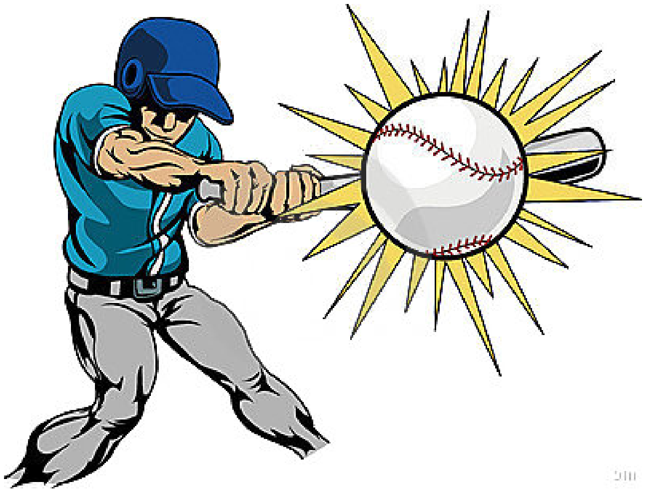
$$
0 = U_C - J_W \qquad \Rightarrow \qquad C^* = U_C^{-1}(J_W)
$$

$$
\mathbf{0} = W^2 J_{WW}\boldsymbol{\Sigma}\mathbf{w} + WJ_W(\boldsymbol{\mu} - r\mathbf{1}) \qquad \Rightarrow \qquad \mathbf{w}^* = -\frac{J_W}{WJ_{WW}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1})
$$



Modeling — Why, What, and How

But this is actually no different than everyday physics.



Pitch speed = 90 mph

baseball weight† = 5.125 oz

bat swing speed‡ = 60 mph

bat weight = 38 oz

For a perfectly elastic collision

$$m_{bat,1}v_{bat,1} + m_{ball,1}v_{ball,1} = m_{bat,2}v_{bat,2} + m_{ball,2}v_{ball,2}$$

But coefficient of restitution

$$\equiv -\frac{v_{ball,1} - v_{ball,2}}{v_{bat,1} - v_{bat,2}} \approx 0.44 \quad \leftarrow$$ empirical fact often ignored by theorists

ball exit speed ≈ 105 mph = 154 feet per second

time to reach fence at 375 feet $= \frac{375}{154} = 2.43$ seconds

$\Rightarrow$ RUN

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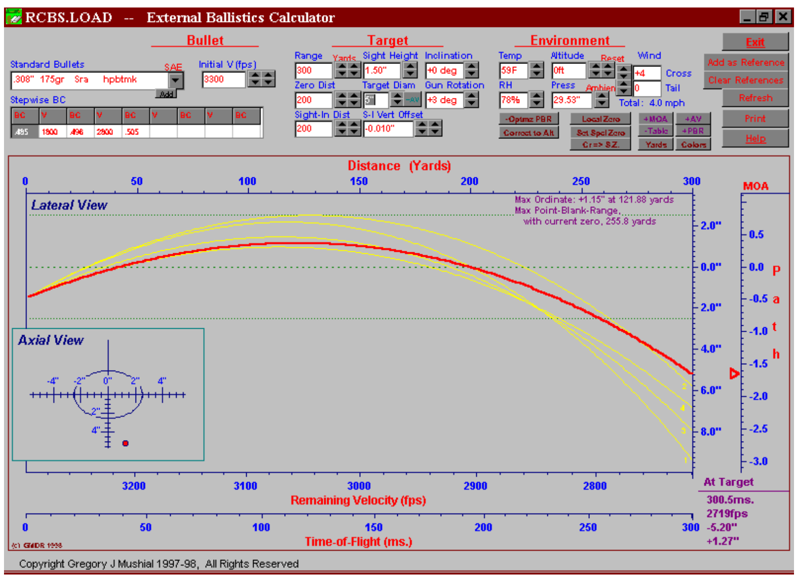
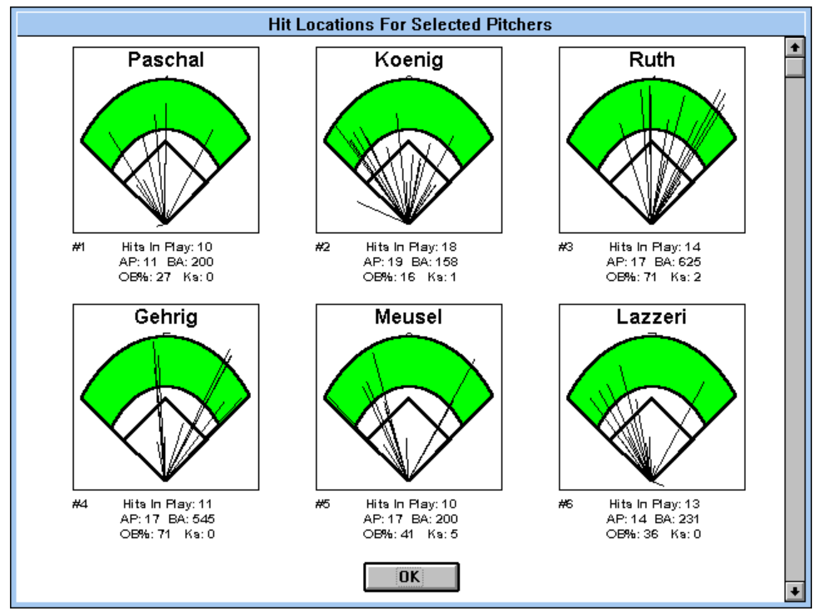
† Should be mass but this is economics not physics

‡ Major league power hitter

Modeling — Why, What, and How

This is a simplified one-dimensional model of the problem. It ignores some “trivial” details like which direction.





Modeling — Why, What, and How

Nevertheless

Modeling — Why, What, and How

A: The physics problem is relatively straight-forward if messy

B: The outfielder does often catch the ball[†]

[†] More often against my team than for my team at a p level of 0.01

Modeling — Why, What, and How

A: The physics problem is relatively straight-forward if messy

B: The outfielder does often catch the ball

$$\text{but Corr}[A,b] \approx 0$$

The same may well be true in Economics and Finance. We don't necessarily need to model exactly what agents are doing to get accurate predictions about the outcomes.

Modeling — Why, What, and How

What is a representative investor?

The representative investor is the average investor. S/he holds the market portfolio and consumes per capita consumption.

I have three concerns with this.

First while we say representative, we use average, should we be using typical?

What do we mean by average?

What do we mean by investor?

Modeling — Why, What, and How

What is a representative investor?

Is the representative investor the average investor or the typical investor?

Average makes our models simple. We can identify the market portfolio and per capita consumption with some caveats.

The representative investor always holds a net-zero position in all derivative assets so all derivative asset prices are shadow price.

But it may well be that idiosyncrasies do not average out and representativeness may well matter more than averageness

- Investors do not have homogeneous beliefs.
- Investors do have non-tradable human capital
- Some investors take large positions for corporate control reasons
- Employees are often compensated with bonuses which cannot be traded and are highly correlated with their human capital
- Investors often have large non-insurable non-financial risks.
- New investors arrive and old investors leave

Modeling — Why, What, and How

What does “representative” mean: Average or typical?

Make a simple (very reduced form) extension of a Lucas-type model with aggregate consumption, C, and a share s_k for investor k

$$dC = \mu(\cdot)dt + \sigma(\cdot)d\omega$$
$$ds_k = \mu_k(\cdot)s_k dt + \sigma_k(\cdot)s_k d\omega_k$$

Any investor’s consumption and utility can be used to determine the stochastic discount factor, Θ. And for any asset price P,

$$\begin{aligned}0 = \mathbb{E}[d(\Theta P)] &= \mathbb{E}[d(u_C^k(C,t)P)] \\ &= [u_{Ct}^k(\cdot)P + u_{CC}^k(\cdot)Ps_kC(\mu_C + \mu_k) + u_C^k(\cdot)\mu_P P + \tfrac{1}{2}u_{CCC}^k(\cdot)Ps_k^2C^2(\sigma_C^2 + \sigma_k^2) \\ &\quad + [u_{CC}^k(\cdot) + u_{CCC}^k(\cdot)s_kC]P\rho_{kC}\sigma_k s_k\sigma_C C + u_{CC}^k(\cdot)Cs_kP(\rho_{kP}\sigma_k\sigma_P + \rho_{PC}\sigma_C\sigma_P)]dt\end{aligned}$$

This also holds for the risk-free asset and standard manipulations give

$$\mu_P - r = \frac{-u_{CC}^k(\cdot)Cs_k}{u_C^k(\cdot)}(\underbrace{\rho_{PC}\sigma_C\sigma_P}_{\text{standard}} + \underbrace{\rho_{kP}\sigma_k\sigma_P}_{\text{non-standard}})$$

Modeling — Why, What, and How

<u>What does "average" mean?</u>

The representative investor holds the market portfolio and consumes per capita consumption.

In a single period model this is trivial.

The representative investor

- starts with average wealth,
- holds the market portfolio,
- has average wealth in period 1
- consumes his wealth which is per capita consumption

But in a multi-period model investors do not consume all their wealth so the last need not be true and

per capita consumption is an equally weighted average across investors while the market portfolio is a wealth-weighted average so it likely will not be true.

Modeling — Why, What, and How

<u>What does "investor" mean?</u>

This would seem to be the easiest question — almost trivial. But as is often true in such cases, it is subtle and difficult.

An investor is

a set of beliefs π_s

an increasing concave utility function $u(\cdot)$

$\Rightarrow$ an efficient (or optimal) portfolio

The first order-condition for the optimal portfolio with gross rate of return $\tilde{R}^*$ is

$$0 = \mathbb{E}[U'(\tilde{R}^* + \alpha(\tilde{r}_i - r_f))(\tilde{r}_i - \tilde{r}_j)]$$

$$\Rightarrow \mathbb{E}[U'(\tilde{R}^*)\tilde{r}_i] = \lambda \quad \forall i \, .$$

Modeling — Why, What, and How

What does “investor” mean?

Consider the four-state three-asset economy

		assets		
State	π	1	2	3
a	0.25	1.5	2.05	1.8
b	0.25	1	1.3	1.2
c	0.25	1.4	0.95	1.2
d	0.25	1.3	1.25	1.2

with two investors endowed with utility functions satisfying

$$u_1'(1.65) = 1 < u_1'(1.3) = 4 < u_1'(1.25) = 59 < u_1'(1.1) = 68$$
$$u_2'(2.05) = 21 < u_2'(1.3) = 28 < u_2'(1.25) = 39 < u_2'(0.95) = 40$$

These are standard utility functions as $u_i' > 0$ and $u_i'(x) > u_i'(y)$ for x $< y$

Modeling — Why, What, and How

What does "investor" mean?

Optimal portfolios for two utility functions are $\mathbf{w}_1 = (\frac{1}{2}, 0, \frac{1}{2})'$ and $\mathbf{w}_2 = (0, 1, 0)'$

		assets			portfolios	
	π	1	2	3	(0.5, 0, 0.5)	(0, 1, 0)
a	0.25	1.50	2.05	1.80	$u_1'(1.65) = 1$	$u_2'(2.05) = 21$
b	0.25	1.00	1.30	1.20	$u_1'(1.10) = 68$	$u_2'(1.30) = 28$
c	0.25	1.40	0.95	1.20	$u_1'(1.30) = 4$	$u_2'(0.95) = 40$
d	0.25	1.30	1.25	1.20	$u_1'(1.25) = 59$	$u_2'(1.25) = 39$

$$\text{investor}_1: \begin{cases} \text{asset}_1: & \frac{1}{4}\left(1.5 \times 1 + 1.0 \times 68 + 1.4 \times 40 + 1.3 \times 59\right) = 50.55 \\ \text{asset}_2: & \frac{1}{4}\left(2.05 \times 1 + 1.3 \times 68 + 0.95 \times 40 + 1.25 \times 59\right) = 50.55 \\ \text{asset}_3: & \frac{1}{4}\left(1.8 \times 1 + 1.2 \times 68 + 1.2 \times 40 + 1.2 \times 59\right) = 50.55 \end{cases}$$

$$\text{investor}_2: \begin{cases} \text{asset}_1: & \frac{1}{4}\left(1.5 \times 21 + 1.0 \times 28 + 1.4 \times 40 + 1.3 \times 39\right) = 41.55 \\ \text{asset}_2: & \frac{1}{4}\left(2.05 \times 1 + 1.3 \times 68 + 0.95 \times 40 + 1.25 \times 39\right) = 41.55 \\ \text{asset}_3: & \frac{1}{4}\left(1.8 \times 1 + 1.2 \times 68 + 1.2 \times 40 + 1.2 \times 39\right) = 41.55 \end{cases}$$

Modeling — Why, What, and How

But the market portfolio, and equal mix of the two is not efficient — it is not optimal for any increasing, concave utility function

	π	portfolios (0.5, 0, 0.5)	(0, 1, 0)	mkt=$(\frac{1}{4},\frac{1}{2},\frac{1}{4})$	(–0.5, –0.4, 1.9)
a	0.25	1.65	2.05	1.85	1.85
b	0.25	1.10	1.30	1.20	1.26
c	0.25	1.30	0.95	1.125	1.20
d	0.25	1.25	1.25	1.25	1.13

The portfolio (–0.5, –0.4, 1.9) first order stochastically dominates the market.

Summary: The efficient set need not be convex so the market portfolio need not be efficient. In this case the average investor cannot be representative. The SDF (or state price density) method cannot be used with the market portfolio.

The average investor may not be an investor at all. And unfortunately this the basic idea is the basis in the vast majority of current pricing models.

Modeling — Why, What, and How

What does "investor" mean?

Moreover, even if a portfolio is efficient in a single period, it might not be in a multi-period context. We've already seen that per capita consumption and the market portfolio are different kinds of averages, but even the portfolio problem alone can lead to difficulties.

	state	π	p	ret on A.D.	$\theta = p/\pi$	R^*
Example:	a	20%	0.1	10.00	0.50	3.722
	b	40%	0.3	1.333	0.75	1.093
	c	40%	0.6	1.167	1.50	0.500

The market is complete and $R_a^* > R_b^* > R_c^*$ and $\theta_a < \theta_b < \theta_c$ so this holding is perfectly consistent with some increasing, concave vNM utility function.

Modeling — Why, What, and How

What does "investor" mean?

Now consider the same assets in a two-period economy

state	π	p	Portfolio		Portfolio	
	$=\pi_1 \cdot \pi_2$	$=p_1 \cdot p_2$	Return	cost	Return	cost
a a	4%	0.01	13.851	0.139	13.851	0.139
a b	8%	0.03	4.067	0.122	4.067	0.122
a c	8%	0.06	1.861	0.112	1.194	0.072
b a	8%	0.03	4.067	0.122	4.067	0.122
b b	16%	0.09	1.194	0.107	1.861	0.167
b c	16%	0.18	0.546	0.098	0.546	0.098
c a	8%	0.06	1.861	0.112	1.194	0.072
c b	16%	0.18	0.546	0.098	0.546	0.098
c c	16%	0.36	0.250	0.090	0.250	0.090
				1.000		0.980

Dominating portfolio has identical returns and costs 2% less.

Modeling — Why, What, and How

<u>What does “investor” mean?</u>

The representative investor might be typical rather than average.
Idiosyncrasies may always be important

The average investor might not even exist
The market portfolio may not be optimal or any risk-averse individual

The average investor might not be the average consumer
These require different averaging (wealth or equally weighted)

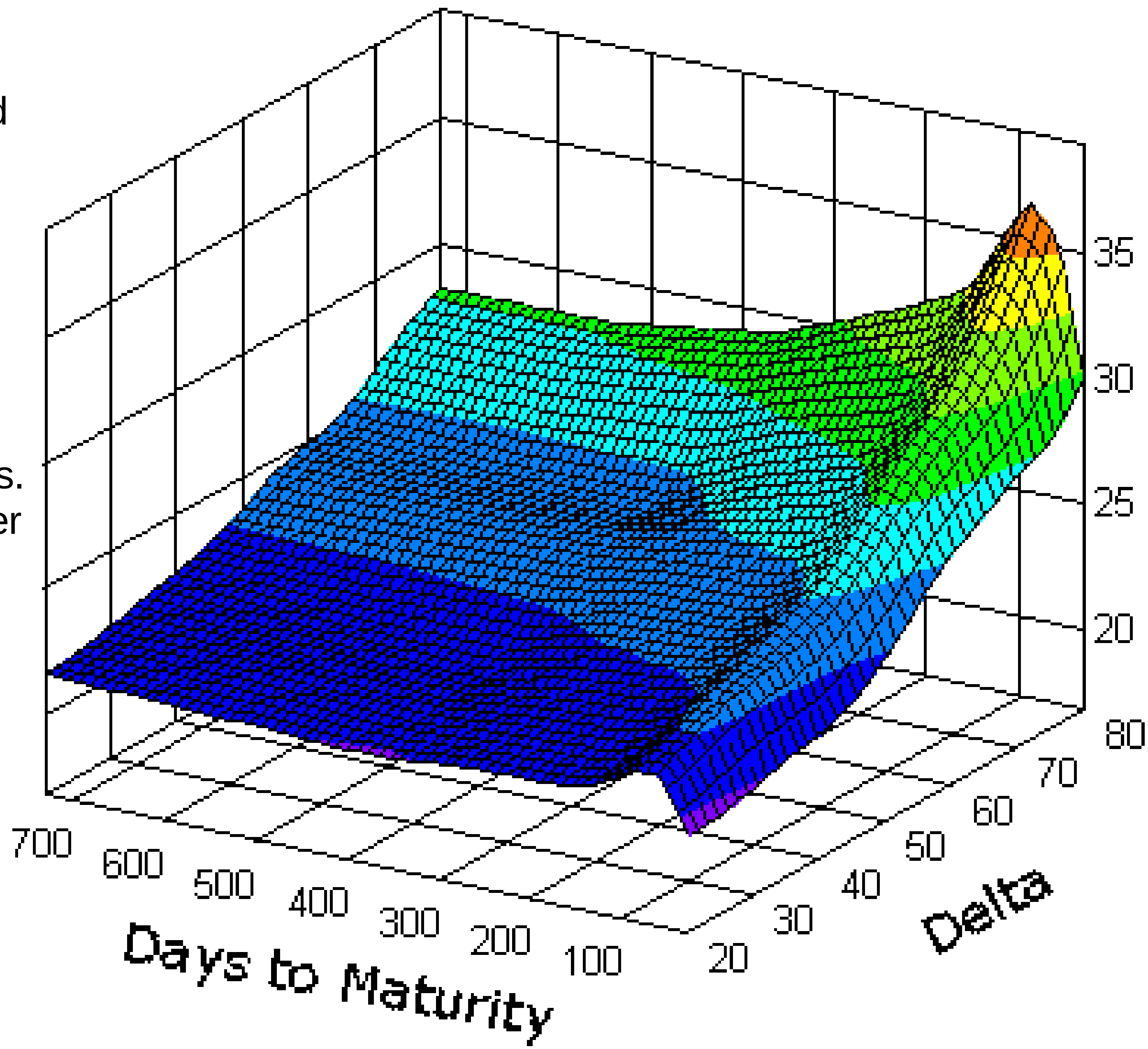
And we haven’t even discussed homogeneous **and correct** beliefs.

Modeling — Why, What, and How

Homogeneous and correct beliefs or How learning hurts us

Implied volatility plotted against maturity (days) and delta (high delta means low strike)

High delta (low strike price) call options have higher implied volatilities. This difference is greater at shorter maturities

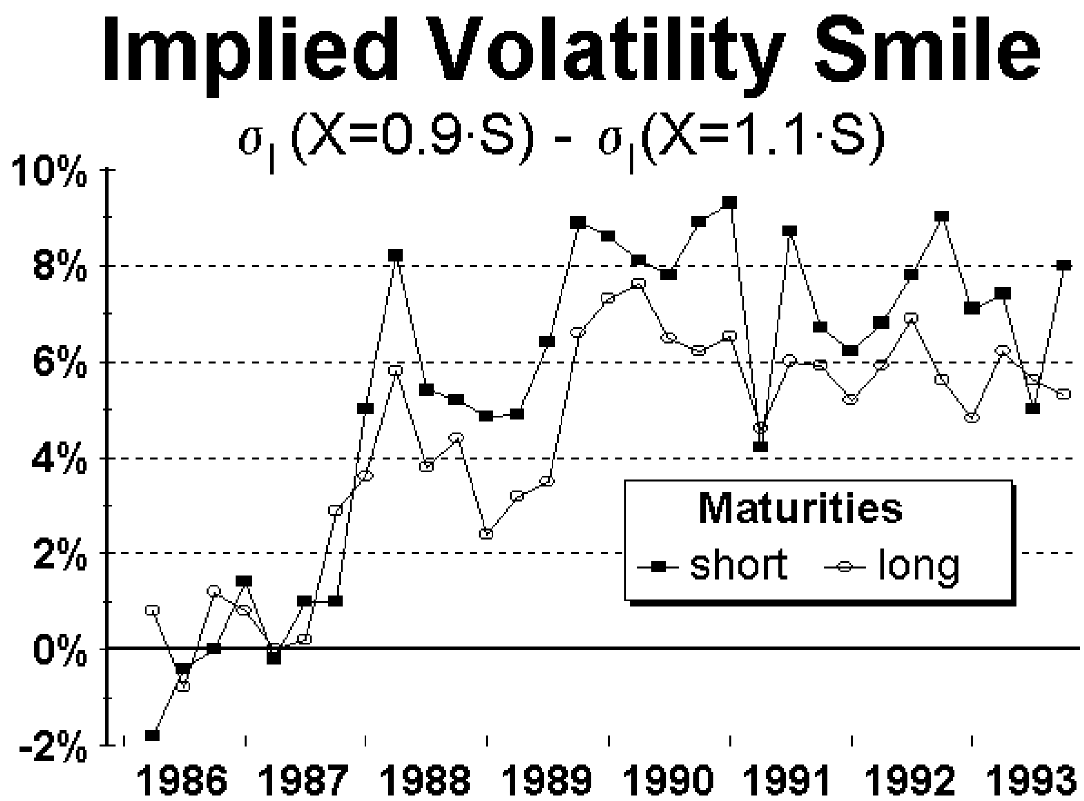


Modeling — Why, What, and How

Homogeneous and correct beliefs or How learning hurts us

Avg. Difference in σ_{Imp}

	short mat	long mat
1986-93	5.5%	4.3%
sub-periods		
1986-87	0.1%	0.7%
1988-93	7.0%	5.4%



This smile or smirk pattern is still seen today as shown in the previous figure

Modeling — Why, What, and How

<u>Homogeneous and correct beliefs or How learning hurts us</u>

What happened in the fall of 1987 to change things?

August 1987: Dow Jones hits a then peak high of 2722

On October 15-16, Iran hits two American ships, the supertanker, *Sungari*, and the *Sea Isle City* with silkworm missiles. US responds on October 19 by shelling and Iranian oil platform in the Persian Gulf

Wednesday October 14:	DJIA drops a record 95.46 points (3.8%) to 2412.70
Thursday October 15:	DJIA drops 57.61 points (2.4%) to 2355.09
Friday October 16:	DJIA drops 108.35 (4.6%) to 2246.74
Monday October 19:	DJIA drops 508 points (22.61%) to 1738.74

Markets were down world-wide as well, New Zealand 60%, Hong Kong 45.5%, Australia 41.8%, Spain 31%, UK 26.45%, Canada 22.5%.

Modeling — Why, What, and How

Homogeneous and correct beliefs or How learning hurts us

Largest percentage price declines in the DJIA since inception (1896)

rank	date	decline	% dec	avg recur (years)	Avg recurrence assumes σ = 25%
1	19-Oct-1987	508	22.61	9×10^{56}	← just WOW
2	28-Oct-1929	38.33	12.82	3×10^{15}	
3	18-Dec-1889	7.94	11.99	19×10^{12}	
4	29-Oct-1929	30.57	11.73	4×10^{12}	← 300 × age universe
5	6-Nov-1929	25.55	9.92	261 million	← great extinction
6	12-Aug-1932	5.79	8.40	329127	
7	14-Mar-1907	6.89	8.29	212748	← 1st homo sapiens
8	26-Oct-1987	156.83	8.04	80752	
9	15-Oct-2008	733.08	7.87	42553	← Neanderthals extinct
10	21-Jul-1933	7.55	7.84	38061	← 320 × life of the DJIA

The crash on Oct 19, 1987 was a rare event, nothing remotely similar had happened in 50 years
But clearly such events are not nearly as rare as they should be if market returns are lognormal

Modeling — Why, What, and How

Homogeneous and correct beliefs or How learning hurts us

How are option traders affected when the stock market plunges?

Clearly out-of-the-money puts are more valuable if sudden sharp declines in stock prices are possible. By put-call parity low strike calls are also more valuable.

As call prices are increasing in implied volatility, implied volatility will be higher for low strike options when sharp market declines are possible.

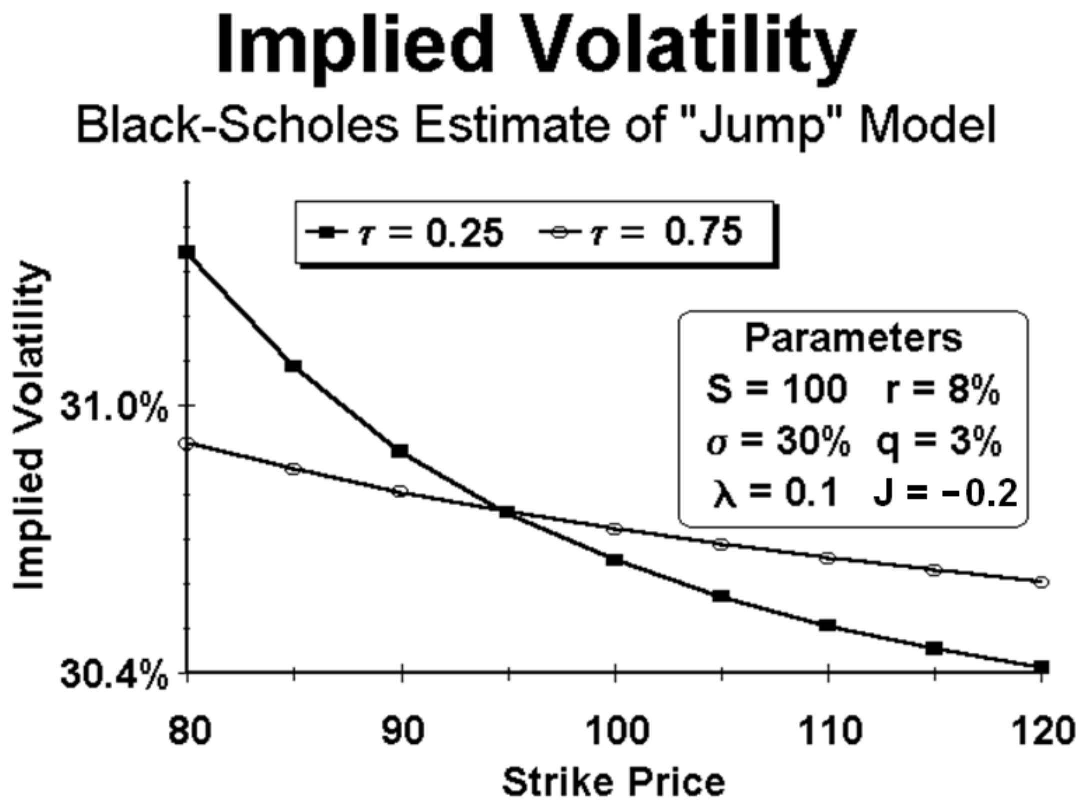
This is exactly what is seen.

Merton’s model can quantify this and shows that the effect is greater for short maturity options.

Modeling — Why, What, and How

<u>Homogeneous and correct beliefs or How learning hurts us</u>

Suppose once every ten years we see a 20% drop in the market

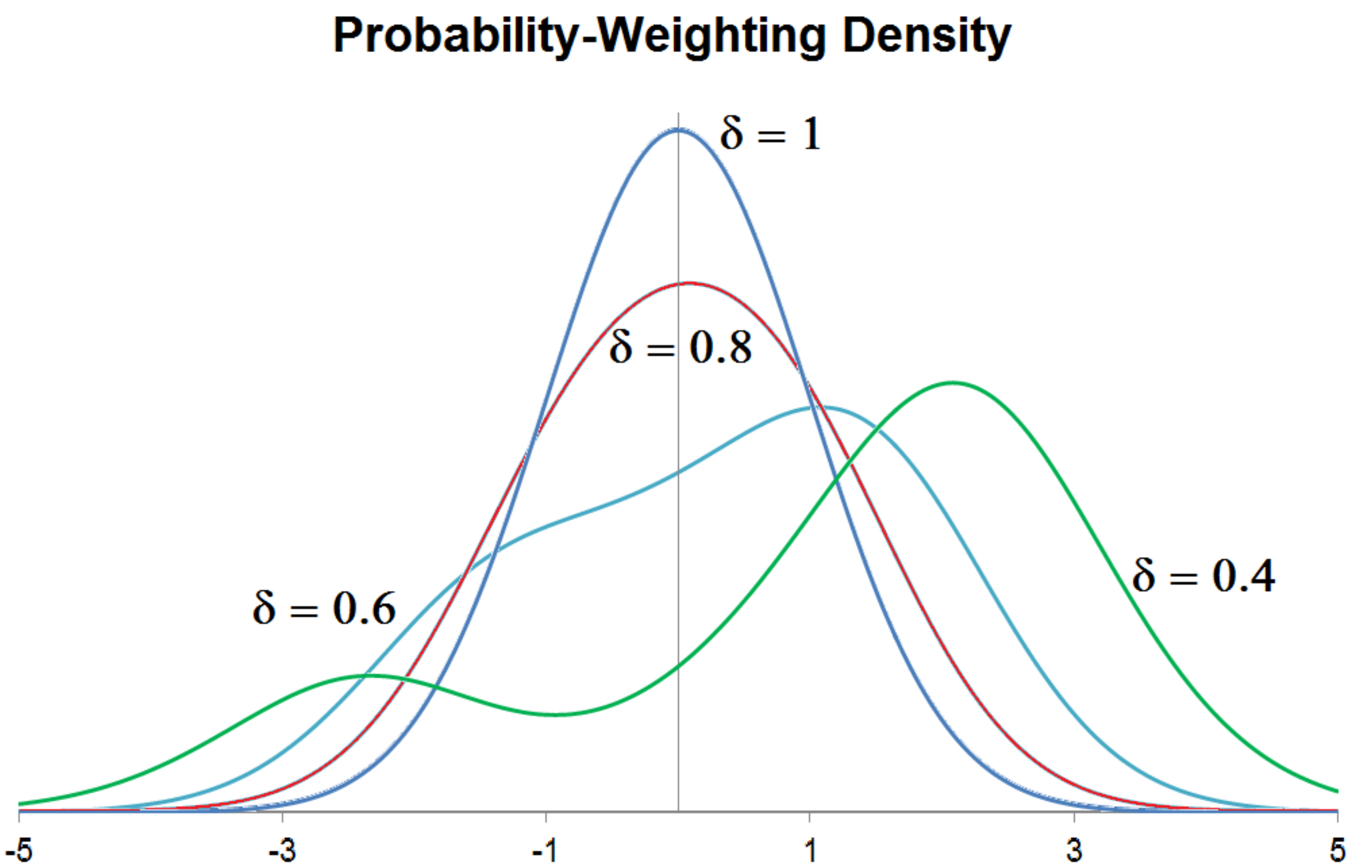


Effect is correct but probably not big enough even though a 20% drop once every ten years is likely an overestimate.

Modeling — Why, What, and How

Homogeneous and correct beliefs or How learning hurts us

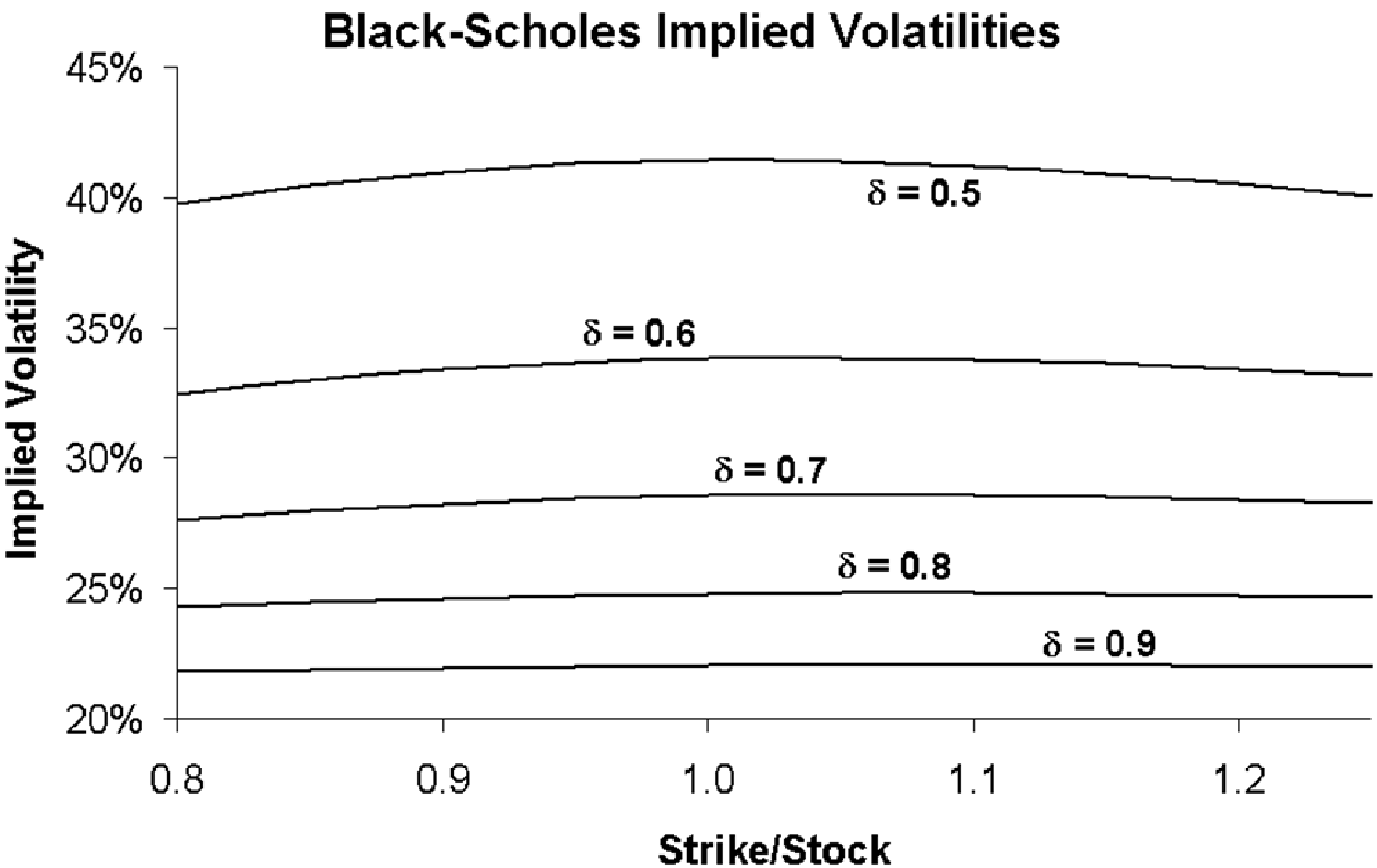
What effects do other models have? CPT preferences
Probability Weighting for Tversky and Kahneman function



Modeling — Why, What, and How

Homogeneous and correct beliefs or How learning hurts us

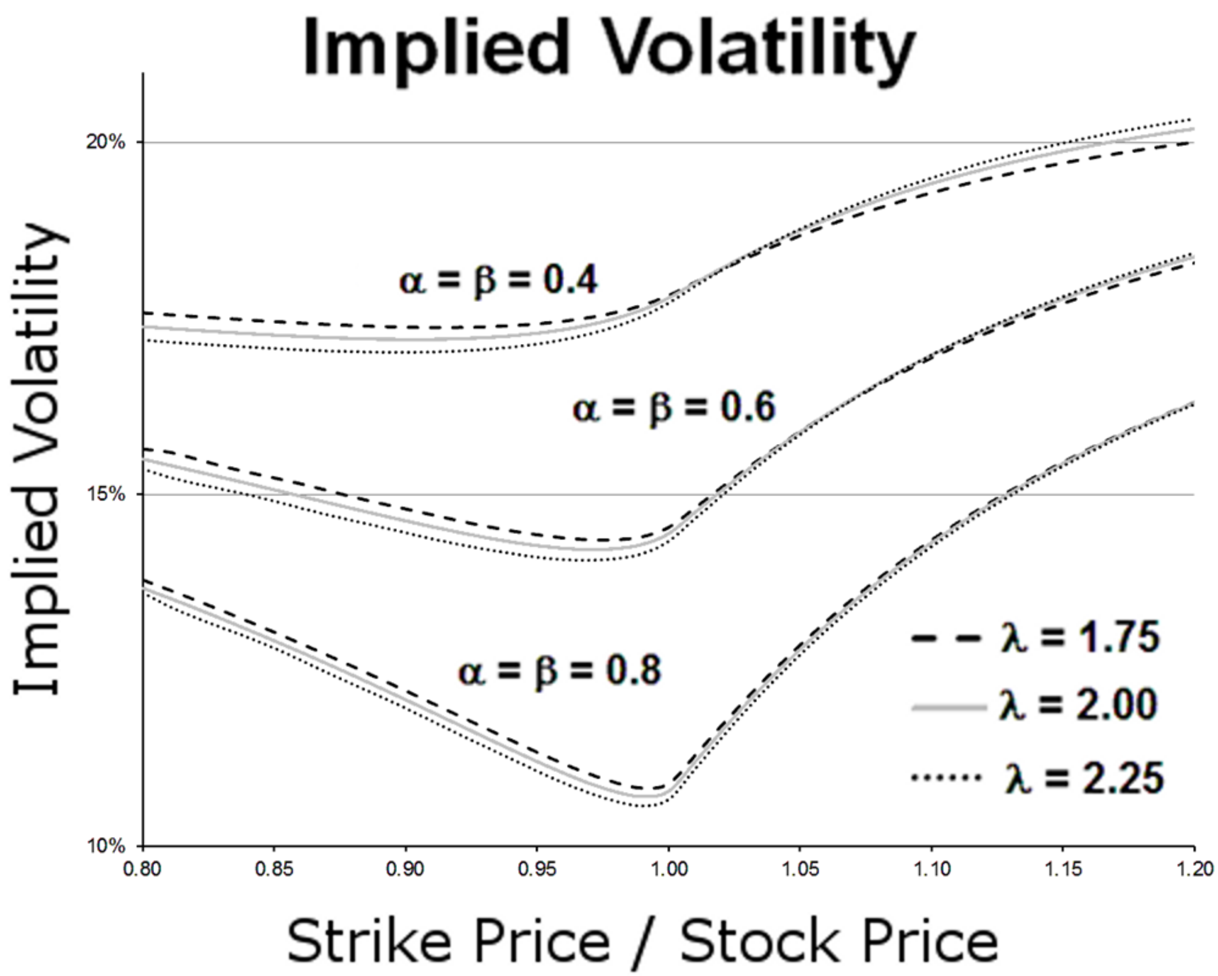
What effects do other models have?: CPT preferences
Probability weighting effect on Black-Scholes implied volatility



Modeling — Why, What, and How

<u>Homogeneous and correct beliefs or How learning hurts us</u>

What effects do other models have? CPT preferences
Tversky and Kahneman utility function effect on implied volatility



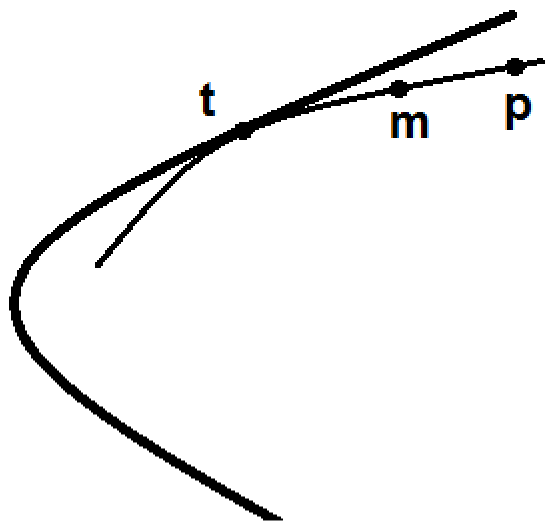
Modeling — Why, What, and How

<u>Perhaps a Physics "trick" — Perturbation Analysis</u>

Assume the CAPM is "close" to correct

Tangency portfolio is not too different from market portfolio, $\mathbf{t} \approx \mathbf{m}$.

The market portfolio can always be described as some combination of the tangency portfolio, **t**, and a residual portfolio, **p**: $\mathbf{m} = (1-w)\mathbf{t} + w\mathbf{p}$



WLOG we can make $w > 0$, then portfolio **p** holds those stocks for which there is excess demand relative to mean-variance analysis.

If money managers are trying to maximize Sharpe ratio, portfolio **p** holds the stocks of the small investors (beyond the same Sharpe ratio demand)

Modeling — Why, What, and How

<u>Perhaps a Physics "trick" — Perturbation Analysis</u>

The tangency portfolio "predicts" all expected returns including the market's

$$\mu_i - r \equiv \beta_i^t(\mu_t - r) \quad \Rightarrow \quad \frac{\mu_t - r}{\mu_m - r} \equiv \frac{1}{\beta_m^t} \equiv \frac{\sigma_t^2}{\sigma_{mt}} \equiv \frac{\sigma_t}{\rho_{mt}\sigma_m} \equiv \frac{\sigma_t}{\rho_{mt}\sigma_m}\frac{\rho_{mt}\sigma_m}{\rho_{mt}\sigma_m} \equiv \frac{\beta_t^m}{\rho_{mt}^2} > \beta_t^m$$

so unsurprisingly the tangency portfolio has a positive market alpha.

As the residual portfolio is a linear combination of the market and the tangency portfolio, we can use an exact second-order Taylor expansion

$$\frac{\mu_p - r}{\mu_m - r} \equiv \beta_p^m - \frac{w s_p^2}{\sigma_m^2(1 - w\beta_p^m)}$$

$$\equiv \beta_p^m \underbrace{\left(1 - \frac{w^2 s_p^2}{\sigma_m^2}\right)}_{\text{flatter in } \beta} - \underbrace{\frac{w}{\sigma_m^2} s_p^2}_{\substack{\text{residual} \\ \text{risk matters}}} - (\beta_p^m)^2 \underbrace{\frac{w\bar{w}^2 s_p^2}{\sigma_m^2}}_{\text{not linear in } \beta} \qquad \bar{w} \in (0, w)$$

s_p^2 is portfolio p's (market) residual variance.

Modeling — Why, What, and How

<u>Perturbation Analysis</u>

$$\frac{\mu_p - r}{\mu_m - r} \equiv \beta_p^m \underbrace{\left(1 - \frac{w^2 s_p^2}{\sigma_m^2}\right)}_{\text{flatter in } \beta} - \underbrace{\frac{w}{\sigma_m^2} s_p^2}_{\substack{\text{residual} \\ \text{risk matters}}} - (\beta_p^m)^2 \underbrace{\frac{w\bar{w}^2 s_p^2}{\sigma_m^2}}_{\text{not linear in } \beta} \qquad \bar{w} \in (0, w)$$

Fama and MacBeth (1973) found

$$\bar{r}_j = \underset{(1.92)}{\underset{.0049}{\gamma_0}} + \underset{(1.79)}{\underset{.0105}{\gamma_1}} \times \hat{\beta}_j + \underset{(-.29)}{\underset{-.0008}{\gamma_2}} \times \hat{\beta}_j^2 + e_j$$

Ang, Hodrick, Xing, and Zhang (2006) found that the highest residual risk portfolios had returns 1.38% lower (t = 4.56) than the lowest residual risk portfolios.

A big question is what stocks go into **p** and why?