Stochastic Volatility and Jumps in Interest Rates: An Empirical Analysis

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Abstract

Daily changes in interest rates display statistical properties that are similar to those observed in other financial time series. The distributions for changes in Eurocurrency interest rate futures are leptokurtic with fat tails and an unusually large percentage of observations concentrated at zero. The implied volatilities for at-the-money options on interest rate futures reveal evidence of stochastic volatility, as well as jumps in volatility. A stochastic volatility model with jumps in both rates and volatility is fit to the daily data for futures interest rates in four major currencies and the model provides a better fit for the empirical distributions. A method for incorporating stochastic volatility and jumps in a complete model of the term structure is discussed.

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1 Introduction

Models of the term structure of interest rates have many applications in financial management. Banks that trade and make markets in interest rate derivatives use models to value and hedge their derivative portfolios. Models for the future evolution of multiple interest rates under the risk-neutral distribution are calibrated daily for the purpose of valuation, and these models are used to implement hedging strategies. Models for changes in interest rates, under real world distributions, are used to monitor market and credit risk in derivative positions. The joint distribution for changes in interest rates, including the covariation across multiple maturities, is critical for all of these applications. A common perception in the market is that interest rates do not vary much in the absence of economic news, but do move in response to the release of new numbers on the state of the economy and to shocks in financial markets. One could argue that interest rate volatility changes as markets react to the arrival of new information. In some cases, the swings in interest rates, particularly short-term interest rates, have been sudden and dramatic. For example, on the day after the stock market crash of October of 1987, short-term interest rates in the U.S. dropped by more than 100 basis points as investors sought safe havens from the stock market: 3 month LIBOR dropped from 9.1875% to 8.25% and the December Eurodollar futures price increased from 90.64 to 91.80. In the British market, short-term interest rates dropped dramatically during the exchange rate crisis of September 1992: on September 17, 1992, 3 month LIBOR in pound sterling dropped from 10.062% to 9.623% after a drop of 50 basis points for the previous day, and the December futures interest rate decreased from 10.80% to 8.72%. The market for interest rate options incorporates stochastic volatility and the potential for large interest rate moves in the form of a volatility smile, or skew. In Figure 1, we have plotted the implied volatility from Black's model for the actively traded Eurodollar futures options. The variation in the implied interest rate volatility is significant across the strikes for options with the same expiration date, and the slope of the skew is largest for the options with near term expirations.[1] The purpose of this paper is to examine empirically the role of stochastic volatility and jumps in short-term interest rates. In the last section of the paper, we use results from Duffie, Pan, and Singleton (2000) to construct a term structure model that is tractable and incorporates these empirical features of interest rate volatility.

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1 The implied volatilities have been computed for the close on Jan. 25, 2002, and only those options with less than 1 year to expiration and volumes of at least 1,000 contracts were used. The options trade at the Chicago Mercantile Exchange and the implied volatility is for the futures rate, not the futures price. The options are American and a lattice model was used to compute the American premium and the implied volatility. The early exercise premiums for these options are small. The largest computed early exercise premium is 0.27 basis points, and the largest as a percentage of the option value is 0.48%.

To motivate the role of stochastic volatility and jumps in interest rates, we begin with a statistical analysis of futures interest rates and implied volatilities from the markets for options on interest rate futures. Figures 2-5 contain time series plots of the LIBOR futures interest rates and the implied volatility for four major currencies.[2] All of the contracts included here are the near delivery futures contracts on 3 month LIBOR, or 3 month TIBOR for the Japanese yen market.[3] To construct the time series, we roll to the next delivery in the quarterly cycle at the beginning of each delivery month. An inspection of these plots reveals that there have been days when there were large, sudden changes in both short-term interest rates and the implied volatility, but it should be noted that some of the apparent jumps are associated with the roll between different futures contracts. Figures 6 to 9 contain scatter plots of changes in the implied volatility versus changes in futures interest rates.[4] The patterns in the scatter plots reveal some interesting features that are common across all four currencies: changes in the futures interest rates and the implied volatilities are randomly scattered around zero and there does not appear to be much, if any, correlation between the two variables. The plots also highlight the discrete nature of changes in the futures rates. The tick size is one basis point for the interest rate futures in German marks (Euros), Japanese yen, and British pounds. The tick size for the U.S. dollar interest rate futures was one basis point up until a few years ago when it was reduced to half of a basis point.

There are some extreme outliers, and it is unlikely that diffusion processes for interest rates and interest rate volatility could produce the time series in Figures 2-5 or the scatter plots in Figures 6-9. These scatter plots are also quite different from what is observed for major stock indexes. Figure 10 contains the scatter plot for changes in the CBOE volatility index (VIX) versus changes in the log of the S&P 100 index. There is a strong negative correlation between stock price changes and changes in implied volatility, for both large changes as well as small changes. The correlation between price changes and volatility changes is -.67 for the S&P 100, and the rank correlation coefficient is -.63. The correlation between changes in interest rates and changes in the implied volatility of interest rates is much smaller, and less significant: -.07 for the U.S. rates, -.12 for Germany, .07 for Japan, and .09 for Great Britain. Several outliers associated with financial shocks can be identified in the plots: the October 1987 crash for the U.S. market and the exchange rate shock in September 1992 for the pound sterling and the European currencies. If these outliers are removed, the correlation coefficients decrease significantly: .008 for the U.S., -.06 for Germany, and .05 for Great Britain. The rank correlation coefficients for

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[2] The markets trade on futures prices with the final settlement based on 100 minus 3 month LIBOR on the delivery date. The futures rate is 100 minus the futures price. The market convention is to quote the implied volatility as a volatility for the log of the futures interest rate.

[3] The data for futures rates and implied volatilities are from the exchanges: the CME for Eurodollars; LIFFE for Eurosterling, Euromarks, and Euribor; TIFFE for the futures on TIBOR. The CME data are from the DRI database, the LIFFE data are from two sources, LIFFE End-of-Day Financial Products and DRI, and the TIFFE data are from the TIFFE web site.

[4] The changes for futures rates and implied volatility are computed by using the same delivery contract, and the delivery month used for computing these changes is the nearest delivery, which is rolled to the next delivery contract at the beginning of each delivery month.

changes in interest rates and changes in the implied volatility are also small: .01 for the U.S. rates, -.04 for Germany, .13 for Japan, and .01 for Great Britain. These observations suggest that there is little or no correlation between interest rate changes and volatility changes, and the direct dependence between jumps in interest rates and jumps in volatility is also weak.

For completeness, we have included some summary statistics for the short-term interest rates of these four currencies in Tables 1-4. There are several common features across all four currencies. The excess kurtosis statistics are large, and there are many days in the samples when the interest rates and the futures settlement rates do not change. For example, in the U.S. dollar market, from March 1985 to December 2000, 3 month LIBOR did not change for 36.5% of the days and the near delivery Eurodollar futures did not change for 19.5% of the days. The autocorrelations for changes in futures rates are all close to zero, and the sample standard deviations for changes in the futures rates are close to the sample standard deviations for changes in 3 month LIBOR. The correlations across interest rate changes within a currency are also significant.

The distribution of interest rate changes is important for both option pricing and value-at-risk analysis. Figure 11 contains a plot of the empirical distribution for daily changes in the Eurodollar futures interest rate, and Figure 12 contains the distribution for the change in the log of this rate. Both graphs include plots of normal distributions that have been fit to the data.[5] Relative to the normal, the actual distributions have fatter tails and more observations closer to zero. The plots of the empirical distributions also highlight the large number of days when there is no change and in Figure 11 the discrete nature of the rate changes, in basis points. The exchanges compute implied volatilities for futures rates by using Black's model with at-the-money put and call options, and the numbers reflect the market's forward looking expectation of volatility in the futures rate. If the futures rate is determined by a diffusion process with stochastic volatility, the implied variance rate computed from at-the-money options should be a good proxy for the expected variance over the remaining life of the option. We have also used the implied volatility from the previous day to compute a conditional standard deviation for daily changes in the log of the futures rate as follows:

$$\text{Std. Dev}(t) = \text{Imp. Vol}(t\text{-}1) \times \sqrt{1/250} \quad .$$

Figure 13 contains a plot of the empirical distribution for the change in the log of the Eurodollar futures rate divided by this proxy for the conditional standard deviation. The sample standard deviation for this time series is .9820, which implies either a small upward bias in the implied volatility as a predictor of future volatility, or some error in the adjustment from the annualized volatility to a daily volatility. Figure 13 includes a plot of a normal distribution with a mean of zero and a standard deviation of .9820, and this normal distribution does not fit the empirical

[5] We have used a mean of zero and a standard deviation equal to the sample standard deviation for the observations.

distribution. This preliminary analysis of the data suggests that a diffusion process with stochastic volatility does not fit the distributions of changes in futures rates.

2 Estimating the Jump Parameters from Daily Changes in Futures Rates

The goal in this section is to fit models to the empirical distributions for the interest rates in the four currencies described in section 1. Jump processes are rare events that are important for characterizing tail behavior, and the estimation of the jump parameters will require estimators that incorporate the tails of the distribution over relatively short time intervals. In addition, it may be difficult to find evidence of jumps in volatility by examining data on interest rates only. Jumps in volatility should be most apparent in option prices, measured relative to the underlying interest rate or futures rate. To estimate the jump parameters, we use daily changes in interest rates as well as daily changes in the option implied volatility.

One approach to this problem has been to specify a model, solve the option pricing function, and develop a complex estimation strategy that utilizes data on both interest rates and option prices. Bakshi, Cao, and Chen (1997) have taken this approach in estimating a model for the S&P 500 stock index and Bates (1996) has taken this approach in estimating models for foreign exchange rates. To apply this approach to the term structure of interest rates, one must apply the model simultaneously to a number of interest rates and option prices. Our approach is to apply simpler econometric techniques to the daily data on futures rates and implied volatilities and test for deviations from diffusion based models. In the final section, we describe a formal model of the term structure, which incorporates jumps in both rates and volatility.

We begin by specifying an ad hoc empirical model for the futures rate as follows:

$$\begin{aligned} d\,FR &= b\sqrt{v}\,dZ_1 + dJ_1 \\ dv &= \kappa(\theta - v)\,dt + \sigma\sqrt{v}\,dZ_2 + dJ_2 \end{aligned} \tag{1}$$

where v is the stochastic variance factor, dZ_1 and dZ_2 are Brownian motion increments, and dJ_1 and dJ_2 are jump processes. The jump in the futures rate is a jump process with a random intensity parameter $\lambda_{J1}(t) = c + d\,v(t)$ and a jump size distributed as a normal with zero mean and a standard deviation σ_{J1}. A mean was initially included in the distribution for the jump magnitude, but the estimates were found to be close to zero and insignificant. The estimates for c in the random intensity parameter were also found to be close to zero and insignificant. The jump in the stochastic variance factor is a Poisson process with intensity parameter λ_{J2} and a jump size that is exponentially distributed with a mean μ_{J2}.

Under the risk neutral distribution, the futures price, or the futures rate in this case, should be a martingale, but these prices are not necessarily martingales under the actual or real world distribution. The autocorrelations for changes in futures rates reported in Tables 1-4 are all close to zero. These results imply that past values of futures rates are not useful in the prediction of changes in the futures rates. One would need to examine other economic variables or deterministic functions of time in order to model a non-zero drift in the *d FR* process. Additional modeling of the drift for changes in the futures rate is not likely to add anything significant to the empirical model.

We have chosen to model the futures rate as a "normal" process with a stochastic volatility, and we use the option implied volatility as a proxy for this stochastic volatility factor. If a futures price is determined by a diffusion process with fixed volatility, $df = \sigma\, dZ$, and the short rate for discounting is constant, the solution for a call on the futures is

$$C \;=\; e^{-rT}\left(f\,[1-N(\tfrac{K-f}{\sigma\sqrt{T}})] \;+\; \tfrac{\sigma\sqrt{T}}{\sqrt{2\pi}}\exp[-\tfrac{1}{2}\left(\tfrac{K-f}{\sigma\sqrt{T}}\right)^2] \;-\; K\,[1-N(\tfrac{K-f}{\sigma\sqrt{T}})]\right).$$

For an at-the-money call, $f = K$, and

$$C \;=\; e^{-rT}\,\frac{\sigma\sqrt{T}}{\sqrt{2\pi}} \;\; .$$

The interest rate futures examined in this paper are all American, but the options traded at the LIFFE (Eurosterling, Euromarks, and the new Euribor and Eurilibor contracts) have futures style margining and it is not optimal to exercise these options early. For call options on futures with futures style margining, the general option pricing solution has the following form:

$$C \;=\; \hat{E}_t\left(\max\left[0,\; f(T)-K\,\right]\right) ,$$

where the expectation is taken under the risk neutral distribution, conditional on information at time t, $t \le T$. The contracts traded in Chicago and Japan do have early exercise premiums, but they are generally quite small for the near term expirations. We find that there is little difference between the implied volatilities from the European pricing model and the American pricing model. Black's model applied to an at-the-money call has the following solution:

$$C \;=\; e^{-rT}\,FR[\,N(\tfrac{1}{2}\sigma\sqrt{T}) - N(-\tfrac{1}{2}\sigma\sqrt{T})\,] \;\approx\; e^{-rT}\,\frac{FR\,\sigma_B\sqrt{T}}{\sqrt{2\pi}} \;\; .$$

As a result, the normal volatility for at-the-money options is approximately equal to the at-the-money implied volatility from Black's model multiplied by the level of the futures rate. The variance rate, $v(t)$, in (1) is the annualized variance rate and we use $\sigma_B^2 \times FR^2$ as the proxy. The scalar parameter b is included to account for the fact that part of the implied volatility in option prices is due to the variance from the jump process. If there are no jumps in interest rates and the implied volatility is equal to expected volatility from near term options, then b = 1.

For data sampled on a daily basis, we estimate the following econometric model.

$$\begin{aligned} \Delta FR(t) \; &= \; b\,\sqrt{v(t-1)}\;\Delta Z_1(t) \; + \; \Delta J_1(t) \\ v(t) \; &= \; \mu \; + \; \rho\, v(t-1) \; + \; \sigma \sqrt{v(t-1)}\;\Delta Z_2(t) \; + \; \Delta J_2(t) \;\; , \end{aligned} \tag{2}$$

where $\rho \;=\; \exp(-\kappa\,\Delta t)$ and $\mu \;=\; \theta(1-\rho)$. We apply maximum likelihood under the assumption that ΔZ_1 is distributed as a normal with mean zero and variance Δt, and ΔJ_1 is a Poisson jump process with intensity $\lambda_{J1}(t) \;=\; c + d\,v(t-1)$ and jump magnitudes that are distributed as normals with mean zero and variance σ_{J1}^2. The distribution for the number of jumps over a daily time interval is approximated as a Poisson distribution with parameter $\lambda_{J1}\Delta t$. The resulting conditional density function for the random variable $\Delta FR(t)$ is

$$\sum_{j=0}^{\infty} \frac{e^{-\lambda_{J1}(t)\Delta t}\left(\lambda_{J1}(t)\Delta t\right)^j}{j!} \; \frac{1}{\sqrt{2\pi\sigma_j^2(t)}} \exp\left\{-\frac{\Delta FR^2(t)}{2\sigma_j^2(t)}\right\} \; ,$$

where $\sigma_j^2(t) \;=\; b^2\,v(t-1)\Delta t + j\,\sigma_{J1}^2$. The Poisson weights decay rapidly so that typically only 20 to 30 terms in the infinite summation are necessary.

We have already noted that rate changes occur in discrete units, basis points, and the percentage of daily changes equal to zero is relatively large. The sample standard deviations reported in Tables 1-4 range from a low of 4.49 basis points for Germany up to a high of 7.85 basis points for Great Britain. The tick size is one basis point for futures on German LIBOR, Japanese TIBOR, and British LIBOR. For most of the sample, the tick size was one basis point for futures on U.S. LIBOR, but it was recently reduced to half of a basis point, and it is a quarter of a basis point for futures with less than a month to delivery. Even though the underlying process driving interest rates may be a continuous random variable, we observe a discrete random variable in the market. A simple model for the discrete process is that the market will round the interest rate to the nearest tick. Given the number of jumps, the model distribution is normal, and the probability of observing $\Delta FR(t) = x$ is

$$\sum_{j=0}^{\infty} \frac{e^{-\lambda_{J1}(t)\Delta t}\left(\lambda_{J1}(t)\Delta t\right)^j}{j!} \left[N\left(\tfrac{x+\delta}{\sigma_j(t)}\right) - N\left(\tfrac{x-\delta}{\sigma_j(t)}\right) \right] \;\; ,$$

where δ is half of the tick size and $N(\cdot)$ is the standard normal distribution function. The resulting log-likelihood function for a sample of observations on $\Delta FR(t)$ is

$$\ln L \; = \; \sum_{t=1}^{T} \ln L(t) \; = \; \sum_{t=1}^{T} \ln\left(\sum_{j=0}^{\infty} \frac{e^{-\lambda_{J1}(t)\Delta t} \, (\lambda_{J1}(t)\,\Delta t)^{j}}{j!} \left[N\left(\tfrac{\Delta FR(t)+\delta_t}{\sigma_j(t)}\right) - N\left(\tfrac{\Delta FR(t)-\delta_t}{\sigma_j(t)}\right) \right] \right). \tag{3}$$

We find that this modification for the discreteness of the observed data has a significant impact on the estimation of the intensity parameters for the jump in interest rates.

To derive a tractable likelihood function for the volatility equation, we need to simplify the second jump process so that at most only one jump can occur each day with a probability of $1 - e^{-\lambda_{J2}\Delta t}$ and if there is a jump, the jump occurs at the end of the period with a jump size that has an exponential distribution with mean μ_{J2}. We need to derive the conditional density function for $v(t)$ given a normal distribution for $\Delta Z_2(t)$ and the assumed distribution for $\Delta J_2(t)$. With probability $e^{-\lambda_{J2}\Delta t}$, the distribution is normal with mean $\mu + \rho v(t-1)$ and variance $\sigma_2^2 \, v(t-1)\,\Delta t$. With probability $1 - e^{-\lambda_{J2}\Delta t}$, the density function is the density for the sum of a normal and an exponential. We derive this density function by using the convolution method, and we get the following log-likelihood function for a sample of observations on $v(t)$.

$$\begin{aligned} \ln L \; = \; \sum_{t=1}^{T} \ln L(t) \; = \; \sum_{t=1}^{T} \ln \Bigg[& \, e^{-\lambda_{J2}\Delta t} \, \frac{1}{\sqrt{2\pi}\;\sigma\,\sqrt{v(t-1)\,\Delta t}} \exp\left\{ - \frac{\left(v(t) - \mu - \rho\, v(t-1)\right)^2}{2\,\sigma^2\, v(t-1)\,\Delta t} \right\} \\ & + \left(1 - e^{-\lambda_{J2}\Delta t}\right) \frac{\left(1 - N(X_t)\right)}{\mu_{J2}} \exp\left\{ \frac{-v(t) + \mu + \rho\, v(t-1)}{\mu_{J2}} + \frac{\sigma^2 \; v(t-1)\,\Delta t}{2\,\mu_{J2}^2} \right\} \Bigg] \end{aligned} \tag{4}$$

where $N(x)$ is the cumulative standard normal distribution function and

$$X_t \; = \; \frac{-v(t) + \mu + \rho\, v(t-1) + \; [\sigma^2 \, v(t-1)\,\Delta t \,] / \mu_{J2}}{\sigma\,\sqrt{v(t-1)\,\Delta t}} \quad .$$

The maximum likelihood estimation for the volatility equation is approximate maximum likelihood because the conditional distribution for volatility has been approximated.

To find the maximum likelihood estimates, we use the algorithm developed by Berndt, Hall, Hall, and Hausman (1974). This method requires analytic first derivatives for the log-likelihood function and an approximation for the information matrix. Let β be the vector of parameters to be estimated and let $\hat{\beta}$ be the maximum likelihood estimator. In large samples, $\hat{\beta}$

has a distribution that is approximately normal with mean β and a covariance matrix that is the inverse of the information matrix. The information matrix is estimated by computing

$$\sum_{t=1}^{T}\left(\frac{\partial \ln L(t)}{\partial \beta}\right)\left(\frac{\partial \ln L(t)}{\partial \beta}\right)' ,$$

and the inverse of the information matrix is used in the algorithm to find the maximum likelihood estimator. This approximation for the information matrix is based on the observation that

$$E\left(-\frac{\partial^2 \ln L(t)}{\partial \beta \partial \beta'}\right) = E\left(\frac{\partial \ln L(t)}{\partial \beta}\right)\left(\frac{\partial \ln L(t)}{\partial \beta}\right)' .$$

The results of the maximum likelihood estimation are summarized in Tables 5–8. All of the parameter estimates are large relative to their standard errors so that the parameter estimates are statistically significant at conventional significance levels. The ρ parameters in the volatility equations are also significantly different from one in all four samples: the t statistics for the null hypothesis that ρ = 1 are -12.54 for volatility in U.S. rates, -8.55 for Germany, -12.94 for Japan, and -10.33 for Great Britain. These results are evidence supporting mean reversion in the implied volatility. The likelihood ratio statistics for testing the null hypotheses of no jumps in rates and no jumps in volatility are all statistically significant. Each one of these statistics has an asymptotic distribution that is chi-squared with 2 degrees of freedom under the null hypothesis. The likelihood ratio statistics for no jumps in rates are 1,105.4 for the U.S., 286.6 for Germany, 383.4 for Japan, and 450.6 for Great Britain. The likelihood ratio statistics for no jumps in volatility are 1,340.6 for the U.S., 729.6 for Germany, 1,166.0 for Japan, and 695.6 for Great Britain. We have also reported the estimated values for σ_{J1}, the standard deviation for the magnitude of the interest rate jump when a jump occurs. For the U.S., the estimate of *d* implies an average value of 206.0 for $\lambda_{J1}(t)$, which corresponds to an average of 206 jumps per year.[6] The σ_{J1} estimate is 0.052%, or 5 basis points, for jumps in interest rates, which can be compared to the daily standard deviation of 6.4 basis points. The estimate of 6.682 for λ_{J2} implies an average of 6 to 7 jumps per year in volatility. The value of μ_{J2} is the expected jump size for the annualized variance of the change in the futures rate. The market quotes the implied volatility as an annualized standard deviation for $\Delta FR/FR$. For the U.S. results, if we use a futures rate of 5% and the estimated average variance level of 6.8198 $\times 10^{-5}$, the estimate of 3.5652×10^{-5} for μ_{J2} implies an expected increase of 3.86% in the annualized implied volatility when a jump in volatility is triggered. The exponential distribution is skewed to the right so that there is a reasonable probability

[6] Here, and below, the mean for the stochastic variance process is computed from the estimates for the model in (1). The mean for the variance process includes the mean reverting level and the effect of the jump.

of observing volatility jumps that are 3 to 4 times the mean. The results are similar for futures interest rates and volatility in the other three currencies. The average number of jumps per year in the rates is, however, much less: 84.7 for Germany, 87.4 for Japan, and 38.8 for Great Britain. The standard deviation for the magnitude of the jump in rates for Great Britain is much larger, 0.12%.

The primary purpose for the maximum likelihood estimation is to fit the model in (1) to the data and the empirical distributions. A natural diagnostic test for the model is to use the maximum likelihood estimates to compute the distribution function for ΔFR and compare this model distribution function to the empirical distribution function. The model in (1) is similar to the double jump model of section 4 in Duffie, Pan, and Singleton (2000). The drift for the change in the futures rate is not the same as the drift for the change in the log of the stock price in their model, and the two jump processes have been modified to account for the empirical features of interest rate variability. We need the unconditional characteristic function for ΔFR over a daily time interval, and we derive it in two steps. The first step is to evaluate the characteristic function for the change in the futures rate at time Δt, $\Delta FR = FR(\Delta t) - FR(0)$, conditional on $v(0)$. This characteristic function is solved by first solving the following conditional expectation.

$$\Psi(u,t,FR,v) \;=\; E\left(e^{iu\,FR(T)} \,\middle|\, FR(t),V(t)\right) .$$

This function must satisfy the following partial differential integral equation:

$$\frac{\partial \Psi}{\partial t} + \frac{1}{2}\frac{\partial^2 \Psi}{\partial FR^2} b^2 v + \frac{1}{2}\frac{\partial^2 \Psi}{\partial v^2}\sigma^2 v + \frac{\partial \Psi}{\partial FR}\cdot 0 \;+\; \frac{\partial \Psi}{\partial v}\,(\kappa\theta - \kappa v)$$

$$+\;(c+dv)\int_{-\infty}^{\infty}\left[\Psi(u,t,FR+x,v) - \Psi(u,t,FR,v)\right]\,\frac{\exp\left(-\frac{1}{2}\frac{x^2}{\sigma_{J1}^2}\right)}{\sqrt{2\pi}\,\sigma_{J1}}\;dx$$

$$+\,\lambda_{J2}\int_{0}^{\infty}\left[\Psi(u,t,FR,v+x) - \Psi(u,t,FR,v)\right]\,\frac{\exp\left(-x/\mu_{J2}\right)}{\mu_{J2}}\;dx \;=\; 0 \quad .$$

And there is a boundary condition that as $t \to T$, $\Psi(u,t,FR,v) = \exp\{iu\,FR(t)\}$. This expectation has an exponential affine solution,

$$\Psi(u,t,FR,v) \;=\; \exp\left\{\alpha^*(\Delta t,u) + iu\,FR(t) + \beta^*(\Delta t,u)\,v(t)\right\},$$

where $\Delta t = T - t$. The characteristic function for the change in the futures rate is

$$\Phi^*(u) \;=\; E\left(e^{iu[FR(\Delta t)-FR(0)]}\,\middle|\, FR(0), v(0)\right) \;=\; e^{-iuFR(0)}\,\Psi(u,0,FR,V)$$

$$=\; \exp\left\{\alpha^*(\Delta t,u) + \beta^*(\Delta t,u)\,v(0)\right\}$$

with

$$\beta^*(\Delta t,u) \;=\; \frac{-a\,(1-e^{-\gamma\Delta t})}{2\gamma - (\gamma-\kappa)(1-e^{-\gamma\Delta t})}$$

$$\alpha^*(\Delta t,u) \;=\; -\kappa\theta\left[\frac{\gamma-\kappa}{\sigma^2}\Delta t + \frac{2}{\sigma^2}\ln\left(1-\frac{\gamma-\kappa}{2\gamma}\left(1-e^{-\gamma\Delta t}\right)\right)\right] \;+\; c\left(e^{-\frac{1}{2}\sigma_{J1}^2u^2}-1\right)\Delta t$$

$$+\;\lambda_{J2}\left[\frac{(\gamma+\kappa)\,\Delta t}{\gamma+\kappa+\mu_{J2}a} - \Delta t - \frac{2\mu_{J2}a}{\gamma^2-(\kappa+\mu_{J2}a)^2}\ln\left(1-\frac{\gamma-\kappa-\mu_{J2}a}{2\gamma}\left(1-e^{-\gamma\Delta t}\right)\right)\right]$$

$\gamma \;=\; \sqrt{\kappa^2+\sigma^2 a}$ and $a \;=\; b^2u^2 \;+\; 2d\,(1-e^{-\frac{1}{2}\sigma_{J1}^2u^2})$.

To get the unconditional characteristic function, we need to take the unconditional expectation of $\Phi^*(u)$:

$$\Phi(u) \;=\; E[\Phi^*(u)] \;=\; e^{\alpha^*(\Delta t,u)}\; E[e^{\beta^*(\Delta t,u)v(0)}]\;.$$

This is done by evaluating first the conditional moment generating function for $v(t+\tau)$, conditional on $v(t)$ and then letting $\tau\to\infty$. This moment generating function,

$$M(u,t,v) \;=\; E[\,e^{\beta^*(\Delta t,u)v(t+\tau)}\,|\,v(t)\,]\;,$$

must satisfy the following partial differential integral equation:

$$\tfrac{1}{2}\sigma^2 v\frac{\partial^2 M}{\partial v^2} + (\kappa\theta-\kappa v)\frac{\partial M}{\partial v} + \frac{\partial M}{\partial t} + \lambda_{J2}\int_0^\infty \left[M(u,t,v+x)-M(u,t,v)\right]\frac{\exp\left\{-x/\mu_{J2}\right\}}{\mu_{J2}}dx \;=\; 0\;.$$

The solution is an exponential affine function, $M(u) \;=\; e^{\alpha(\tau)+\beta(\tau)v(t)}$, where

$$\beta(\tau) \;=\; \frac{2\kappa e^{-\kappa\tau}\,\beta^*(\Delta t,u)}{2\kappa - \beta^*(\Delta t,u)\,\sigma^2\,(1-e^{-\kappa\tau})}$$

$$\alpha(\tau) \;=\; \frac{2\kappa\theta}{\sigma^2}\ln\left(\frac{2\kappa}{2\kappa-\beta^*(\Delta t,u)\sigma^2(1-e^{-\kappa\tau})}\right)$$

$$+\frac{2\lambda_{J2}\,\mu_{J2}}{\sigma^2-2\kappa\mu_{J2}}\ln\left(\frac{2\kappa(1-\mu_{J2}\beta^*(\Delta t,u))}{2\kappa-\beta^*(\Delta t,u)\sigma^2(1-e^{-\kappa\tau})+e^{-\kappa\tau}\,\beta^*(\Delta t,u)\,2\kappa\mu_{J2}}\right) .$$

Let $\tau \to \infty$ and the unconditional characteristic function is

$$\Phi(u) \;=\; E\left(e^{iu\,\Delta FR}\right) \;=\; \exp\left\{-\frac{\kappa\theta(\gamma-\kappa)\Delta t}{\sigma^2}-\frac{2\kappa\theta}{\sigma^2}\ln\left[1-\frac{\gamma-\kappa}{2}\left(\frac{1-e^{-\gamma\Delta t}}{\gamma}\right)\right]\right.$$

$$-\frac{2\kappa\theta}{\sigma^2}\ln\left[1-\frac{\sigma^2\,\beta^*(\Delta t,u)}{2\kappa}\right]+\lambda_{J2}\,\frac{(\gamma+\kappa)\Delta t}{\gamma+\kappa+\mu_{J2}a}$$

$$-\lambda_{J2}\,\Delta t-\frac{2\lambda_{J2}\mu_{J2}}{\sigma^2-2\kappa\mu_{J2}-\mu_{J2}^2\,a}\ln\left[1-\frac{\gamma-\kappa-\mu_{J2}a}{2}\left(\frac{1-e^{-\gamma\Delta t}}{\gamma}\right)\right]$$

$$\left.+\;\frac{2\lambda_{J2}\mu_{J2}}{\sigma^2-2\kappa\mu_{J2}}\ln\left[\frac{2\kappa-2\kappa\,\mu_{J2}\,\beta^*(\Delta t,u)}{2\kappa\;-\sigma^2\,\beta^*(\Delta t,u)}\right]\right\} \;+\; c\,(e^{-\frac{1}{2}\sigma_{J1}^2u^2}\;-1)\,\Delta t \;\; .$$

The distribution has a mean of zero and the density function is symmetric around zero. Applying these results, we compute the cumulative distribution function for the model by using Fourier inversion of the characteristic function. For $x > 0$,

$$F(x) \;=\; \frac{1}{2} \;+\; \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(1-e^{-iux})\,\Phi(u)}{iu}\,du \quad ,$$

and for $x < 0$, $F(x) = 1 - F(-x)$. The numerical integration is performed by applying the Poisson summation formula.[7]

In Figures 14-17, we have plotted the empirical distribution functions with the model distribution functions for the stochastic volatility, jump model and the stochastic volatility model without jumps. The stochastic volatility, jump model fits the empirical distribution for the interest rates in all four currencies. The stochastic volatility model without jumps provides a good fit

[7] See Feller (1972, Chps. 15, 19) for a discussion of Fourier inversion formulas and the Poisson summation formula. An application of the Poisson summation formula in an option pricing model can be found in Chen and Scott (1995).

for most of the distributions, but there are noticeable deviations. The model without jumps does not fit the middle part of the distributions for the U.S., German, and Japanese interest rates. It is, however, difficult to distinguish the model without jumps from the stochastic volatility, jump model in Figure 17 for British interest rates. There are, however, numerous extreme outliers, as shown in Figures 6-9. Figures 18-21 focus on the tails of the distribution, and here one can see the differences between the two models, particularly in the cases of the U.S. and British interest rates. In every currency, the stochastic volatility model without jumps misses either the middle part of the distribution or the tails of the distribution, and in some case, it misses both.

3 Summary and Extensions

The empirical analysis in section 2 highlights the importance of jumps in both interest rates and interest rate volatility. The empirical model for the analysis is an ad hoc empirical model in which jump processes are added to the diffusion equations for futures interest rates. A complete model of the term structure of interest rates would be necessary for the management of a portfolio of interest rate derivatives that includes options on multiple maturities. Duffie, Pan, and Singleton (2000) have recently shown how to extend exponential affine models to incorporate multiple jumps, and they develop a model for valuing stock options with stochastic volatility and simultaneous jumps in both the stock price and volatility. Their results can be used to develop a multi-factor model of the term structure with jumps in both interest rates and volatility. An example would be the following three factor model with a stochastic mean, y_1, and a stochastic volatility, y_2:

$$\begin{aligned} dr &= \left[\kappa_0 y_1 - \kappa_0 r\right] dt + \sqrt{y_2}\, dZ_0 + dJ_0 \\ dy_1 &= \left[\kappa_1\theta_1 - \kappa_1 y_1\right] dt + \sigma_1 \sqrt{y_1}\, dZ_1 \\ dy_2 &= \left[\kappa_2\theta_2 - \kappa_2 y_2\right] dt + \sigma_2 \sqrt{y_2}\, dZ_2 + dJ_2 \end{aligned} \tag{5}$$

where $r(t)$ is the instantaneous short-term interest rate.[8] The two jump processes can have the same structure as the jump processes used in the empirical model for futures rates. A model of this form has exponential affine solutions for discount bond prices and futures on simple interest rates such as LIBOR.[9]

[8] See Anderson and Lund (1996) and Dai and Singleton (2000) for examples of three-factor models with stochastic mean and stochastic volatility, without the jumps.
[9] The solutions for a term structure model are developed in the Appendix.

Strategies for calibrating a model of this form are numerous. Because the bond pricing function and the model for futures rates are exponential affine, one can work with transformations that are linear functions of the factors and develop estimators based on moments. There are analytic solutions for the means, variances, covariances of linear combinations of the factors and one could fit the model to a set of sample moments estimated from discount rates and futures rates. One could alternatively develop estimators based on the empirical characteristic function, as in Chacko (1999) and Singleton (2001). Another strategy would be to solve the option pricing functions and calibrate the model to an entire initial term structure of rates as well as option prices for various maturities. The model could be calibrated to match an initial term structure of interest rates (discount rates, futures rates, and swap rates) by introducing a deterministic function of time in the drift of the *dr* equation. To match the model to a term structure of implied volatilities, which is essentially a set of at-the-money option prices for different expirations, one could make σ_1 a deterministic function of time.[10]

Stochastic volatility and jumps are important empirical features for stock prices, foreign exchange rates, and interest rates. The empirical results presented in section 2 support the case for stochastic volatility and jumps in interest rates, but there are some subtle, but important differences for interest rate models. There is no evidence of correlation between changes in interest rates and changes in interest rate volatility. Aside from one or two observations out of several thousand, there is little or no evidence of correlation between large changes in interest rates and large changes in interest rate volatility. These results can be contrasted with the evidence of strong negative correlation between stock price changes and changes in stock market volatility, for both large changes and small changes.

[10] In this case, the bond pricing function is still exponential affine and the deterministic function for the volatility of the stochastic mean factor is incorporated in the numerical solutions for the ODE's.

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Appendix

The formal model, under the risk neutral distribution, is

$$dr \;=\; \left[\kappa_0\, y_1 - \kappa_0 r - \lambda_0 y_2\right] dt \;+\; \sqrt{y_2}\; dZ_0 \;+\; dJ_0$$

$$dy_1 \;=\; \left[\kappa_1 \theta_1 - (\kappa_1 + \lambda_1) y_1\right] dt \;+\; \sigma_1 \sqrt{y_1}\; dZ_1 \tag{A1}$$

$$dy_2 \;=\; \left[\kappa_2 \theta_2 - (\kappa_2 + \lambda_2) y_2\right] dt \;+\; \sigma_2 \sqrt{y_2}\; dZ_2 \;+\; dJ_2$$

where $r(t)$ is the instantaneous short-term interest rate, $y_1(t)$ is a stochastic mean factor, and $y_2(t)$ is a stochastic volatility factor. The jump in the short-term rate is a jump process with a random intensity parameter $\lambda'_{J0}(t) = a' y_2(t)$ and a jump size distributed as a normal with mean μ'_{J0} and a standard deviation σ_{J0}. The jump in the stochastic volatility factor is a Poisson process with intensity parameter λ'_{J2} and a jump size that is exponentially distributed with a mean μ'_{J2}. The two jumps can be modeled as independent processes for interest rates, except that the intensity of the interest rate jump is a function of y_2. The parameters with primes are jump parameters that should be adjusted when moving from the real world distribution to the risk neutral distribution.

The model in (A1) can be solved to produce a pricing function for discount bonds, as well as futures on 3 month LIBOR. The discount bond pricing function is the solution to the following expectation under the risk-neutral distribution:

$$P(t, r, y_1, y_2; T) \;=\; \hat{E}\left[\, \exp\!\left(- \int_t^T r(s)\, ds \right) \;\middle|\; r(t),\, y_1(t),\, y_2(t) \right].$$

This function must satisfy the following partial differential integral equation:

$$\frac{\partial P}{\partial t} + \tfrac{1}{2}\frac{\partial^2 P}{\partial r^2} y_2 + \tfrac{1}{2}\frac{\partial^2 P}{\partial y_1^2}\sigma^2\, y_1 + \tfrac{1}{2}\frac{\partial^2 P}{\partial y_2^2}\sigma^2\, y_2 + \frac{\partial P}{\partial r}\left[\kappa_0\, y_1 - \kappa_0\, r - \lambda_0\, y_2\right]$$

$$+\; \frac{\partial P}{\partial y_1}\left[\kappa_1\, \theta_1 - (\kappa_1 + \lambda_1)\, y_1\right] + \; \frac{\partial P}{\partial y_2}\left[\kappa_2\, \theta_2 - (\kappa_2 + \lambda_2)\, y_2\right] \;-\; r\, P$$

$$+ \ a' y_2 \int_{-\infty}^{\infty} \left[P(t, r + x, y_1, y_2) - P(t, r, y_1, y_2) \right] \frac{\exp\left\{ -\frac{1}{2} \left(\frac{x - \mu'_{J0}}{\sigma_{J0}} \right)^2 \right\}}{\sqrt{2\pi}\, \sigma_{J0}} \, dx$$

$$+ \lambda'_{J2} \int_{0}^{\infty} \left[P(t, r, y_1, y_2 + x) - P(t, r, y_1, y_2) \right] \frac{\exp\left(-x / \mu'_{J2} \right)}{\mu'_{J2}} \, dx \ = \ 0 \qquad .$$

The solution for the bond pricing function is an exponential affine function of the state variables,

$$P(t, r, y_1, y_2; T) \ = \ \exp\left\{ -A(\tau) - B_0(\tau) r(t) - B_1(\tau) y_1(t) - B_2(\tau) y_2(t) \right\} ,$$

where $\tau = T - t$ and the coefficients are solutions to the following system of ordinary differential equations.

$$\frac{dB_0}{d\tau} \ = \ 1 \ - \kappa_0 \, B_0(\tau)$$

$$\frac{dB_1}{d\tau} \ = \ -\tfrac{1}{2} \sigma_1^2 \, B_1^2(\tau) \ + \ \kappa_0 \, B_0(\tau) \ - \ (\kappa_1 + \lambda_1) B_1(\tau)$$

$$\frac{dB_2}{d\tau} \ = \ -\tfrac{1}{2} \sigma_2^2 \, B_2^2(\tau) \ - \ \tfrac{1}{2} B_0^2(\tau) \ - \ \lambda_0 B_0(\tau) \ - \ (\kappa_2 + \lambda_2) B_2(\tau)$$

$$- \ a' \left[\exp\left\{ -B_0(\tau) \mu'_{J0} + \tfrac{1}{2} B_0^2(\tau) \sigma_{J0}^2 \right\} - 1 \right]$$

$$\frac{dA}{d\tau} \ = \ \kappa_1 \theta_1 \ B_1(\tau) \ + \ \kappa_2 \theta_2 \ B_2(\tau) \ - \ \lambda'_{J2} \left(\frac{1}{1 + \mu'_{J2} \, B_2(\tau)} - 1 \right)$$

The boundary conditions are $A(0) = B_0(0) = B_1(0) = B_2(0) = 0$. The first equation can be solved analytically:

$$B_0(\tau) \ = \ \frac{1 - e^{-\kappa_0 \tau}}{\kappa_0} \quad .$$

The other equations must be solved numerically.[11] The continuously compounded yields for discount bonds, $-\ln P / (T - t)$, are linear functions of the state variables, which are useful for empirical analysis.

[11] Standard ODE solvers, such as Runge-Kutta, produce extremely accurate solutions very quickly.

The futures contracts on 3-month LIBOR are set up so that the final settlement is equal to 100 - 3 month LIBOR. LIBOR is a simple interest money market rate quoted on an annual basis so that

$$3\,\text{month LIBOR} \;=\; \frac{360}{90}\times\left(\frac{1}{P(t,r,y_1,y_2\,;\,t+{}^{90}\!/_{365})}-1\right).$$

The futures price is a martingale under the risk neutral distribution and it is found by solving the following risk-neutral expectation conditional on the current values for $r(t)$, $y_1(t)$, and $y_2(t)$:[12]

$$F(t,T) \;=\; \hat{E}\left[100-100\times\tfrac{360}{90}\times\left(\exp\{A(\tau')+B_0(\tau')r(T)+B_1(\tau')\,y_1(T)+B_2(\tau')\,y_2(T)\}-1\right)\right],$$

where $\tau' \;=\; {}^{90}\!/_{365}$. The futures rate is 100 minus the futures price:

$$FR(t,T) \;=\; 100\times\tfrac{360}{90}\times\left[\exp\left\{\alpha(T-t)+\beta_0(T-t)\,r(t)+\beta_1(T-t)\,y_1(t)+\beta_2(T-t)\,y_2(t)\right\}-1\right],$$

where the coefficients must satisfy the following system of ordinary differential equations.

$$\frac{d\beta_0}{d\tau} \;=\; -\,\kappa_0\,\beta_0(\tau)$$

$$\frac{d\beta_1}{d\tau} \;=\; \tfrac{1}{2}\,\sigma_1^2\,\beta_1^2(\tau)\;+\;\kappa_0\,\beta_0(\tau)\;-\;(\kappa_1+\lambda_1)\,\beta_1(\tau)$$

$$\frac{d\beta_2}{d\tau} \;=\; \tfrac{1}{2}\,\sigma_2^2\,\beta_2^2(\tau)\;+\;{}^{1}\!/_{2}\,\beta_0^2(\tau)-\lambda_0\,\beta_0(\tau)\;-\;(\kappa_2+\lambda_2)\,\beta_2(\tau)$$

$$+\;a'\left[\exp\left\{\beta_0(\tau)\mu'_{J0}+\tfrac{1}{2}\beta_0^2(\tau)\sigma_{J0}^2\right\}-1\right]$$

$$\frac{d\alpha}{d\tau} \;=\; \kappa_1\,\theta_1\,\beta_1(\tau)\;+\;\kappa_2\,\theta_2\,\beta_2(\tau)\;+\;\lambda'_{J2}\left(\frac{1}{1-\mu'_{J2}\,\beta_2(\tau)}-1\right)$$

The boundary conditions for this system are $\alpha(0)=A(\tau')$, $\beta_0(0)=B_0(\tau')$, $\beta_1(0)=B_1(\tau')$, and $\beta_2(0)=B_2(\tau')$. The first equation has an analytic solution, $\beta_0(\tau) \;=\; e^{-\kappa_0\tau}\,B_0(\tau')$, and the other three equations must be solved numerically.

[12] See Cox, Ingersoll, and Ross (1981).

Table 1
Summary Statistics, U.S. LIBOR
Sample Period, March 1985 to December 2000, *T* = 3985

	Changes in Interest Rates			
	Futures Rate	3 Month LIBOR	6 Month LIBOR	12 Month LIBOR
Sample Mean	-0.0027%	-0.0010%	-0.0010%	-0.0012%
Standard Deviation	0.0641%	0.0682%	0.0725%	0.0798%
Excess Kurtosis	32.30	14.45	17.47	18.07
% of Days, No Change	19.5%	36.5%	34.4%	30.5%
Autocorrelations:				
1	0.07	-0.09	-0.11	-0.09
2	0.02	0.02	0.04	0.03
3	-0.01	-0.02	-0.01	-0.01
4	-0.02	0.02	-0.01	0.00
Correlation Matrix				
	1.00	0.38	0.41	0.43
	0.38	1.00	0.70	0.61
	0.41	0.70	1.00	0.72
	0.43	0.61	0.72	1.00

Table 2
Summary Statistics, German LIBOR
Sample Period, August 1990 to February 1998, *T* = 1898

	Changes in Interest Rates			
	Futures Rate	3 Month LIBOR	6 Month LIBOR	12 Month LIBOR
Sample Mean	0.0000%	-0.0026%	-0.0027%	-0.0028%
Standard Deviation	0.0449%	0.0630%	0.0620%	0.0653%
Excess Kurtosis	22.23	9.10	8.11	6.09
% of Days, No Change	15.4%	30.5%	30.9%	30.5%
Autocorrelations:				
1	-0.01	-0.19	-0.24	-0.20
2	0.07	0.02	0.04	0.00
3	-0.05	-0.03	-0.03	0.00
4	0.03	-0.02	0.02	0.01
Correlation Matrix				
	1.00	0.38	0.40	0.35
	0.38	1.00	0.58	0.48
	0.40	0.58	1.00	0.58
	0.35	0.48	0.58	1.00

Table 3
Summary Statistics, Japan TIBOR
Sample Period, July 1989 to December 2000, T = 2808

	Changes in Interest Rates			
	Futures Rate	3 Month LIBOR	6 Month LIBOR	12 Month LIBOR
Sample Mean	-0.0008%	-0.0018%	-0.0018%	-0.0018%
Standard Deviation	0.0371%	0.0353%	0.0358%	0.0357%
Excess Kurtosis	9.30	15.10	11.20	10.95
% of Days, No Change	21.4%	42.7%	38.9%	36.4%
Autocorrelations:				
1	0.08	0.14	0.12	0.16
2	-0.01	0.02	0.05	0.07
3	0.02	0.02	0.02	0.07
4	0.06	0.01	0.03	0.03
Correlation Matrix				
	1.00	0.54	0.60	0.61
	0.54	1.00	0.78	0.69
	0.60	0.78	1.00	0.79
	0.61	0.69	0.79	1.00

Table 4
Summary Statistics, British LIBOR
Sample Period, August 1990 to December 2000, T = 2620

	Changes in Interest Rates			
	Futures Rate	3 Month LIBOR	6 Month LIBOR	12 Month LIBOR
Sample Mean	-0.0019%	-0.0035%	-0.0035%	-0.0034%
Standard Deviation	0.0785	0.0836%	0.0920%	0.0947%
Excess Kurtosis	202.53	29.75	24.14	21.74
% of Days, No Change	11.53%	28.9%	26.9%	28.0%
Autocorrelations:				
1	0.07	-0.23	-0.18	-0.18
2	-0.04	0.03	-0.02	0.01
3	0.09	-0.01	0.05	0.05
4	0.01	0.05	0.03	0.01
Correlation Matrix				
	1.00	0.42	0.53	0.52
	0.46	1.00	0.59	0.48
	0.54	0.59	1.00	0.68
	0.54	0.48	0.68	1.00

Table 5
Maximum Likelihood Estimation of Jump Parameters, U.S.
Sample Size, T = 3985

A. $\Delta FR(t)$ Equation		B. Stochastic Volatility Equation	
ln L	-12,135.6	ln L	41,170.4
Restricted ln L ($d = \sigma_{J1} = 0$)	-12,688.3	Restricted ln L ($\lambda_{J2} = \mu_{J2} = 0$)	40,500.1
b^2	0.15944 (0.00884)	μ	6.2259×10^{-7} (1.4823×10^{-7})
d	3.0211×10^{6} (0.2068×10^{6})	ρ	0.97706 (0.00183)
σ^2_{J1}	2.6530×10^{-7} (0.1631×10^{-7})	σ^2_2	2.1298×10^{-4} (0.0331×10^{-4})
σ_{J1}	0.052%	λ_{J2}	6.6823 (1.0183)
		μ_{J2}	3.5652×10^{-5} (0.2228×10^{-5})

Table 6
Maximum Likelihood Estimation of Jump Parameters, Germany
Sample Size, T = 1905

A. $\Delta FR(t)$ Equation		B. Stochastic Volatility Equation	
ln L	-5,093.8	ln L	19,692.6
Restricted ln L ($d = \sigma_{J1} = 0$)	-5,237.1	Restricted ln L ($\lambda_{J2} = \mu_{J2} = 0$)	19,327.8
b^2	0.33817 (.02445)	μ	1.0895×10^{-6} (0.2335×10^{-6})
d	1.3024×10^{6} (0.2473×10^{6})	ρ	0.96742 (0.00381)
σ^2_{J1}	2.8340×10^{-7} (0.3439×10^{-7})	σ^2_2	2.4176×10^{-4} (0.0581×10^{-4})
σ_{J1}	0.053%	λ_{J2}	12.896 (2.611)
		μ_{J2}	2.0265×10^{-5} (0.0221×10^{-5})

Note: Standard errors in parentheses.

Table 7
Maximum Likelihood Estimation of Jump Parameters, Japan
Sample Size, T = 2302

A. $\Delta FR(t)$ Equation

ln L	-5,383.2
Restricted ln L ($d = \sigma_{J1} = 0$)	-5,574.9
b^2	0.33198 (.02092)
D	2.7926×10^{6} (0.4278×10^{6})
σ_{J1}^2	1.7606×10^{-7} (0.2317×10^{-7})
σ_{J1}	0.042%

B. Stochastic Volatility Equation

ln L	24,344.8
Restricted ln L ($\lambda_{J2} = \mu_{J2} = 0$)	23,761.8
μ	1.6495×10^{-6} (0.2151×10^{-6})
ρ	0.91281 (0.00674)
σ_2^2	3.1403×10^{-4} (0.0547×10^{-4})
λ_{J2}	10.083 (1.604)
μ_{J2}	3.1957×10^{-5} (0.4506×10^{-5})

Table 8
Maximum Likelihood Estimation of Jump Parameters, Great Britain
Sample Size, T = 2640

A. $\Delta FR(t)$ Equation

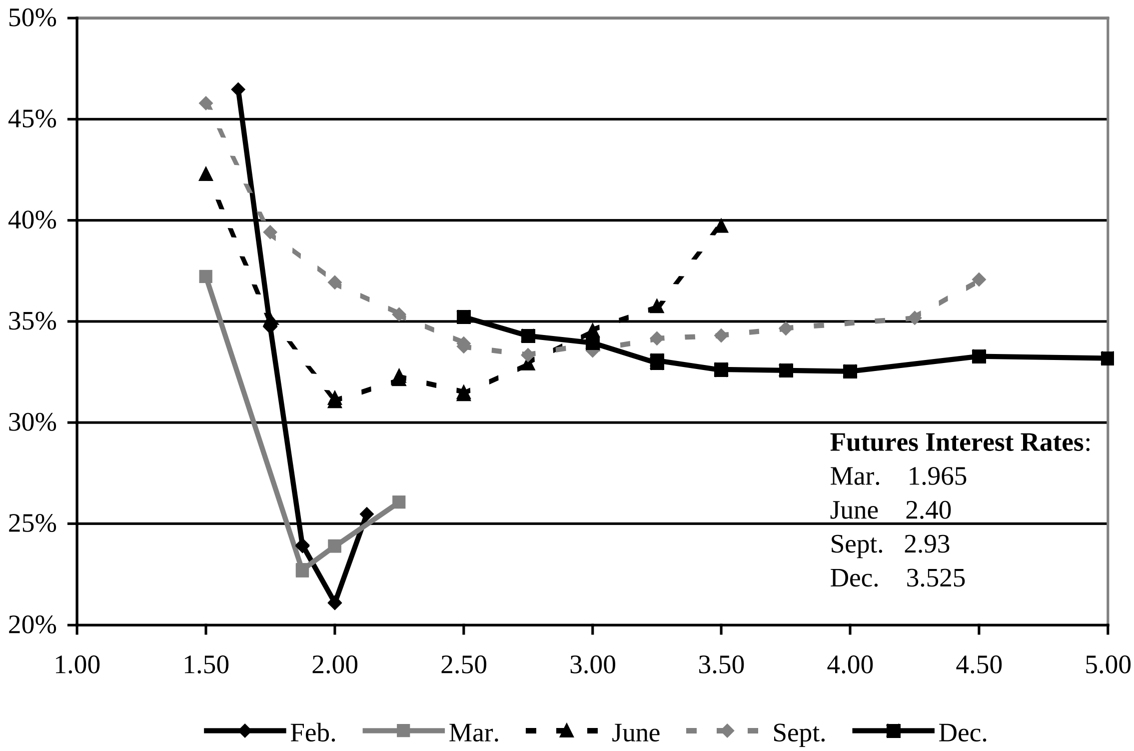
ln L	-7,846.3
Restricted ln L ($d = \sigma_{J1} = 0$)	-8,142.4
b^2	.54115 (.02167)
D	4.3279×10^{5} (0.6178×10^{5})
σ_{J1}^2	1.4210×10^{-6} (0.1355×10^{-6})
σ_{J1}	0.119%

B. Stochastic Volatility Equation

ln L	26,458.5
Restricted ln L ($\lambda_{J2} = \mu_{J2} = 0$)	26,018.7
μ	1.5771×10^{-6} (0.2996×10^{-6})
ρ	0.96357 (0.00314)
σ_2^2	3.7018×10^{-4} (0.0460×10^{-4})
λ_{J2}	5.6627 (1.0091)
μ_{J2}	7.5966×10^{-5} (0.8966×10^{-5})

Note: Standard errors in parentheses.

Figure 1
Implied Volatility from Eurodollar Futures Options
January 25, 2002



Note: The February options exercise into the March 2002 futures.

Figure 2
Futures Interest Rate and Implied Volatility, United States

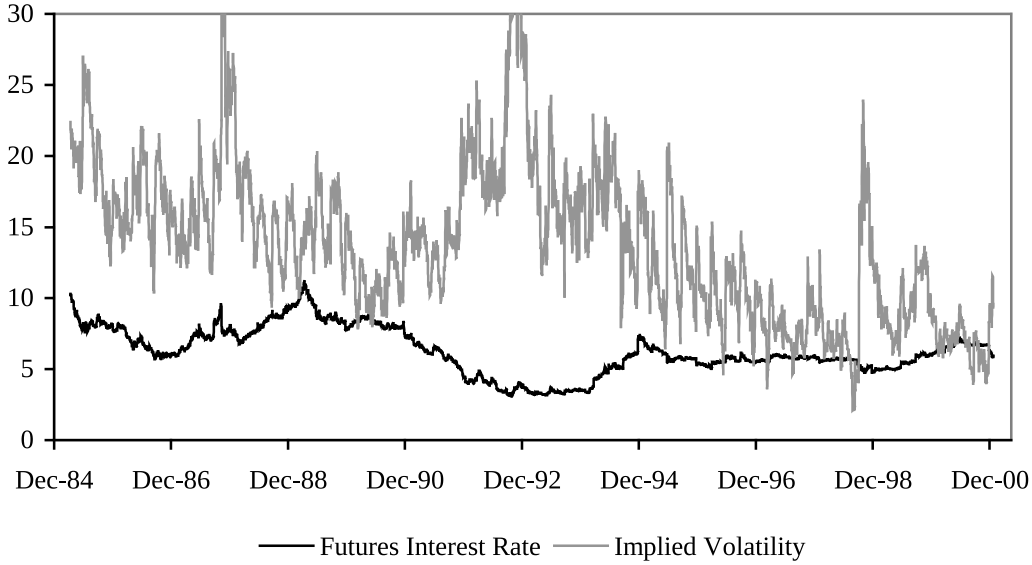


Figure 3
Futures Interest Rate and Its Implied Volatility, Germany

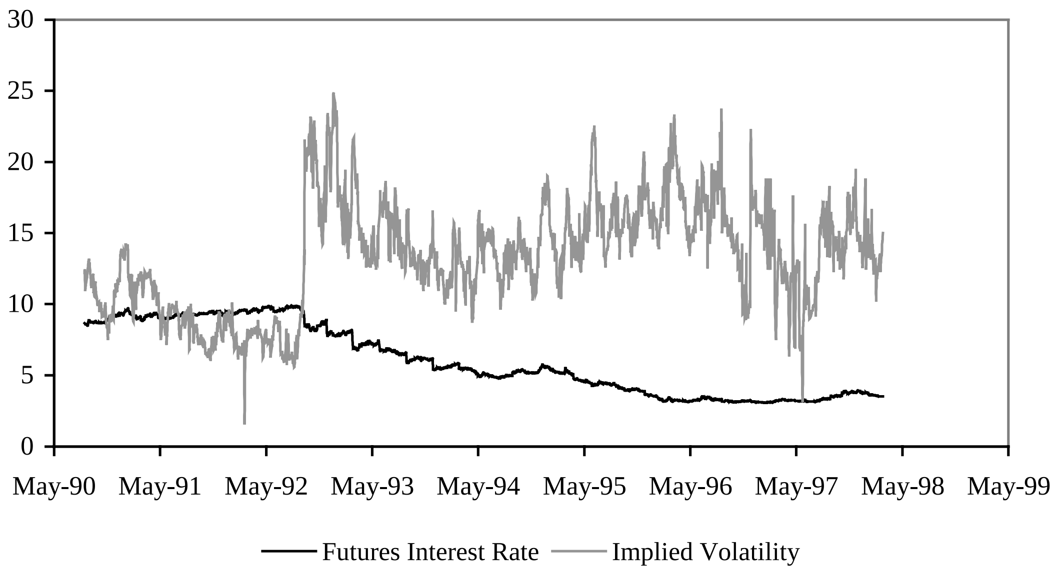


Figure 4
Futures Interest Rate and Implied Volatility (Rate Level), Japan

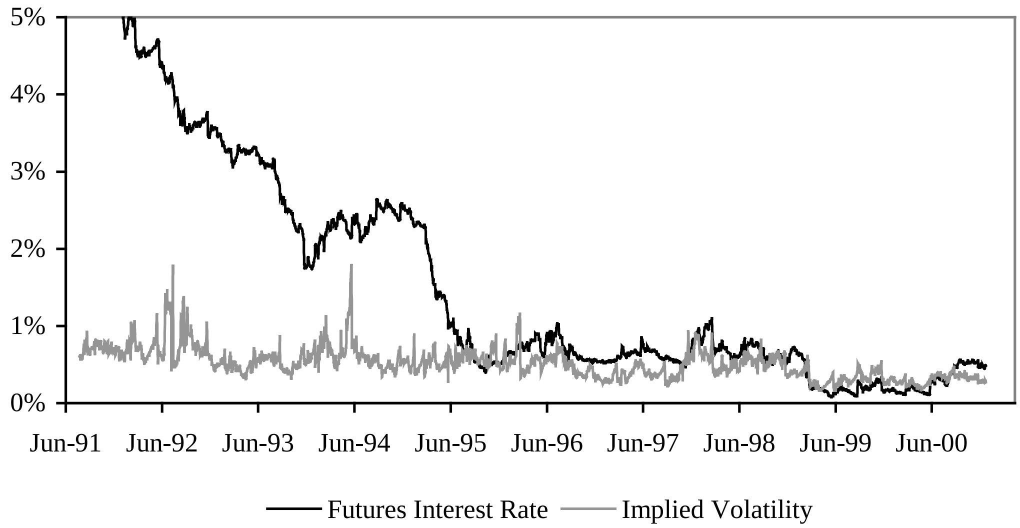


Figure 5
Futures Interest Rate and Its Implied Volatility, Great Britain

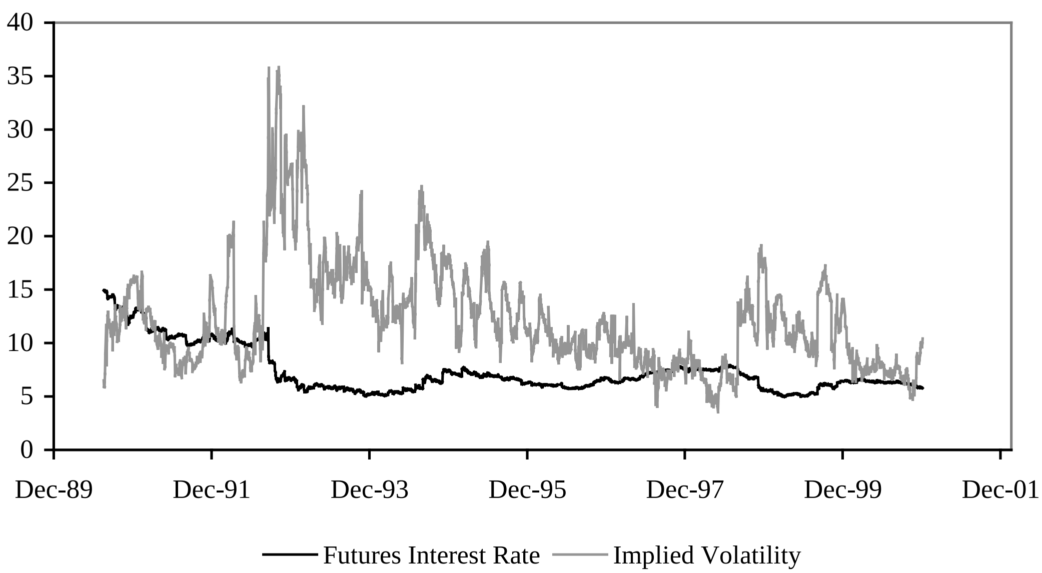


Figure 6
Change in Implied Volatility vs. Change in Futures Interest Rate, U.S.

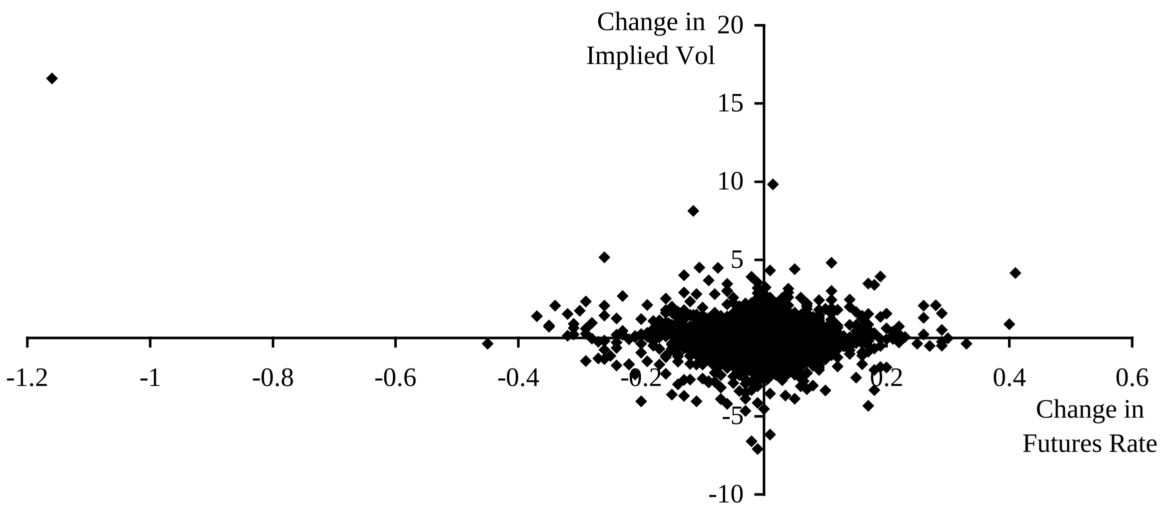


Figure 7
Change in Implied Volatility vs. Change in Futures Interest Rate, Germany

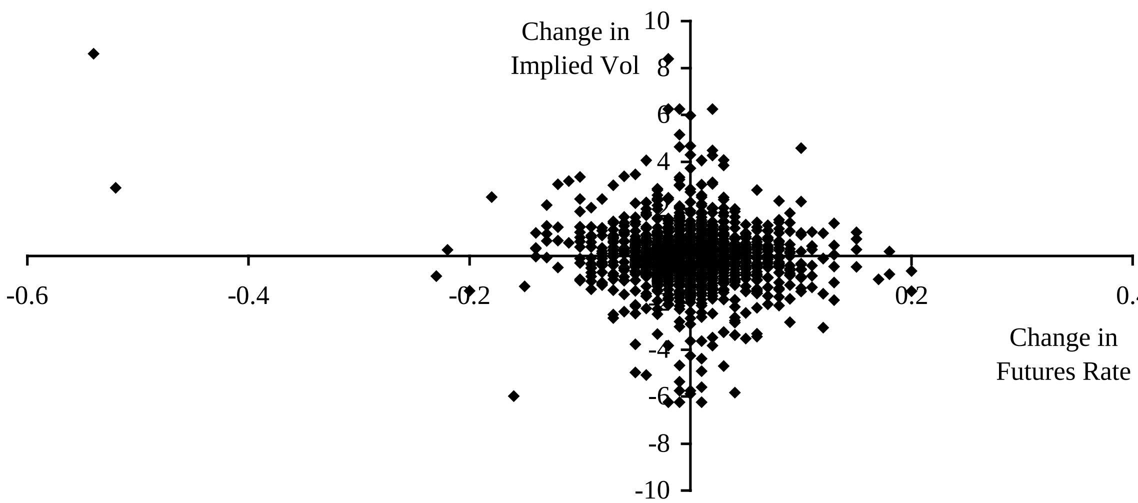


Figure 8
Change in Implied Volatility (Levels) vs. Change in Futures Interest Rate, Japan

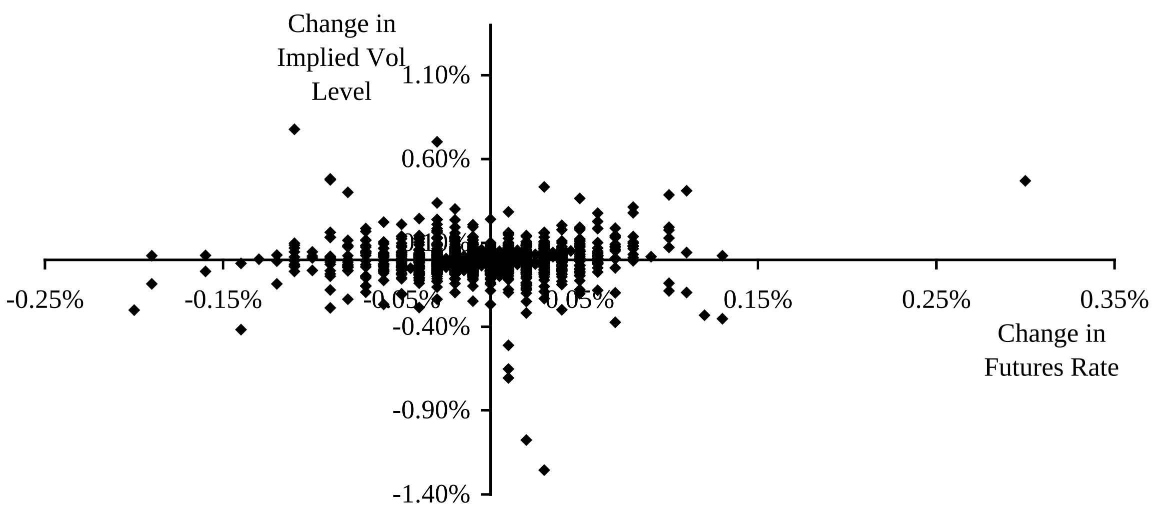


Figure 9
Change in Implied Volatility vs. Change in Futures Interest Rate, Great Britain

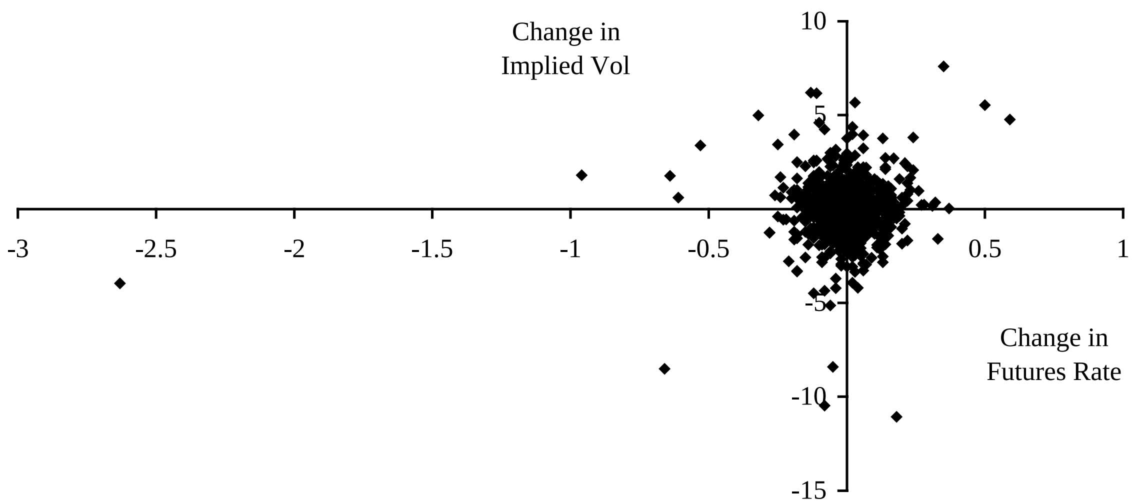


Figure 10
Change in VIX vs. Change in Log of S&P100 Index

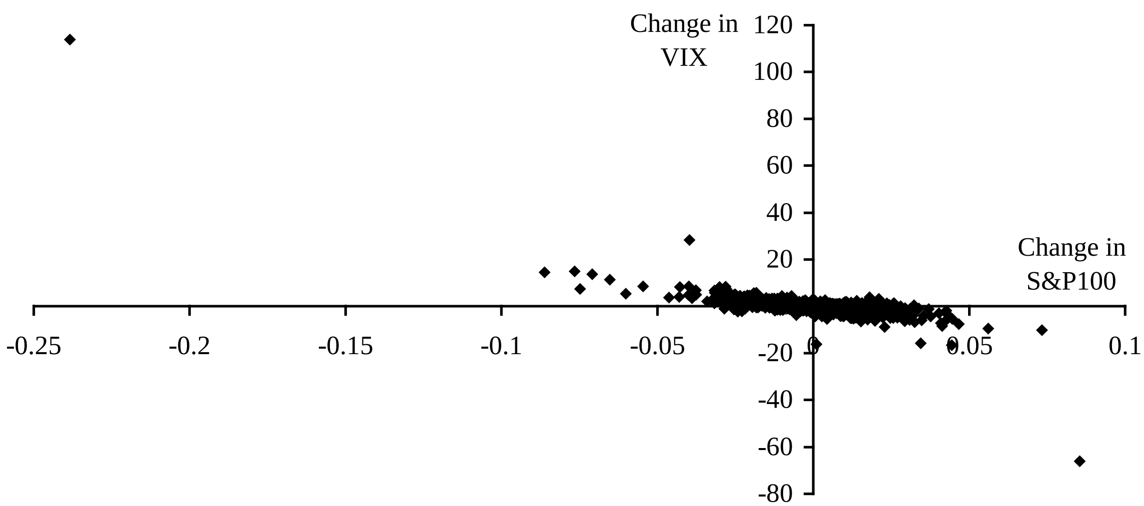


Figure 11
Distribution of Changes in Futures Rate, United States

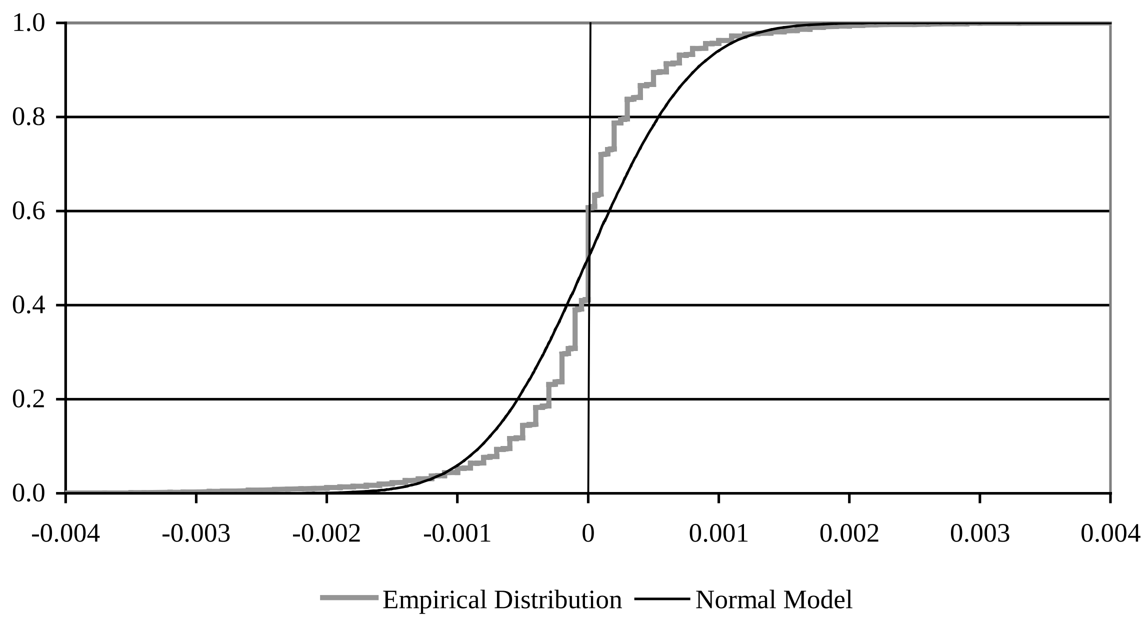


Figure 12
Distribution of Changes in Log of Futures Rate, United States

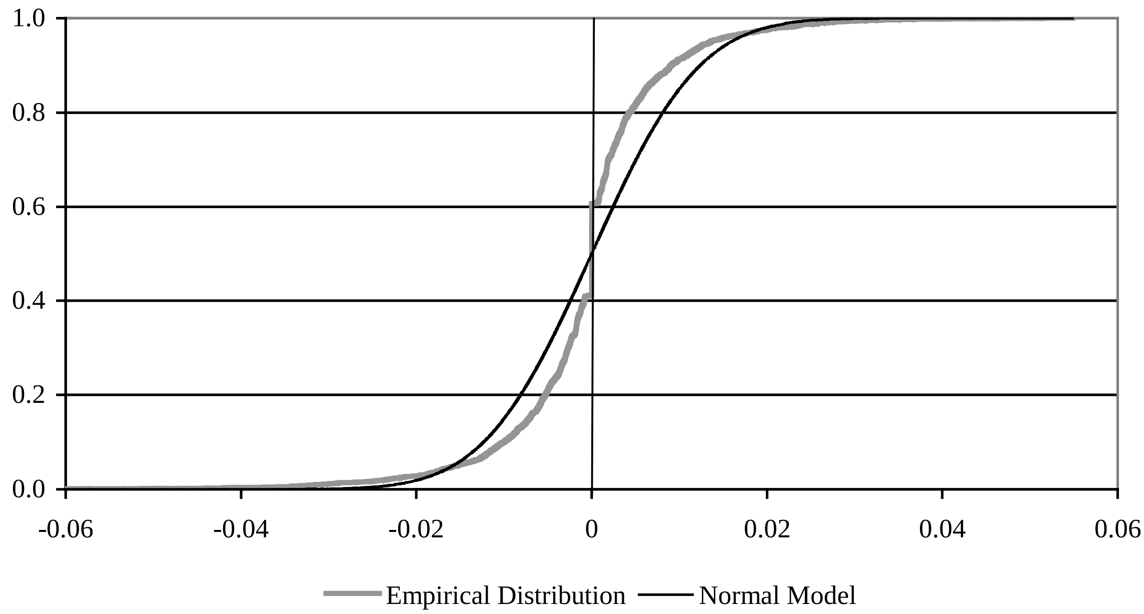


Figure 13
Distribution of Changes in Log of Futures Rate Scaled by Implied Volatility, United States

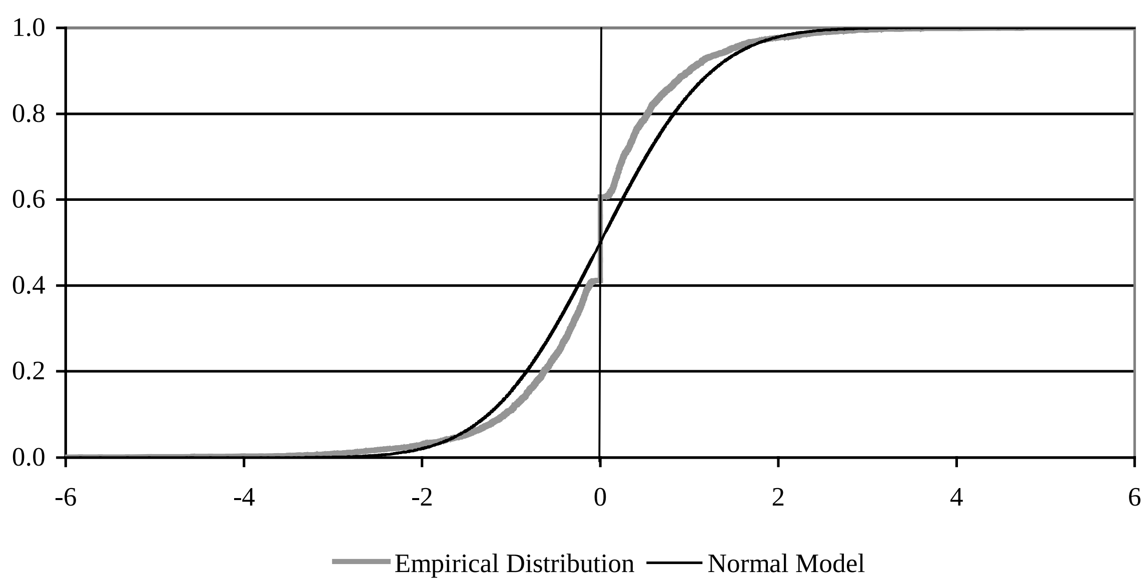


Figure 14
Distribution of Changes in Futures Rate, United States

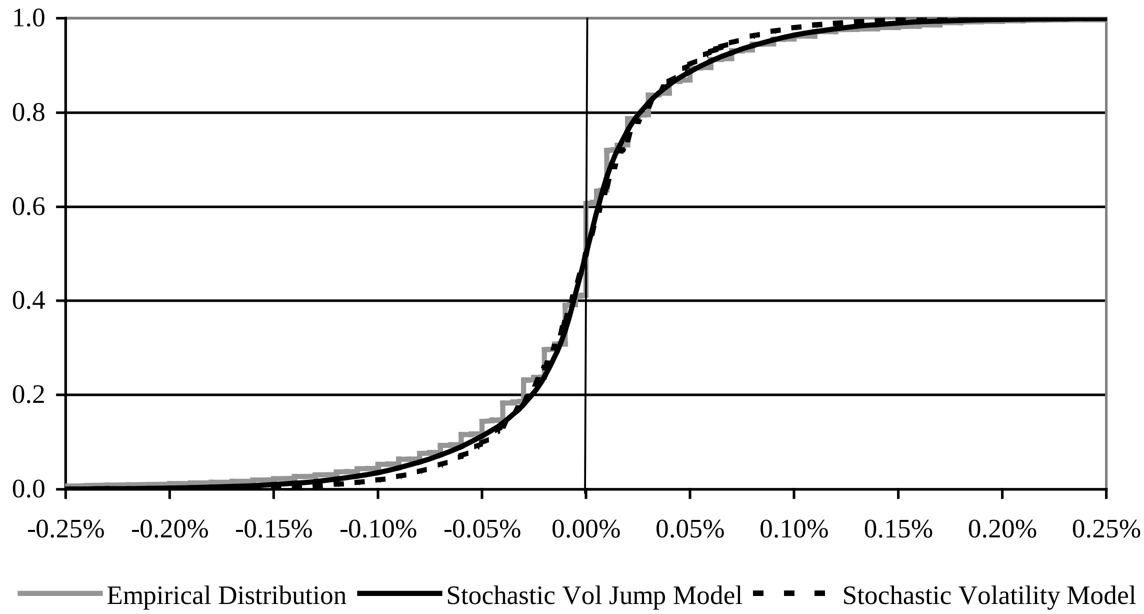


Figure 15
Distribution of Changes in Futures Rate, Germany

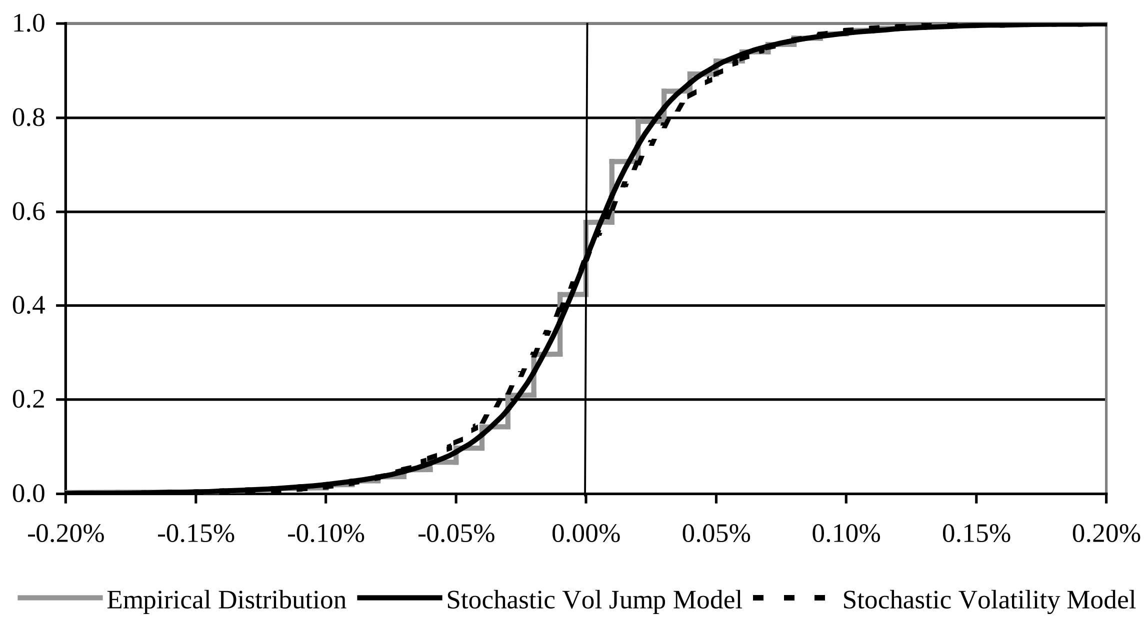


Figure 16
Distribution of Changes in Futures Rate, Japan

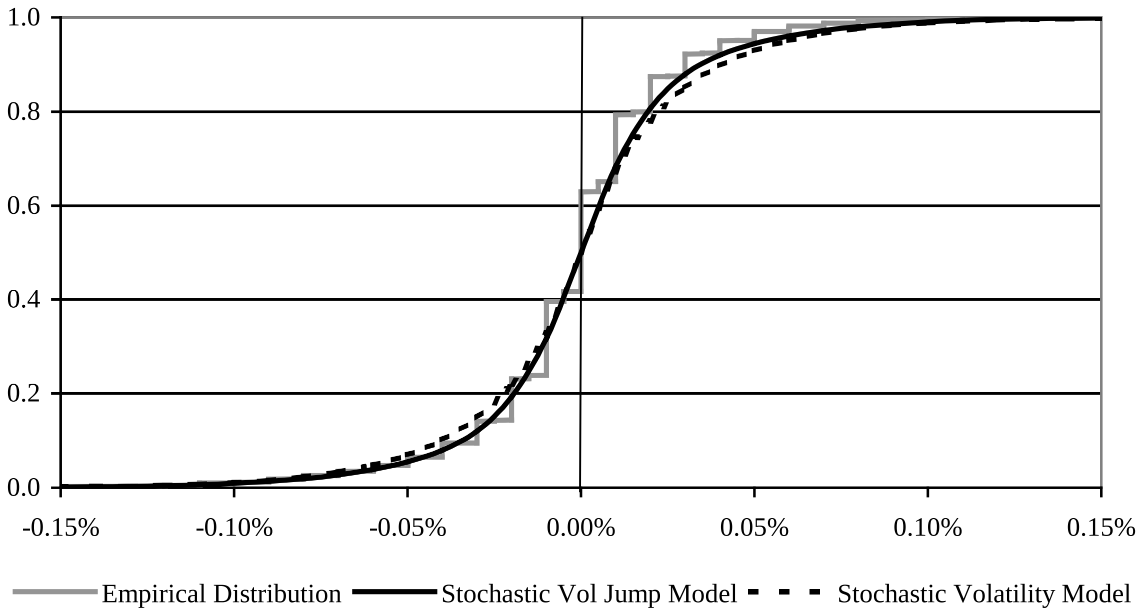


Figure 17
Distribution of Changes in Futures Rate, Great Britain

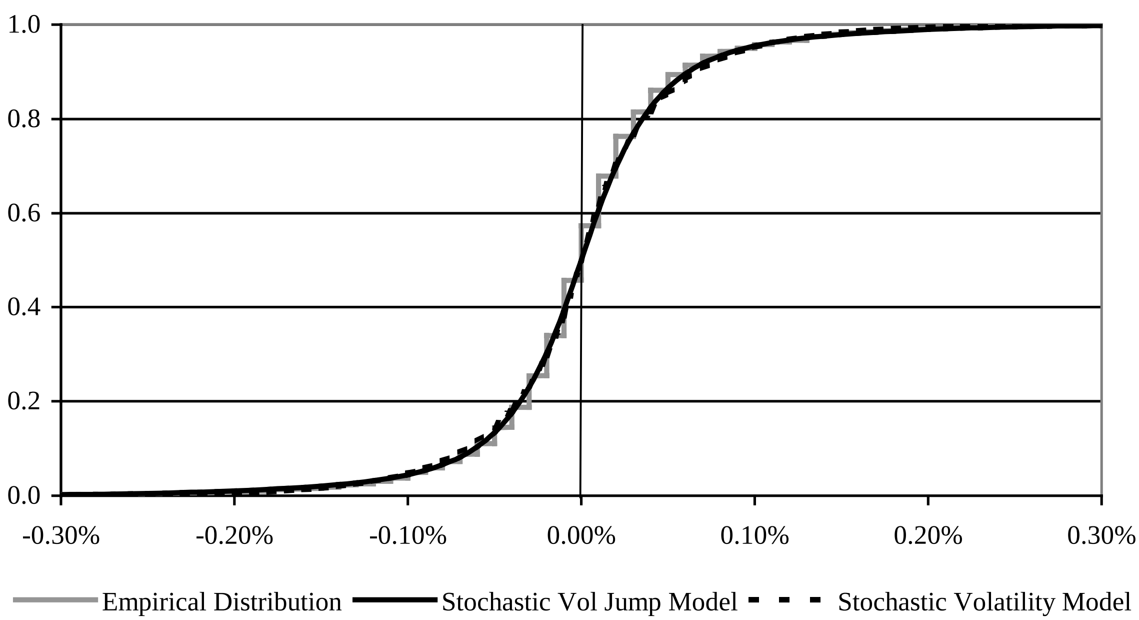


Figure 18
Left and Right Tails of the Distribution for Changes in the Futures Rate, USD

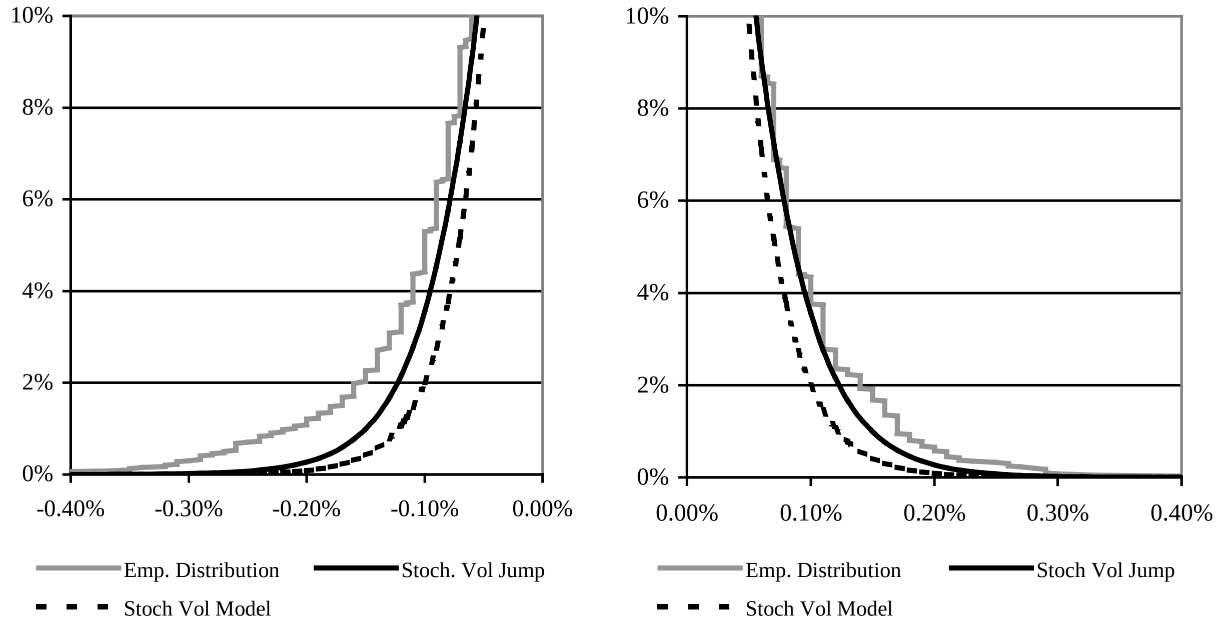


Figure 19
Left and Right Tails of the Distribution for Changes in the Futures Rate, Germany

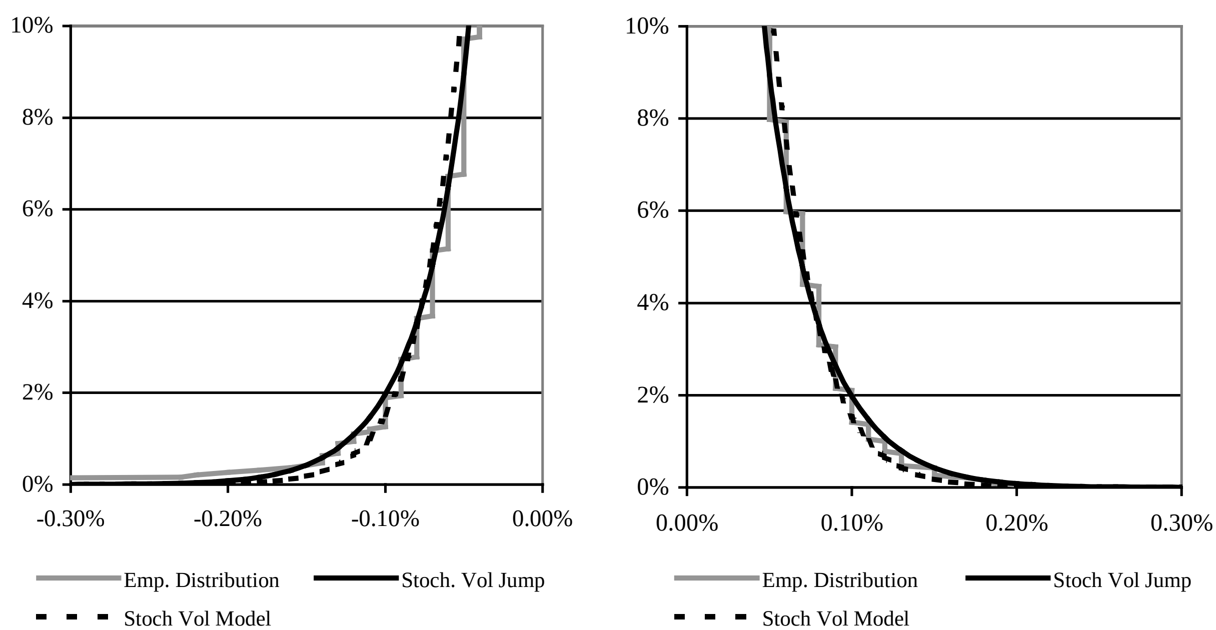


Figure 20
Left and Right Tails of the Distribution for Changes in the Futures Rate, Japan

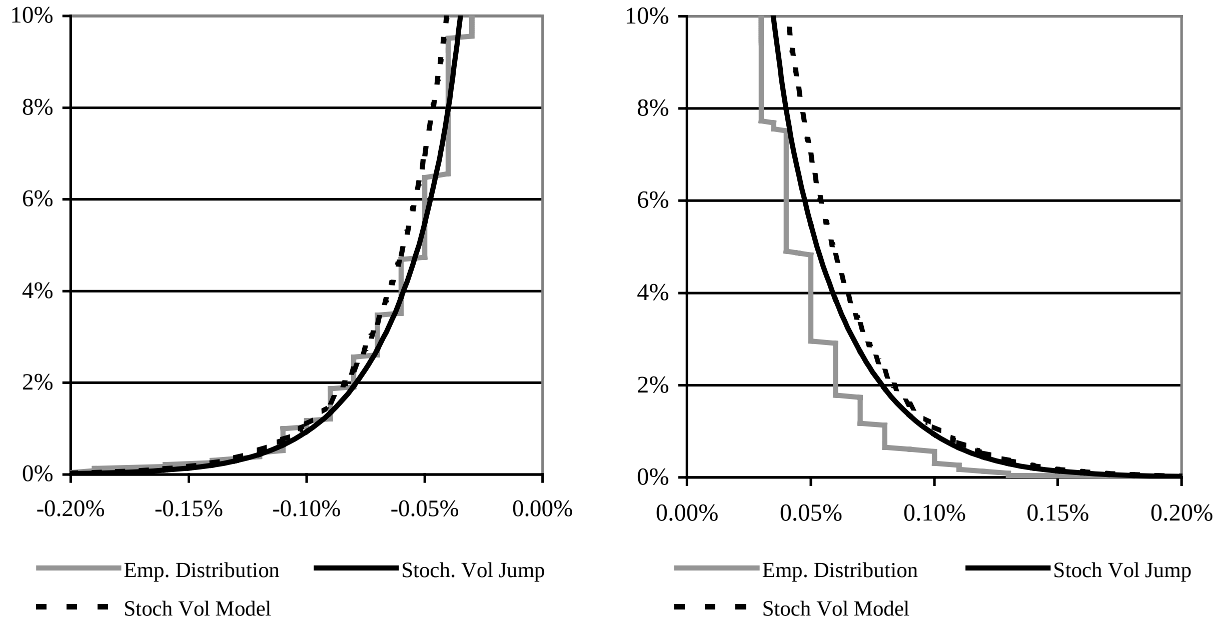


Figure 21
Left and Right Tails of the Distribution for Changes in the Futures Rate, Great Britain