Ultra Treasury Bond Futures

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KEY FINDINGS

- This article is the initial study which rigorously evaluates the quality option value of Ultra Treasury bond futures contracts as the introduction of the Ultra futures.
- We use the Ho-Lee model for the evaluation because the normality assumption of this model is more consistent with the interest rate dynamics in low interest rate times. Moreover, it can prevent those factor-based models from producing accurate pricing results.
- This article is one of the few which clarify whether or not the viewpoint about the “dry spell” phenomenon is supported.

ABSTRACT

In this article, we determine the quality option value of Ultra Treasury bond futures contracts which allow deliverable bonds between 25 and 30 years to maturity and compare them with the new regular Treasury bond futures which allow deliverable bonds between 15 and 25 years to maturity. We use the arbitrage-free Ho-Lee model for the valuation. Using weekly data from March 25, 2011 until April 16, 2021 after the Ultra futures contract was introduced, we discover that: (1) that quality option value is higher for the Ultra futures than the new regular futures; (2) the Ho-Lee model consistently underprices the market; and (3) the “dry spell” period predicted by Ben-Abdallah and Breton (2017) is only partially supported.

Before 2010, the Treasury bond futures contract permitted delivery of at least 15 years to maturity Treasury bonds. In 2010, this contract became known as the “classical” Treasury bond futures contract and it was replaced by the “Ultra” and the new “regular” Treasury bond futures contracts which differ only by their deliverable bonds. The deliverable basket for the Ultra T-Bond futures comprises Treasury bonds with at least 25 years to maturity, while the deliverable basket for the new “regular” Treasury bond futures contract comprises Treasury bonds with remaining terms to maturity of 15~25 years. Presumably the CME Group felt that segmenting the deliverable basket in this way would help hedgers manage their long-term interest rate risks.[1,2] In all other respects, the specifications for the Ultra T-Bond futures resemble those for the regular CME Group Treasury bond contract.

[1] See a CME publication (http://futures.hexun.com/upload/TreasuryFuturesOptions.pdf).

[2] https://www.marketswiki.com/wiki/CME_Group_Ultra_T-Bond reports January 11, 2009 as the introduction of the Ultra futures. Ben-Abdallah and Breton (2017) report March 2011 (Section 2.1 An Alignment of Four Concurrent Conditions) as the starting date of Ultra T bond futures.

They are identical in terms of their notional value, minimum tick size, contract critical dates, and notional coupon.[3]

In a recent paper, Ben-Abdallah and Breton (2017) connect the introduction of the Ultra Treasury bond futures to a shortage of Treasury bonds in between 2031 and 2036. Due to a budget surplus in the Clinton administration, the Treasury department suspended issuing 30-year Treasury bonds in 2001~2006, causing a disappearance of deliverable bonds in between 2031 and 2036. The lack of these Treasury bonds is a source of a 5-year gap for the futures contracts expiring in the end of 2015.[4] Ben-Abdallah and Breton compare this delivery shortage of Treasury bonds to the same liquidity shortage due to the cancellation of the callable Treasuries in November 2009 (causing a similar 5-year "dry spell" in the futures contracts from December 1994 until September 1999).[5] The dry spell phenomenon explained in Ben-Abdallah and Breton is summarized in the Appendix.[6]

In this article, we evaluate the Ultra Treasury bond futures contracts that have deliverable bonds ranging from 25~30 years to maturity and compare them with the regular futures contracts that have deliverable bonds from 15~25 years. Our data contain both regular and Ultra weekly futures prices and their deliverable bonds from March 25, 2011 until April 16, 2021 which cover the dry spell period. We use the arbitrage-free Ho-Lee model (1986) for the evaluation in that: (1) the shape of the yield curve changes drastically during the sample period that prevents those factor-based models (e.g., Cox-Ingersoll-Ross model) from producing accurate pricing results; and (2) the normality assumption of the Ho-Lee model is more consistent with the interest rate dynamics in low interest rate times.[7]

Our empirical evidence suggests that the quality option value (which equals the difference between the cost-of-carry price and the theoretical Ho-Lee futures price) is substantially higher for Ultra futures than for regular futures. This is expected in that the deliverable bonds to the Ultra contracts are longer-term bonds and hence, their prices are more volatile (and volatility fuels the option value). Furthermore, the range in years to maturity for the deliverables is half as large for the ultra-futures as it is for the regular futures, that is, 5 years vs 10. Secondly, the Ho-Lee model consistently under-prices the futures contracts for both Ultra and regular futures reflecting a negative timing option value. Unfortunately, the timing option is not self-financing and hence cannot be arbitraged away. Lastly, our quality option valuation result contradicts the Ben-Abdallah and Breton (2017) argument of the dry spell period between 2015–2020.

[3] On October 18, 2015, CME Group announced the launch of the Ultra 10-Year US Treasury Note futures and options for early in the first quarter of 2016. The new Ultra 10-Year US Treasury Note futures will allow for delivery of original issue 10-year US Treasury notes with remaining terms to maturity at delivery of at least 9 years 5 months and not more than 10 years.

[4] The Treasury bond of 4¼%, February 2036 stays as the CTD for all regular bond futures contracts over the period (with the 5% of May 2037 as the only close contender).

[5] The dry spell refers to the 1994–1999 period where the only deliverable option is the 11¼%, February 2015 bond, which had been the shortest duration bond for the period. It has not changed for the entire duration of 5 years.

[6] Ben-Abdallah and Breton (2017) specifically cite CME's concern about this liquidity issue in 2013. Details are provided in the Appendix.

[7] Although surprising, empirical evidence suggests that low interest rates in recent years fit the normality assumption better. See, for example, Grasselli and Lipton (2018). Also see a federal reserve bank report on October 17, 2016 by Stanley Fischer: "Why Are Interest Rates So Low? Causes and Implications" on https://www.federalreserve.gov/newsevents/speech/fischer20161017a.htm.

THE COST-OF-CARRY MODEL

The cost-of-carry (COC) model is a common industry practice to roughly estimate what the futures price needs to be. It decides the cheapest-to-deliver (CTD) bond today and uses it for delivery at the settlement date. It is an upper bound of the "correct" futures price in that the possibility for the CTD bond to switch between now and the settlement date is the option value (known as the quality option, see Hemler 1990) that needs to be deducted from the COC price.[8]

The COC model maximizes the delivery profit by using today's CTD bond. Formally, the delivery profit is calculated as:

$$\begin{aligned}&\max_i \left\{\Phi(t)q_i + a_i(T_f) - \tfrac{1}{P(t,T_f)}[Q_i(t) + a_i(t)]\right\}\\ &\quad = \max_i \left\{\Phi(t)q_i - \left(\tfrac{1}{P(t,T_f)}[Q_i(t) + a_i(t)] - a_i(T_f)\right)\right\}\end{aligned} \tag{1}$$

where T_f is the settlement date of the futures contract, $\Phi(t)$ is the market futures price today, $a_i(t)$ is the current accrued interest of bond i, $Q_i(t)$ is the quoted price (a.k.a. clean price) of a bond that pays c_i as a coupon rate. Note that in our implementation, we ignore the timing option (i.e., the delivery flexibility in the delivery month) and assume the last day of the delivery month as the delivery date.[9]

The idea behind this COC model is that the short side of the futures contract can buy the CTD bond now and hold it until the settlement date for delivery. However, "correct" futures price $\Phi(t)$ should reflect the CTD bond at maturity. Hence in theory, the correct futures price must consider all the possibilities (i.e., states of economy) that result in different CTD bonds at the settlement date of the futures contract.

On the settlement date T_f, Carr (1988) shows that the futures price can be computed as follows:

$$\Phi(t) = \mathbb{E}_t\left[\min\left\{\frac{Q_i(T_f)}{q_i}\right\}\right] \tag{2}$$

where $\mathbb{E}_t$ is the risk-neutral expectation taken at the current time t, $Q_i(T_f)$ is the quoted (clean) price of the ith eligible bond for delivery and q_i is the corresponding conversion factor.[10] Note that $\Phi(t)$ is the quoted (clean) futures price today.

A SHORT REVIEW AND IMPLEMENTATION OF THE HO-LEE MODEL

To evaluate the quality option properly, we need an interest rate model to predict the interest rates at the delivery date so that the CTD bond can be determined in each prediction. Early valuation of the quality option uses Margrabe's (1978) exchange option formula but the pricing formula becomes intractable as the number of deliverable bonds increases.[11] Carr (1988) was the first to use a term structure model

[8] A formal no-arbitrage proof of this upper bond can be found in Chen and Yeh (2012).

[9] An analysis of the timing options in Treasury bond futures can be found in Chen and Yeh (2012).

[10] The discount rate used for conversion factors was 6%. That is, Treasury futures equalize the deliverable bonds by pricing them to a 6% yield. Thus, when yields are below 6%, the cheapest-to-deliver bond will have a short duration; when they are above 6%, a long-duration bond will be cheapest to deliver. See https://www.thestreet.com/investing/fixed-income/what-makes-a-bond-cheapest-to-deliver-against-the-futures-contract-769619.

[11] See the seminal work by Hemler (1990).

to price the quality option.[12] Carr and Chen (1997) test the extended version of the Carr model. Chen and Yeh (2012) derive an upper bound of other delivery options.

There are numerous choices of an interest rate model. Of the factor models, the typical choices are the Cox-Ingersoll-Ross model by Chen and Yeh (2012) and the Hull-White model by Lin, Chen and Chou (1999) and Kiryazov (2015).[13] On the arbitrage-free models, numerous authors use the Heath, Jarrow and Morton (1992) model (e.g., Ritchken and Sankarasubramanian 1992 and Nunes and Oliveira 2007).

We use the arbitrage-free Ho-Lee model for our empirical work in that factor models perform poorly in a volatile interest environment. Since arbitrage-free models take the current yield curve as given, they perform better than the factor models. Secondly, the normality assumption adopted by the Ho-Lee model is more suitable in recent years when interest rates are low. Lastly, the Ho-Lee model is a special case of the Heath-Jarrow-Morton model with constant volatility. Lacking data on vanilla interest rate options to calibrate to the volatility surface required by the Heath-Jarrow-Morton model, the Ho-Lee model is a better choice.

In this section, we briefly review the widely known Ho-Lee model (1986) and how we implement it in the context of our empirical work. In particular, we modify the standard Ho-Lee model which is usually explained in equal time intervals to variable time intervals in that our deliverable bonds have various coupon payments and maturity dates.

Forward Rates and Forward Prices

Define a zero-coupon bond price (or the present value at time t of \$1 paid at time T, a.k.a. discount factor) as $P(t,T)$ and its yield (a.k.a. spot rate) as:

$$y(t,T) = -\frac{1}{T-t}\ln P(t,T) \tag{3}$$

Then, a discrete forward price for $t < T < s$ is defined as:

$$\Psi(t,T,s) = \frac{P(t,s)}{P(t,T)} \tag{4}$$

which is ratio of two zero-coupon bond prices and hence the forward rate is:

$$\begin{aligned} f(t,T,s) &= -\frac{1}{s-T}\ln \Psi(t,T,s) \\ &= \frac{(s-t)y(t,s)-(T-t)y(t,T_i)}{s-T} \end{aligned} \tag{5}$$

For an n-period, standard (i.e., equal time intervals) Ho-Lee model is described as follows:

$$\begin{aligned} P(i+1,i+\ell,j) &= \Psi(i,i+\ell,j)d(\ell-1) = \frac{P(i,i+\ell,j)}{P(i,i+1,j)}d(\ell-1) \\ P(i+1,i+\ell,j+1) &= \Psi(i,i+\ell,j)u(\ell-1) = \frac{P(i,i+\ell,j)}{P(i,i+1,j)}u(\ell-1) \end{aligned} \tag{6}$$

[12] An early discussion of the valuation of the quality option appears in Cox, Ingersoll, and Ross (1981) in which they state that their valuation can be applied to futures with the quality option when the single spot bond price is replaced with the minimum from the deliverable set.

[13] A recent review can be found in Kiryazov (2015).

where i is current time, ℓ is steps into the future, and j is the state and finally

$$u(k) = \frac{1}{p + (1-p)\delta^{\ell}}$$
$$d(k) = \frac{\delta^{\ell}}{p + (1-p)\delta^{\ell}} \tag{7}$$

are the up and down perturbation functions in which p is the up risk-neutral probability and δ is known as the volatility parameter.[14] At the current time, the initial yield curve is represented by a series of equally spaced zero-coupon bond prices $P(0,i,0)$ where $i = 1\ldots n$. Details can be found in Chen (2013).

In our implementation, the time intervals are not equal. For example, a bond with a maturity time 15 years, 3 months (or 7 months), and 5 days from now has the next coupon date in 3 months (or 1 month) and 5 days; and the following coupon date 9 months (or 7 months) and 5 days from now. As a result, it is not feasible to use equally spaced Ho-Lee model. Hence, we generalize the Ho-Lee model in the following manner:

$$\begin{aligned} P(t_1, T_i, j+1) &= \Psi(t, T_i, j)u(T_i - t_1) \\ P(t_1, T_i, j) &= \Psi(t, T_i, j)d(T_i - t_1) \end{aligned} \tag{8}$$

where $t_1 \ldots t_m$ are the time intervals of the Ho-Lee model and $T_1^{(k)} \cdots T_{n(k)}^{(k)}$ are the coupon dates of bond k that pays a coupon rate c_k.

While there is no need for an equal partition in the Ho-Lee model, it is more convenient to define $t_i = \frac{i}{m} T_f$ so that the last period is when the futures contract settles. A bond k pays $n(k)$ coupons of c_k (adjusted by proper frequency, usually twice a year) at $T_1^{(k)} \cdots T_{n(k)}^{(k)}$ coupon dates.

As a result, our Ho-Lee model does not have equal time intervals. Between now t and the settlement date T_f (which is equal to t_m), the partition is equal. Afterwards, the partition depends on when the next coupon date is, and then it will be semi-annual in order to match the coupon dates. It can be shown that the Ho-Lee model of unequal intervals shall still recombine just like the standard Ho-Lee model with equal intervals. The risk-neutral probability is then applied to price any American-style derivative backwards. A four-period Ho-Lee model is explained in details in the Appendix.

A key input to the Ho-Lee model is the current yield curve. In this article, we let the yield to maturity to be a continuous function of the time to maturity. We use CMT (constant maturity Treasury rates) to construct the yield curve and we assume the function to be piece-wise flat.[15]

From the Appendix, we know that we can derive the yield curve at the settlement date T_f directly from today's yield curve and the two perturbation functions of Equation (7).

$$\begin{aligned} y_j(T_f, T_i^{(k)})(T_i^{(k)} - T_f) = {} & y(t, T_i^{(k)})(T_i^{(k)} - t) - y(t, T_f)(T_f - t) \\ & + \sum_{i=1}^{j} (\ln u(T_f - t_i) - \ln u(T_i^{(k)} - t_i)) \\ & + \sum_{i=j+1}^{m} (\ln d(T_f - t_i) - \ln d(T_i^{(k)} - t_i)) \end{aligned} \tag{9}$$

[14] Note that $0 \le \delta \le 1$ and higher is d, lower is volatility. When $\delta = 1$, $u(\ell) = d(\ell) = 1$ and there is no volatility. When $\delta = 0$, $d(\ell) = 0$ reaches maximum volatility.

[15] One can use linear interpolation or quadratic/cubic spline. In this article, the simplest piece-wise flat function serves the purpose well. The other more complex methods do not generate qualitatively different results.

where j represents the state of economy at the settlement date T_f. Equation (8) allows us to discount any coupon from any bond and compute the bond price at each state

$$\Pi_k(T_f;\underline{T}^{(k)},c_k) = \frac{c_k}{2}\sum_{i=1}^{n(k)} P(T_f,T_i^{(k)}) + P(T_f,T_{n(k)}^{(k)}) \tag{10}$$

which is the dirty price of the bond. The quoted bond price (clean) is:

$$Q_{k,j} = \Pi_k(T_f;\underline{T}^{(k)},c_k) - a_k(T_f) \tag{11}$$

From Equation (2), we know that the payoff of the futures contract at the settlement time T_f is the CTD bond quoted price adjusted by its conversion factor. In the Ho-Lee model, we re-labeled it as: at each state can be calculated as:

$$\min\left\{\frac{Q_i(T_f)}{q_i}\right\} = \min\left\{\frac{Q_{k,j}}{q_k}\right\} \tag{12}$$

and the current futures price can be computed via the Ho-Lee lattice as follows:

$$\begin{aligned}\Phi(t) &= \mathbb{E}_t\left[\min\left\{\frac{Q_i(T_f)}{q_i}\right\}\right]\\ &= \Pi_{j=1}^{m} C_j^m p^j (1-p)^{m-j} \min\left\{\frac{Q_{k,j}}{q_k}\right\}\end{aligned} \tag{13}$$

where C_j^m is a combination function. We note that the bonds that are eligible for delivery must be 15 years or longer at the settlement date for regular futures contracts and 25 years or longer for the Ultra futures contracts.

Estimation of the Ho-Lee Model

There are two parameters in the Ho-Lee model: p the risk-neutral probability and δ the volatility parameter. We follow the regression methodology proposed by Chen and Yang (1995) to estimate these two parameters.

Chen and Yang (1995) argue that under the approximation that today's short rate (e.g., 3- and 6-month) do not change dramatically over time, then:

$$D_{t+1}(\tau) = \begin{cases} D_t(\tau)\dfrac{\delta^\tau}{p+(1-p)\delta^\tau} & \text{if rate rises}\\ D_t(\tau)\dfrac{1}{p+(1-p)\delta^\tau} & \text{if rates falls}\end{cases} \tag{14}$$

where $D_t(\tau) = e^{-y_t(\tau)\tau}$ is the discount factor at the current yield $y_t(\tau)$ with a time to maturity τ. Then,

$$\begin{aligned}\ln D_{t+1}(\tau) - \ln D_t(\tau) &= -\ln\left[p+(1-p)\delta^\tau\right] + \tau \ln \delta \mathbb{I}_{t+1} + u_{t+1}\\ y_{t+1}(\tau) - y_t(\tau) &= \ln\left[p+(1-p)\delta^\tau\right] - \tau \ln \delta \mathbb{I}_{t+1} + v_{t+1}\end{aligned} \tag{15}$$

where $\mathbb{I}_{t+1}$ is an indicate function equal 1 when rate rises at $t + 1$ and 0 otherwise. This methodology suits particularly well for the CMT (constant maturity Treasury) rates whose maturities are rolling constant at τ. For a 3-month CMT $\tau = 0.25$.

A simple regression can be performed and

$$\delta = \exp\left(-\frac{b}{\tau}\right) \tag{16}$$

and

$$p = \frac{e^{a} - e^{-b}}{1 - e^{-b}} \tag{17}$$

where *a* and *b* are regression constant and slope respectively.

DATA

We collect prices of two futures contracts: regular and Ultra futures contracts from March 25, 2011 (Friday) until December 18, 2020 (Friday). These are weekly data every Friday, a total of 496 observations.

The days to maturity of these futures contracts range from 7 days to 70 days. Contracts that have more than 70 days to maturity or less than 7 days to maturity are excluded due to liquidity concerns. When a contract is less than 7 days, we then drop that futures contract and roll onto the next futures contract in line.

The regular and Ultra futures prices are plotted in Exhibit 1. The futures prices of the Ultra contract are higher than those of the regular futures contract. Also, the differences are larger more recently (toward the end of the sample period) than at the beginning. When the Ultra futures contract was first introduced in 2011, it was priced very similarly to the regular futures contract.

The summary statistics of the two types of futures prices are reported in Exhibit 2. Exhibit 2 contains averages of the two types of futures prices by their settlement date in that different settlement dates represent different contracts. The average Ultra futures price of the entire sample is \$165.73 and the average regular futures price is \$150.27. The futures prices trended higher in our sample period. Besides the dollar

EXHIBIT 1
Futures Prices – Regular and Ultra

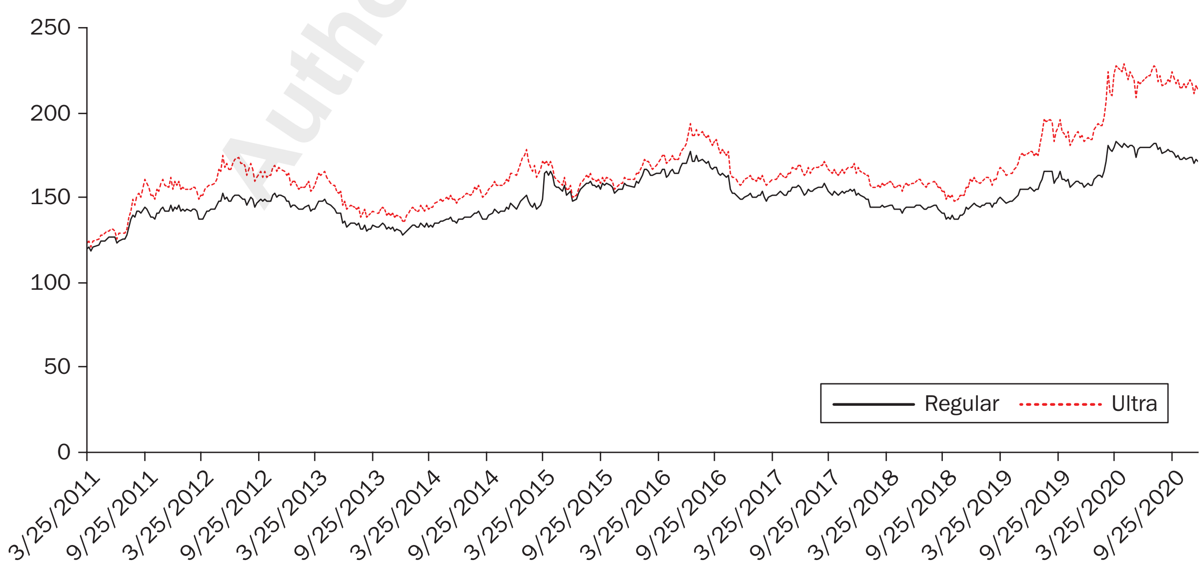


EXHIBIT 2

Futures Prices – Regular and Ultra

Panel A: 2011/03~2015/12				Panel B: 2016/03~2020/12			
Settlement	Regular	Ultra	Ultra/Reg	Settlement	Regular	Ultra	Ultra/Reg
2011/06	123.5443	127.3152	1.0305	2016/03	162.5653	167.6335	1.0312
2011/09	132.8102	138.9855	1.0465	2016/06	165.9297	174.2604	1.0502
2011/12	141.5095	156.2117	1.1039	2016/09	171.5987	186.5457	1.0871
2012/03	142.5144	155.6972	1.0925	2016/12	158.9494	170.6924	1.0739
2012/06	144.5265	161.0793	1.1145	2017/03	150.7475	160.4013	1.0640
2012/09	148.8582	168.1322	1.1295	2017/06	153.3438	163.7864	1.0681
2012/12	149.4735	165.0505	1.1042	2017/09	155.0649	167.3028	1.0789
2013/03	144.7164	157.9663	1.0916	2017/12	153.3798	166.2162	1.0837
2013/06	144.8931	159.2083	1.0988	2018/03	147.3317	160.4038	1.0887
2013/09	133.3702	142.9424	1.0718	2018/06	143.9583	157.2709	1.0925
2013/12	132.6155	141.0264	1.0634	2018/09	144.3102	158.4062	1.0977
2014/03	132.0745	141.5048	1.0714	2018/12	139.7212	151.8653	1.0869
2014/06	135.5262	147.737	1.0901	2019/03	145.8749	160.3341	1.0991
2014/09	138.5576	152.0745	1.0976	2019/06	150.1538	168.4713	1.1220
2014/12	142.2019	158.423	1.1141	2019/09	159.1298	183.8652	1.1554
2015/03	146.8725	169.4881	1.1540	2019/12	160.0529	187.8919	1.1739
2015/06	158.3798	163.0866	1.0297	2020/03	164.8616	195.6919	1.1870
2015/09	155.3073	158.8542	1.0228	2020/06	179.7552	221.5417	1.2325
2015/12	156.1972	159.2331	1.0194	2020/09	179.2006	221.2449	1.2346
				2020/12	173.4567	216.8439	1.2501
Average	142.3542	153.9881		Average	157.8633	177.0062	
Grand Average	150.2651	165.7291					

NOTE: We split the sample in two sub-periods because in the Ho-Lee model, the first half of the data are used for in-sample estimation.

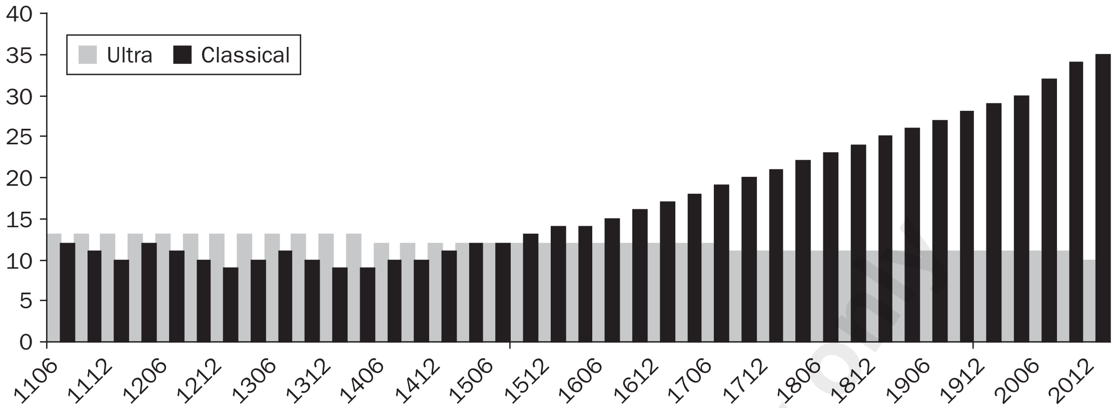
differences, it can be seen from Exhibit 2 that the percentage differences are small at the beginning (10% or so) and large (over 20%) toward the end.

We split the sample into two halves for the estimation of the Ho-Lee model (as explained later). In the Ho-Lee model, the first half is an in-sample test and the second half is an out-of-sample test. If we split the sample in two halves, then the two types of futures prices have averages in the first half of \$142.35 and \$153.99 for regular and Ultra, respectively and in the second half of \$157.86 and \$177.01 for regular and Ultra, respectively.

Besides futures prices, in order to evaluate the futures contracts, we collected the prices for the deliverable bonds. For regular futures contracts, the eligible bonds for delivery are 15 years ~ 25 years to maturity or the first call date. For Ultra futures, the bonds must be at least 25 years to maturity or the first call date. Exhibit 3 plots the number of the deliverable bonds for each day.

The number of deliverable bonds for the Ultra futures is between 13 and 10 (and decreases over time), while for the regular futures it ranges between 9 and 35 (and increases over time). The disappearance of the 30-year Treasury bonds between February 2031 and February 2036 (the dry spell) which could impact the regular futures contracts between December 2015 and December 2020 did not cause any visible reduction of the deliverable bonds. On the contrary, the number of deliverable bonds for the regular futures contracts has increased in this period. More discussion will be provided later in the empirical section.

EXHIBIT 3
Number of Deliverable Bonds



NOTES: The number of deliverable bonds for the Ultra futures is between 13 and 10 and for the regular futures ranges between 9 and 35. The x-axis is labeled as yymm, where 1106 represents the June 2011 futures contract.

Lastly, to evaluate the quality option, we collected interest rate data. The interest rate data we use are the CMT (constant maturity Treasury) rates from the St. Louis Fed (fred2). These are daily 3M, 6M, 1Y, 2Y, 3Y, 5Y, 7Y, 10Y, 20Y, and 30Y. The results are plotted in Exhibit 4.

The short-term interest rates (e.g., 3M and 6M) do not present much volatility for the first half of the sample and yet become extremely volatile in the second half of the period. In addition, in the period between the end of 2017 and the end of 2019, these short rates are high (and hence the yield curve becomes flat). However, the long-term interest rates (e.g., 20Y and 30Y) present more regular patterns although trend downwards from 4% or so to less than 2%.

Understandably, interest rate volatility has an enormous impact on the value of the quality option. We see both the market futures prices (which we assume the market wisdom already prices in the volatility) and the model prices (which specifically includes the volatility) of the Ho-Lee model deviate from the COC prices in the second half of the sample.

EMPIRICAL RESULTS

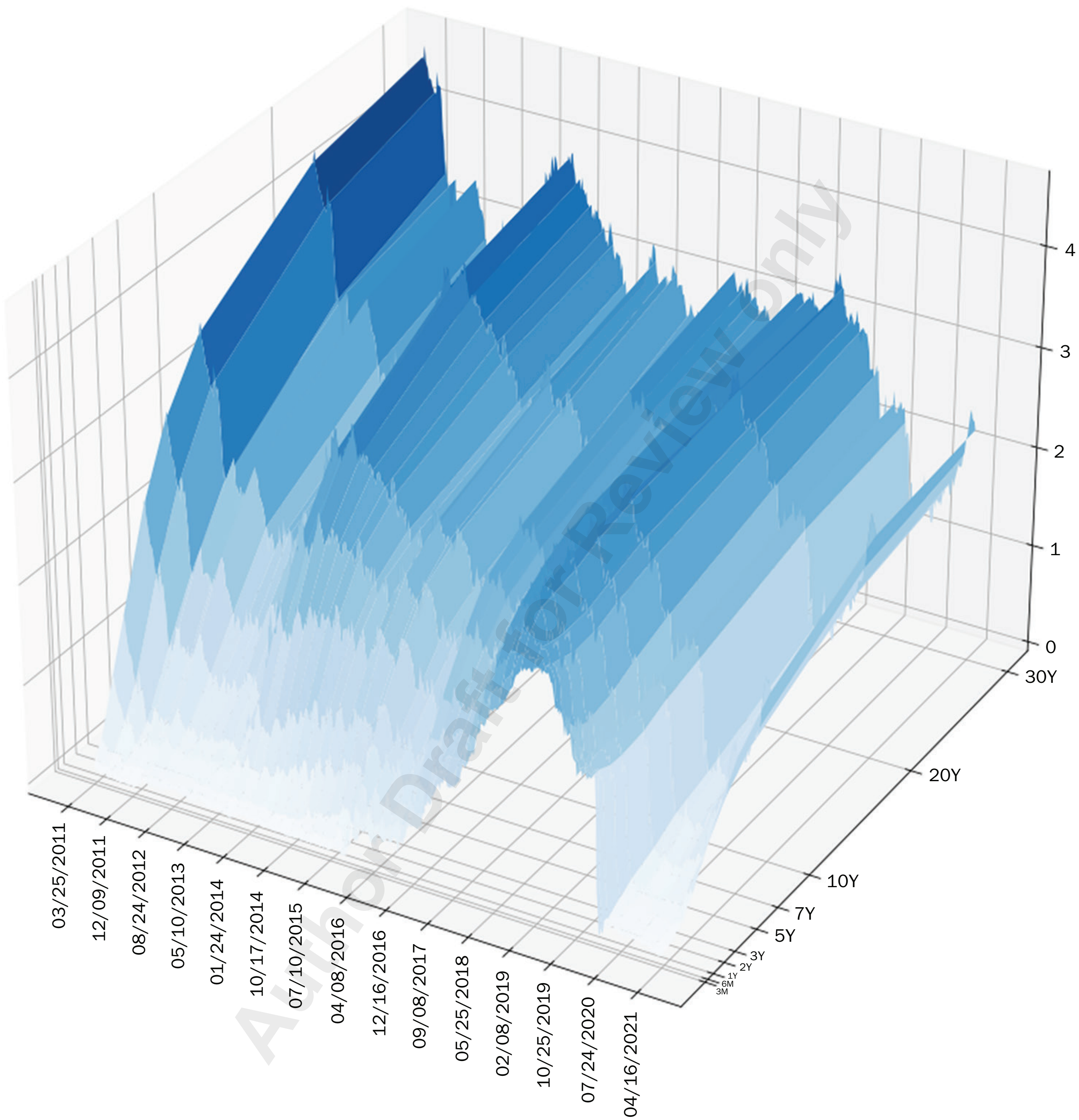
Estimating the Ho-Lee Model

To properly evaluate the quality option embedded in the futures price, we must employ an interest rate model of term structure. In this paper, the no-arbitrage Ho-Lee model[16] is used because: (1) at low interest rates, normal models are better; and (2) the current yield curve is used as an input.

As mentioned in the previous section, there are two parameters: (up) risk-neutral probability and volatility of the Ho-Lee model. These parameters can be estimated via a regression as described by Chen and Yang (1995). The estimates are reported in Exhibit 5.

[16] "No-arbitrage" here refers to those models that take the current yield curve as given. In addition to the Ho-Lee model, popular choices include the Black-Derman-Toy model (1990), Hull-White model (1990) and the Heath-Jarrow-Morton model (1992).

EXHIBIT 4
US Treasury (CMT) Yield Curve



We use daily rates for the estimation in that a sharp change in the short rate is unlikely and hence the Chen-Yang assumption is satisfied. We estimate the parameters for the first half (March 25, 2011 until December 31, 2015) of the sample and for the whole sample in order to see if there is a regime shift. As noted earlier in Exhibit 4, there is a sharp change in the short rates in the second half of the sample period. The former is reported on the left side of the table and the latter on the right side.

EXHIBIT 5

Parameter Estimates of the Ho-Lee Model

Panel A: Parameter Estimates

	Until 12/31/2015		Whole Sample	
	Coef	t	Coef	t
a	–0.0048	–17.4929	–0.0079	–22.6859
b	0.0183	34.4054	0.0250	40.3161
delta	0.9294		0.9049	
p	0.7366		0.6813	

The data used here are daily 3-month CMT rates from March 15, 2011 until December 14, 2021.

Panel B: Risk-Neutral Probability Distribution at the Settlement Date

State	0	1	2	3	4
Probability	0.294324	0.421081	0.225911	0.053867	0.004817

These risk-neutral probabilities of the binomial distribution are calculated as: $p_j = C_j^4 p^j (1-p)^{4-j}$ using (18).

The estimates of the first half of the sample are $p = 0.7365$ for the risk-neutral probability and $\delta = 0.9294$ for the volatility parameter; they are compared to those of the whole sample where $p = 0.6813$ and $\delta = 0.9149$. The volatility parameter δ for the whole sample is smaller indicating a larger volatility of the yield curve (note that the value of δ is reversely related to the volatility of the yield curve, see the previous discussion for details).

Besides δ, it is also important to understand that the shape (i.e., slope and curvature) of the yield curve also embeds information about the expectation of future interest rate volatility. A steeper and more curved yield curve reflects high future volatility and vice versa. Hence, in those arbitrage-free interest rate models that take the initial yield curve as an input, δ is not the only source of volatility. To be shown later, δ is rather stable throughout the whole sample period and there is little difference between δ for the 3M CMT and δ for the 6M CMT.

The Ho-Lee model can illustrate the yield curve under a different economic state at the settlement date and then find the cheapest-to-deliver (CTD) bond at the settlement date. A 4-period Ho-Lee model is given in the Appendix. The initial yield curve (i.e., the CMT rates) is translated into a set of zero-coupon bond prices (or discount factors) as $P(0, t_i)$.

For the sake of convenience and easy notation, we let $t = 0$. At the last period t_4, the futures contract settles. In other words, $t_4 = T_f$ which is the settlement date. As a result, for convenience, we set each time interval as $t_i = \frac{i}{4} T_f$.

On the left, the current time is labeled from 0 to t_4. At t_1, there are two boxes, the left for the down movement of the bond prices and the right for the up movement. Consequently, each bond price in the left box is multiplied by the corresponding down perturbation function $d(t_1 - t)$ as described in Equation (7). Note that a subscript is used (d_{t_1-t}) to conserve space. Similarly, an up perturbation function u_{t_1-t} is used for up movements.

The same process is applied at each time t_i until the settlement date T_f (which is t_4). Note that the maturity of any bond k is $T_{n(k)}^{(k)}$ and the bond has $n(k)$ coupon payments and hence $T_{n(k)}^{(k)}$ represents the maturity time of the bond. As a result, our version of the Ho-Lee model has unequal time intervals. Between now t and the settlement date T_f, the partition is equal. Afterwards, the partition depends on when the next coupon date is, and then it will be semi-annual in order to match the coupon dates.

The risk-neutral probability is then applied to price any American-style derivative by computing backwards discounted expected payoffs (known as backward induction). The futures contract is a European contract and hence backward induction is not necessary. As we can see in the Appendix, recursive substitutions give rise to a very simple answer to the bond price at the settlement date of any maturity.

As a result, we can directly compute the expected value of the payoffs at the settlement date using the binomial probability distribution:

$$\pi_j = C_j^4 p^j (1-p)^{4-j} \tag{18}$$

where $j = 0, \ldots, 4$ to be used in Equation (13): The values of the probabilities are given in Exhibit 5.

As described in Equation (13), the futures price is merely a risk-neutral expectation of the CTD bond prices (after it is adjusted by the corresponding conversion factors) at the settlement date:

$$\Phi(t) = \sum_{j=0}^{4} \pi_j \min_k \left\{ \frac{Q_{k,j}}{q_k} \right\} \tag{19}$$

where $k = 1, \ldots, K$ indicates a bond and $j = 0, \ldots, 4$ indicates a state. In the remainder of this section, we use the Ho-Lee futures prices to compute the quality and various timing option values.

Cost-of-Carry Futures Price and the Quality Option Value

The cost-of-carry (COC) method is widely used in practice to crudely estimate the futures price. Chen and Yeh (2012) prove rigorously that the COC futures price is an upper bond to the true futures price. The COC method assumes the current cheapest-to-deliver (CTD) bond to be the delivery bond at the settlement date. Hence, it ignores the option to switch (which has a non-negative value) to a new CTD bond between current date and the settlement date.

The Ho-Lee model can identify the CTD bond in a specific economic state at the settlement date, then the difference between the COC price (that captures the CTD bond at the current date) and the Ho-Lee model price can properly estimate the value of the quality option. The value of the quality option is reported in Exhibit 6 where the COC future prices are computed using Equation (2).

Again, we report the quality option value by contract. The average value of the quality option is \$31.76 for the regular futures and is \$47.94 for the Ultra futures. The quality option values of the first half sub-sample (in-sample) are much smaller than the second half (out-of-sample) by a 1 to 4 ratio. The values are \$7.25 for the regular futures and \$8.46 for the Ultra in the first half sub-sample and \$55.30 and \$85.84 for the regular and Ultra futures respectively in the second half.

There are two possible reasons to explain the large differences between the two sub-samples. The first reason could be larger mispricing in the out-of-sample sub-period. The second reason could be due to larger volatility in the out-of-sample sub-period. The time series plot of COC futures prices and the Ho-Lee model futures prices are presented alongside of the market prices in Exhibit 7.

We can see three time series in Exhibit 7. The COC futures prices are rising for both regular and Ultra futures contracts toward the end of 2017. They continue to rise till mid-2020 and reach their peaks at beginning of 2019 (the peak for the Ultra is a little later than the regular futures contract). During this period, the market futures prices of both regular and Ultra futures contracts fall, causing an enormous gap between the two. The model prices computed by the Ho-Lee model are the lowest of all three.

EXHIBIT 6
Quality Option Value (COC Analysis)

Panel A: In-sample			Panel B: Out-of-sample		
	Regular	Ultra		Regular	Ultra
2011/06	7.2801	3.4191	2016/03	18.5489	30.0013
2011/09	5.9323	1.3384	2016/06	32.2552	30.8853
2011/12	6.4276	2.9725	2016/09	37.9016	43.0768
2012/03	7.2290	6.1443	2016/12	33.2456	39.0889
2012/06	7.5433	6.3262	2017/03	33.0967	40.5384
2012/09	7.9427	6.6998	2017/06	47.3004	62.7195
2012/12	7.1268	7.7072	2017/09	51.9731	74.2027
2013/03	7.0163	7.2374	2017/12	56.3872	76.8248
2013/06	6.7778	5.1675	2018/03	59.9667	93.0701
2013/09	6.3081	5.9077	2018/06	69.6577	106.0279
2013/12	6.7304	6.0218	2018/09	72.8338	118.0495
2014/03	7.5164	5.7773	2018/12	79.3547	130.6035
2014/06	8.0849	8.2486	2019/03	85.7530	150.8058
2014/09	7.1176	8.4419	2019/06	90.6384	158.8287
2014/12	7.4913	11.9503	2019/09	83.9725	154.4266
2015/03	6.4902	19.6323	2019/12	71.8738	131.1327
2015/06	6.7876	17.2160	2020/03	60.9524	110.0408
2015/09	6.9967	14.0175	2020/06	39.8433	59.0829
2015/12	10.8919	16.2085	2020/09	38.5056	46.7038
			2020/12	34.7761	46.3060
Average	7.2463	8.4559	Average	55.2985	85.8387
		Average	31.7568	47.9403	

Violation of Arbitrage

Given that the COC futures price is an upper bond, it cannot fall below the market futures price, or an arbitrage can take place and profits can be made. We find 120 violations (out of 496 days in the sample period) in the regular futures contract. The summary is reported in Exhibit 8.

The average amount of arbitrage is \$4.40 with a standard deviation of \$4.14. The median is \$1.91; the minimum is \$0.02 and the maximum is \$13.51. The most violations happen in 2013 (19 violations), 2014 (18 violations), and 2020 (17 violations). The least violations happen in 2017 (3 violations) and 2016 (5 violations). When a violation occurs, it suggests an arbitrage opportunity. However, it may not really exist due to market frictions such as liquidity, bid-offer gaps, and transaction costs. In the Panel (b) of Exhibit 8, we list the violations that exceed \$10 at which we believe the arbitrage profit exists. Except for 2016– 2019, there are opportunities in other years with 2015 having the most opportunities (7 days).

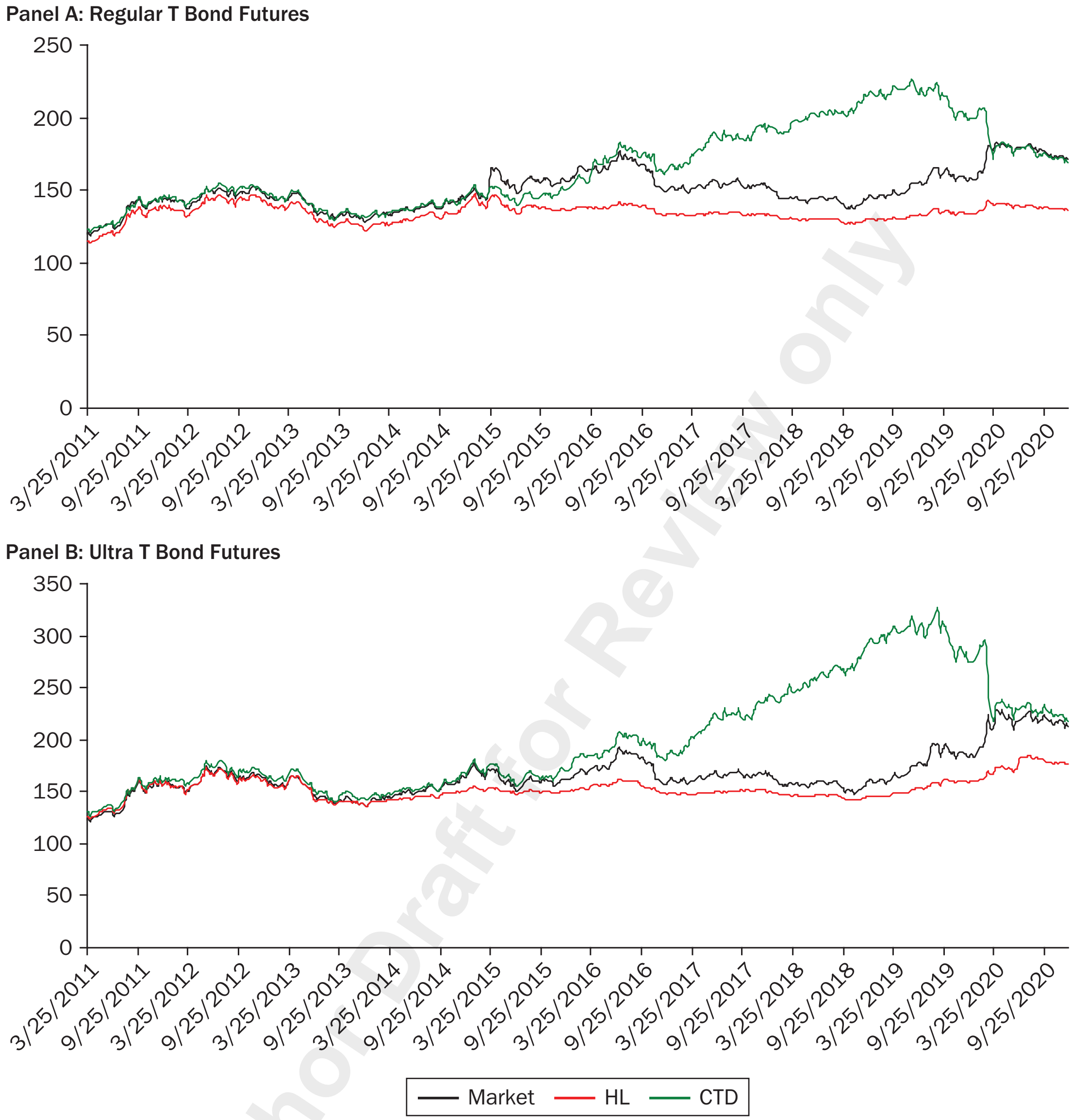
However, we do not find any violation in the Ultra contract.

The Timing Option Value

The timing options refer to the various flexibilities in timing of the delivery (hence options) the short side of the futures contract owns. As a result, the short side can choose to deliver when the interest rates are high (and the bond prices are low). The most valuable timing option in Treasury bond futures is the end-of-month timing

EXHIBIT 7

Comparison of Futures Prices



option which refers to the last 7 business days of the delivery month when the futures market is closed.[17]

While the quality option value can be measured by the difference between the COC price and the model price, the value of timing option (total value of all three timing options) is hard to gauge. This is because the timing options are American-style and furthermore require recursive substitutions within the lattice. If the market is efficient, then the market futures price represents the true futures price. In this case,

[17] The general evaluation of the timing options can be found in the seminal paper by Boyle (1989). The timing option (end-of-month) regarding Treasury bond futures can be found in Chen, Ju, and Yeh (2009). Given that the futures market is closed in this period (and hence the futures price is fixed at the beginning of the period), Chen, Ju, and Yeh (2009) show that a recursive valuation of the lattice is necessary. Due to the fact that recursive valuation is computationally expensive, Chen and Yeh (2012) derive an upper bound of value of various timing options.

EXHIBIT 8
Violation of Arbitrage

Panel A: Violation Summary

Year	# of Violations	Average Amount
2011	9	–4.6683
2012	9	–8.0444
2013	19	–5.2115
2014	18	–4.6150
2015	14	–7.3328
2016	5	–2.3077
2017	3	–2.6914
2018	12	–1.4761
2019	14	–1.5045
2020	17	–4.1265
Total #	120	
Mean		–4.3975
Median		–1.9165
Std.dev.		4.1448
Min		–13.5081
Max		–0.0228

Panel B: Largest Violations

Date	Arb Amt
6/12/2020	–13.5081
10/23/2015	–13.1893
12/30/2011	–12.9835
7/10/2015	–12.9204
6/8/2012	–12.7285
5/31/2013	–12.2543
12/18/2015	–12.1097
5/18/2012	–12.0612
4/10/2015	–12.0223
11/29/2013	–11.1897
3/13/2020	–10.9985
12/28/2012	–10.7469
5/22/2015	–10.4503
10/9/2015	–10.4496
5/30/2014	–10.4414
2/27/2015	–10.2086
7/11/2014	–10.124
11/16/2012	–10.1166

the value of the timing options can be measured by the difference between the model price and the actual market price. The result is reported in Exhibit 9.

To our surprise, the values of the timing option are mostly negative. We only find positive value for three contracts: 2011/06, 2011/09, and 2011/12 of the Ultra contract and the remaining contracts all have negative timing option values. For the regular futures, the values are unanimously negative. Not only are the values of timing option negative, they are immense. Toward the end of the sample period, for example 2020/06, 2020/09, and 2020/12 contracts, the negative values are as high as nearly $40 for both regular and Ultra contracts.

Similar to the quality option result, the magnitudes of the violation are higher in the out-of-sample sub-period than the in-sample period. Unfortunately, unlike the arbitrage violations in the COC case, these violations cannot be arbitraged. This is because the valuation of the time options is a complex process. Unlike the usual options where delta hedging generates a perfectly self-financing trading strategy (at least in theory), the timing options are different. The value of the timing options is built upon the actual delivery date and time in the delivery month which then depends upon the opening futures price on each day in the delivery month; and yet this the futures price that needs to be evaluated. Secondly and typically, the Ho-Lee model can be mis-specified, either on the distribution assumption or the parameterization of the model; or both. This results in incorrect prices that are erroneously lower than the market futures prices.[18]

Analysis of the Deliverable Bonds

The key determining factor to the futures price is to find the CTD bond in the deliverable bond basket at the settlement date of the futures contract. At a different state of economy, the CTD bond may be different. Hence, a crucial step in valuing the futures contract is to identify all the deliverable bonds.

We rank and label the delivery bonds by their maturities. We then record the current CTD bond (for the COC calculation) and the CTD bond in each Ho-Lee state. Because we use a 4-period Ho-Lee model, there are 5 economic states (labeled 0~4). These CTD bonds are plotted in Exhibit 10.

As discussed earlier, if the current CTD bond is not expected to switch before the settlement date, then the COC price should be very close to the actual market price. Yet, in a volatile environment, such the CTD bond is expected to switch (as the yield curve at the settlement date will shift substantially). As a result, by inspecting if the CTD should switch, we can gauge roughly how high the quality option value (which measures the option value of switching) is.

Panel (a) of Exhibit 10 presents the deliverable bonds of the regular futures contracts. At the bottom (dashed line) is current CTD bond (for the COC calculation).

[18] To investigate whether there is an arbitrage opportunity it is simply a model error, we would need the data on actual deliveries. Without such data, we cannot test if the CTD bonds predicted by the Ho-Lee model are accurate.

EXHIBIT 9

Timing Option Value

Panel A: In-sample			Panel B: Out-of-sample		
	Regular	Ultra		Regular	Ultra
2011/06	–6.0362	2.1849	2016/03	–24.7245	–15.4420
2011/09	–5.7041	2.5795	2016/06	–27.7422	–17.5552
2011/12	–5.3607	0.8495	2016/09	–31.4010	–26.8670
2012/03	–5.8994	–0.8971	2016/12	–22.8678	–18.5984
2012/06	–5.6470	–0.2240	2017/03	–17.4070	–12.1432
2012/09	–5.1503	–1.6584	2017/06	–19.7930	–15.2370
2012/12	–4.7899	–2.0528	2017/09	–20.7026	–17.1470
2013/03	–5.7186	–2.0966	2017/12	–20.1875	–15.1144
2013/06	–6.2744	0.5029	2018/03	–15.7753	–12.0861
2013/09	–5.4889	–2.0177	2018/06	–14.0373	–11.4156
2013/12	–5.3469	–0.9416	2018/09	–14.0786	–11.5889
2014/03	–6.5182	–1.7505	2018/12	–11.8973	–9.0104
2014/06	–7.2050	–4.9515	2019/03	–15.9230	–14.6157
2014/09	–5.6826	–6.6877	2019/06	–19.0109	–18.7706
2014/12	–7.4832	–9.4655	2019/09	–24.5461	–28.6983
2015/03	–4.8911	–16.7684	2019/12	–25.3571	–27.4668
2015/06	–17.1308	–11.7979	2020/03	–28.2219	–33.1318
2015/09	–17.3758	–8.7641	2020/06	–39.8499	–49.1347
2015/12	–19.2424	–9.7767	2020/09	–40.6553	–38.5944
			2020/12	–36.2114	–38.9962
Average	–7.7095	–3.8992	Average	–23.4496	–21.5910
	Average	–15.7382	–12.8762		

As we can see, the bonds with the shortest maturities are the cheapest to deliver (these are bond #1 ~ bond #3). However, the CTD bonds in various states vary drastically. The most noticeable is state #4 (where the rates are the lowest) labeled by ×. At the beginning of the sample period, the CTD bond in the state is the same as the current CTD (bond #1 or bond #2 which are the shortest maturities). However, in the second half, the CTD bond shifts to a long-term bond (#15). In other states (e.g., state #3 labeled ×), bonds with mid maturities are the CTD bond.

From Panel (a) of Exhibit 10, it is quite apparent that the CTD bond will almost definitely switch (except for state #4 when rates are extremely low). As a result, we can conclude that the quality option has a non-trivial value. We note that state #4 has the lowest probability of reaching and hence, even though it is the state where the CTD bond does not switch, it has the least impact on the quality option price.

Panel (b) of Exhibit 10 depicts the CTD bonds for the Ultra futures contracts. We see quite a different picture. State #4 is no longer the only state where the CTD bond does not switch. We see that in state #0 (where rates are extremely high) occasionally the CTD bond will not switch. Note that the probability reaching state #0 is high and hence the quality option value should be impacted. Also, we observe a quite even distribution of the CTD bonds across maturities.

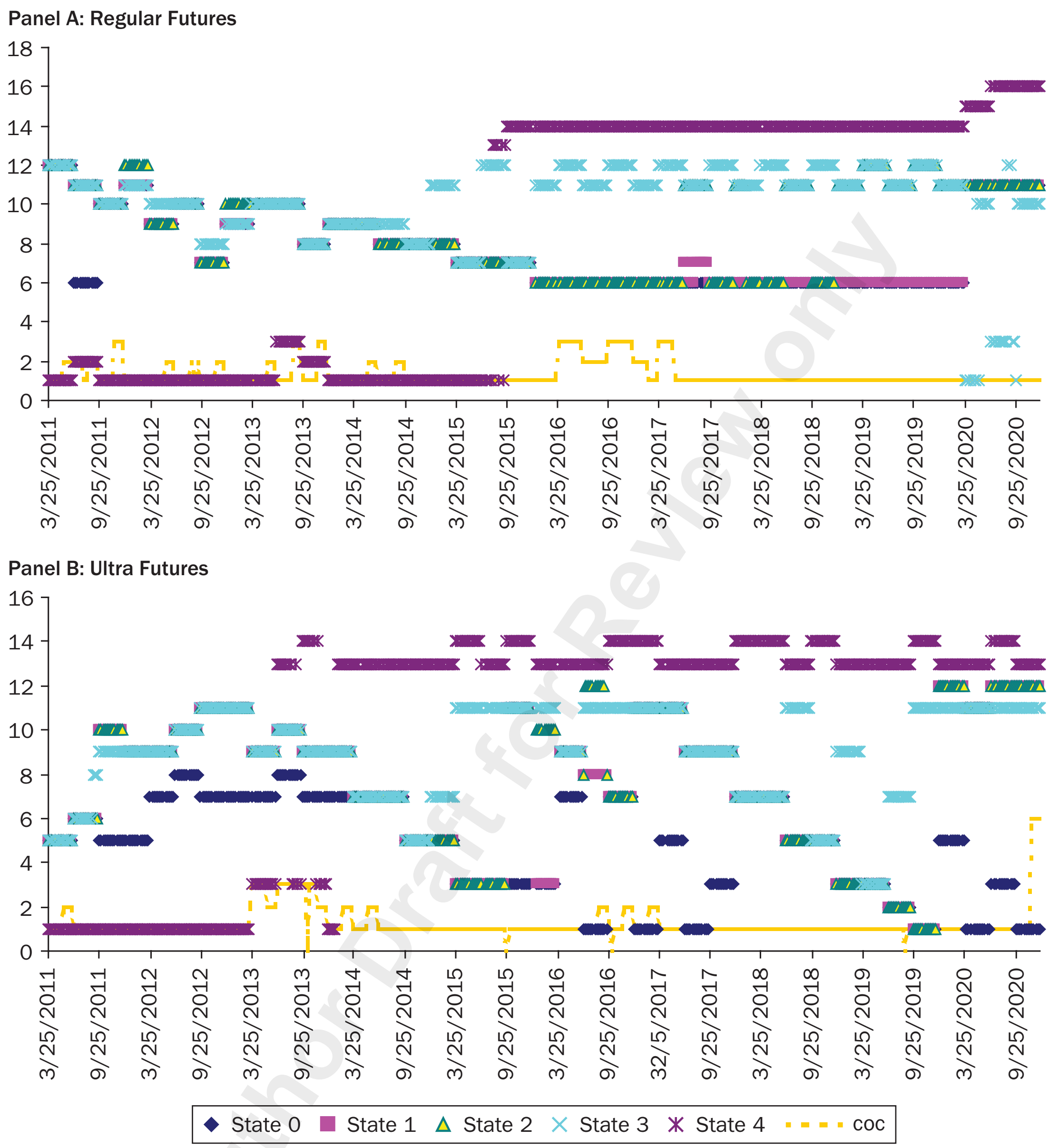
In conclusion, Exhibit 10 portraits a picture where the quality option should have a non-trivia value. As reported in Exhibit 6, the quality option value is indeed substantial – \$31.76 for the regular futures and \$47.94 for the Ultra futures.

Robustness Check

While the 3M CMT rates are a natural choice for the parameters of the Ho-Lee model, we are concerned about how this short rate can fairly evaluate long term bonds

EXHIBIT 10

Regular versus Ultra Deliverable Bonds



The probability of each state is

State	0	1	2	3	4
Probability	0.294324	0.421081	0.225911	0.053867	0.004817

used for the delivery of the futures contract. It would be biased if long rates behave very differently from the short rates. Furthermore, as Exhibit 4 shows, we observe a sharp increase in the short rates in the 2018~2019 period. This also raises an issue of parameter stability. We are concerned with whether the volatility parameter δ of the Ho-Lee model is affected by this large move in the short rates.

Hence, in this sub-section, we estimate the Ho-Lee model with different interest rates and using different time periods. The various estimation results are summarized in Exhibit 11.

EXHIBIT 11
Ho-Lee Parameters

Panel A: 6M Estimates

	Tenor = 6M Till 12/31/2015		Whole Sample	
	Coef	t	Coef	t
a	–0.00463	–16.3099	–0.00746	–22.6994
b	0.018866	33.8473	0.024069	40.72665
delta	0.962971		0.953003	
p	0.753014		0.68755	

Panel B: Estimates of other Tenors

Tenor	Delta	p
1Y	0.97425	–0.07612
2Y	0.97904	–0.25043
3Y	0.98325	–0.34692
5Y	0.98719	–0.40014
7Y	0.98983	–0.41883
10Y	0.99283	–0.42898
20Y	0.99630	–0.44084
30Y	0.99755	–0.45035

Panel C: Rolling 3M Estimates

	Delta	p
1512	0.9294	0.7366
1603	0.9263	0.7298
1606	0.9253	0.7294
1609	0.9216	0.7190
1612	0.9174	0.7157
1703	0.9141	0.7160
1706	0.9117	0.7125
1709	0.9097	0.7069
1712	0.9063	0.7055
1803	0.9015	0.7038
1806	0.8993	0.7022
1809	0.8985	0.6939
1812	0.8966	0.6920
1903	0.8948	0.6807
1906	0.8919	0.6678
1909	0.8856	0.6570
1912	0.8828	0.6456
2003	0.8725	0.6061

First, we estimate the parameters using 6M CMT rates. We perform estimation on the whole sample and the first sub-sample (in-sample) and the results are reported in Panel (a) of Exhibit 11. As we can see, the estimates are very similar to those of the 3M. The volatility parameter δ is estimated as 0.9630 (versus 0.9294 for 3M) for the sub-sample and 0.9530 (versus 0.9049 for 3M) for the whole sample. The risk-neutral probability is also similar. It is 0.7530 (versus 0.7366 for 3M) for the sub-sample and 0.6876 (versus 0.6813 for 3M) for the whole sample.

Secondly, we estimate the parameters for all the longer-dated CMTs. The results are reported in Panel (b) of Exhibit 11. The volatility parameter δ is increasing as the tenor increases, indicating lower volatility in the longer term rates, which is consistent with the data we observe. However, the risk-neutral probabilities are all negative indicating that these estimates are not reliable.[19] This confirms the assumption made by Chen and Yang (1995) that the estimation of the Ho-Lee model is best for the short rates.

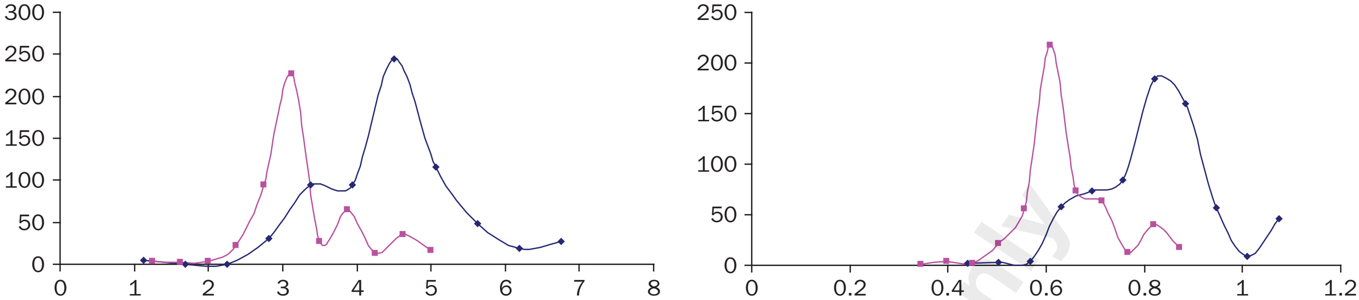
Lastly, we estimate the 3M CMT on a rolling basis in order to see if the parameters (especially δ) are unstable. The results are reported in Panel (c) of Exhibit 11. The volatility parameter δ is mildly decreasing reflecting a rising volatility over time but within a narrow range of 0.88 and 0.93. The risk-neutral probability is also stable over time, ranging from 61% to 73%. This is an interesting result in that the sharp increase in the short rates in the 2018~2019 does not impact δ (the estimates of δ is 0.90 in this period and is not very different from that in other periods). This provides great confidence using the Ho-Lee model for valuation.

[19] Theoretically speaking, risk-neutral probabilities are "pseudo probabilities" which can be negative. They have no meaning but can be still used to compute derivative prices. See Burgin and Meissner (2012) for an excellent discussion. Empirically, many researchers have found negative probabilities (e.g., Chen, Hsieh, and Huang 2018).

EXHIBIT 12

Distributions of Coupons and Conversion Factors of Deliverable Bonds

Panel A: Coupon Distributions **Panel B: Conversion Factor Distributions**



Coupon	Regular	Ultra
Mean	4.2764	3.1821
Std.dev.	0.9204	0.6445
Median	4.375	3

Conv Fact	Regular	Ultra
Mean	0.7914	0.6195
Std.dev.	0.1149	0.0887
Median	0.8078	0.5984

Regular vs. Ultra – A Story of the Conversion Factor

Finally, we provide a comparison analysis between the regular and the Ultra futures contracts. We note that the Ultra futures prices are unanimously higher than the regular futures prices. This implies that the deliverable bonds of the Ultra contracts are more expensive than those of the regular contracts. Moreover, this observation is confirmed by the COC analysis in which the COC prices of the Ultra futures contracts are higher than those of the regular futures contracts.

Given that the deliverable bonds of the Ultra futures have longer terms to maturity than those of the regular futures, it would be natural for the deliverable bonds under the Ultra contracts to have higher coupons. Panel (a) of Exhibit 12 plots the coupon distributions of the two contracts (and summary statistics are provided below).

Interestingly, the coupons of the Ultra contracts are lower than those of the regular contracts. The average coupon for the regular contracts is 4.2764 (median is 4.375) and 3.1821 (median is 3) for the Ultra contract. A lower coupon (with a longer time to maturity) should lead to a lower bond price.

However, delivery profits are adjusted by the conversion factor. Panel (b) of Exhibit 12 plots the distributions of the conversion factors of the two contracts. Indeed, the conversion factor values of the regular contracts are higher than those of the Ultra contracts. The mean is 0.7614 (median is 0.8078) for the regular futures and 0.6195 (median is 0.5984) for the Ultra futures.

US Treasuries are auctioned at the par value. Hence, their coupons reflect the market interest rates at the time of issuance. The deliverables of the Ultra futures have longer terms to maturity, meaning that they are issued at least 5 years after the deliverables of the regular futures. This indicates that interest rates have fallen during the period of our sample, which is what we observed in the market.

Conversion factors are designed to adjust raw bond prices in a hope that they are more comparable (i.e., have a lower quality option value). In comparing the deliverable bonds of the two futures contracts, we can easily conclude that the conversion factor does do a good job (in fact it over-corrects) in adjusting the bond prices for a fair delivery.

A Comment on the Dry Spell

From Exhibit 10, it is clear that the CTD bond under the COC model tends to be the shortest maturity in the period of 2015~2020. This is consistent with the dry spell prediction by Ben-Abdallah and Breton (2017). However, the Ho-Lee model predicts differently. In the same period, none (except for extremely few in end of 2020) of the CTD bonds is short maturity. In fact, very long-term bonds are predicted to be delivered in states 3 and 4 (labeled as × and × respectively) when rates are low.[20] None of the states predict short maturities.

Note that the Ho-Lee model chooses the CTD bond in every state in order to maximize the quality option value (see Carr 1988). And these choices are different from the choice by the COC model in a very large way resulting a high-quality option value. This indicates that the choice of the shortest maturity bonds is not optimal. Furthermore, the fact that the Ho-Lee model value for the futures price is close to the market indicates that the COC futures price is too exaggerated, further proving that the choice of the shortest maturity bonds is not optimal.

CONCLUSION

In this article, we evaluate the Ultra futures contracts and compare them with the regular futures contracts. We use the Ho-Lee model for the yield curve dynamics. The Ho-Lee model is an arbitrage-free interest rate model that takes the current yield curve as an input and hence incorporates market expectation of future volatility.

Using the Ho-Lee model, we find non-trivial quality option values for both the regular and Ultra futures contracts – \$31.76 for the regular futures and \$47.94 for the Ultra futures. Looking at sub-periods, we find that the first half sub-period (in-sample) has a substantially smaller quality option value than the second half sub-period (out-of-sample) by 8~10 times. This attributes partially to model error but also partially to a higher volatility environment in the second sub-period.

Interestingly (yet disturbingly), the timing option values are negative. Without the information on actual bonds that are delivered, we cannot gauge the model error of the Ho-Lee model. Hence, these negative option values do not necessarily represent arbitrage opportunities. To investigate this further, we must employ alternative interest rate models.[21]

While negative timing option values are not necessarily arbitrageable, situations when the COC price falls below the actual market futures price are. We have identified a handful of dates when such violations occurred. However, the magnitudes of most of the violation are small and could be explained by liquidity and market frictions.

Lastly, we compare the CTD bonds for the regular futures versus those for the Ultra futures. The CTD bonds for the Ultra contracts are longer and hence, they will be easily highly/lowly priced if their coupons are high/low. We discover that high/low coupon bonds are adjusted downwards/upwards by the high/low conversion factors and as a result even though the CTD bonds for the Ultra contracts are more expensive, the futures prices are lower.

[20] We note that the probability of state 4 is low (48.17 basis points).

[21] We did also look into the two-factor Cox-Ingersoll-Ross model as in Chen and Yeh (2012). The parameters estimated do not give rise to reasonable future yield curve scenarios. Results are available upon request.

APPENDIX

HO-LEE MODEL

0	t_1	$P(0,t_1)$					
	t_2	$P(0,t_2)$					
	t_3	$P(0,t_3)$					
	t_4	$P(0,t_4)$					
	$\vdots$	$\vdots$					
	t^*	$P(0,t^*)$					
t_1	t_2	$\frac{P(0,t_2)}{P(0,t_1)} d_{t_2-t_1}$	$\frac{P(0,t_2)}{P(0,t_1)} u_{t_2-t_1}$				
	t_3	$\frac{P(0,t_3)}{P(0,t_1)} d_{t_3-t_1}$	$\frac{P(0,t_3)}{P(0,t_1)} u_{t_3-t_1}$				
	t_4	$\frac{P(0,t_4)}{P(0,t_1)} d_{t_4-t_1}$	$\frac{P(0,t_4)}{P(0,t_1)} u_{t_4-t_1}$				
	$\vdots$	$\vdots$	$\vdots$				
	t^*	$\frac{P(0,t^*)}{P(0,t_1)} d_{t^*-t_1}$	$\frac{P(0,t^*)}{P(0,t_1)} u_{t^*-t_1}$				
t_2	t_3	$\frac{P(0,t_3)}{P(0,t_2)} \frac{d_{t_3-t_1}}{d_{t_2-t_1}} d_{t_3-t_2}$	$\frac{P(0,t_3)}{P(0,t_2)} \frac{d_{t_3-t_1}}{d_{t_2-t_1}} u_{t_3-t_2}$	$\frac{P(0,t_3)}{P(0,t_2)} \frac{u_{t_3-t_1}}{u_{t_2-t_1}} u_{t_3-t_2}$			
	t_4	$\frac{P(0,t_4)}{P(0,t_2)} \frac{d_{t_4-t_1}}{d_{t_2-t_1}} d_{t_4-t_2}$	$\frac{P(0,t_4)}{P(0,t_2)} \frac{d_{t_4-t_1}}{d_{t_2-t_1}} u_{t_4-t_2}$	$\frac{P(0,t_4)}{P(0,t_2)} \frac{u_{t_4-t_1}}{u_{t_2-t_1}} u_{t_4-t_2}$			
	$\vdots$	$\vdots$	$\vdots$	$\vdots$			
	t^*	$\frac{P(0,t^*)}{P(0,t_2)} \frac{d_{t^*-t_1}}{d_{t_2-t_1}} d_{t^*-t_2}$	$\frac{P(0,t^*)}{P(0,t_2)} \frac{d_{t^*-t_1}}{d_{t_2-t_1}} u_{t^*-t_2}$	$\frac{P(0,t^*)}{P(0,t_2)} \frac{u_{t^*-t_1}}{u_{t_2-t_1}} u_{t^*-t_2}$			
t_3	t_4	$\frac{P(0,t_4)}{P(0,t_3)} \frac{d_{t_4-t_1}}{d_{t_3-t_1}} \frac{d_{t_4-t_2}}{d_{t_3-t_2}} d_{t_4-t_3}$	$\frac{P(0,t_4)}{P(0,t_3)} \frac{d_{t_4-t_1}}{d_{t_3-t_1}} \frac{d_{t_4-t_2}}{d_{t_3-t_2}} u_{t_4-t_3}$	$\frac{P(0,t_4)}{P(0,t_3)} \frac{d_{t_4-t_1}}{d_{t_3-t_1}} \frac{u_{t_4-t_2}}{u_{t_3-t_2}} u_{t_4-t_3}$	$\frac{P(0,t_4)}{P(0,t_3)} \frac{u_{t_4-t_1}}{u_{t_3-t_1}} \frac{u_{t_4-t_2}}{u_{t_3-t_2}} u_{t_4-t_3}$		
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
	t^*	$\frac{P(0,t^*)}{P(0,t_3)} \frac{d_{t^*-t_1}}{d_{t_3-t_1}} \frac{d_{t^*-t_2}}{d_{t_3-t_2}} d_{t^*-t_3}$	$\frac{P(0,t^*)}{P(0,t_3)} \frac{d_{t^*-t_1}}{d_{t_3-t_1}} \frac{d_{t^*-t_2}}{d_{t_3-t_2}} u_{t^*-t_3}$	$\frac{P(0,t^*)}{P(0,t_3)} \frac{d_{t^*-t_1}}{d_{t_3-t_1}} \frac{u_{t^*-t_2}}{u_{t_3-t_2}} u_{t^*-t_3}$	$\frac{P(0,t^*)}{P(0,t_3)} \frac{u_{t^*-t_1}}{u_{t_3-t_1}} \frac{u_{t^*-t_2}}{u_{t_3-t_2}} u_{t^*-t_3}$		
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
$t_4 (= T_f)$	t^*	$\frac{P(0,t^*)}{P(0,t_4)} \frac{d_{t^*-t_1}}{d_{t_4-t_1}} \frac{d_{t^*-t_2}}{d_{t_4-t_2}} \frac{d_{t^*-t_3}}{d_{t_4-t_3}} d_{t^*-t_4}$	$\frac{P(0,t^*)}{P(0,t_4)} \frac{u_{t^*-t_1}}{u_{t_4-t_1}} \frac{d_{t^*-t_2}}{d_{t_4-t_2}} \frac{d_{t^*-t_3}}{d_{t_4-t_3}} d_{t^*-t_4}$	$\frac{P(0,t^*)}{P(0,t_4)} \frac{u_{t^*-t_1}}{u_{t_4-t_1}} \frac{u_{t^*-t_2}}{u_{t_4-t_2}} \frac{d_{t^*-t_3}}{d_{t_4-t_3}} d_{t^*-t_4}$	$\frac{P(0,t^*)}{P(0,t_4)} \frac{u_{t^*-t_1}}{u_{t_4-t_1}} \frac{u_{t^*-t_2}}{u_{t_4-t_2}} \frac{u_{t^*-t_3}}{u_{t_4-t_3}} d_{t^*-t_4}$	$\frac{P(0,t^*)}{P(0,t_4)} \frac{u_{t^*-t_1}}{u_{t_4-t_1}} \frac{u_{t^*-t_2}}{u_{t_4-t_2}} \frac{u_{t^*-t_3}}{u_{t_4-t_3}} u_{t^*-t_4}$	

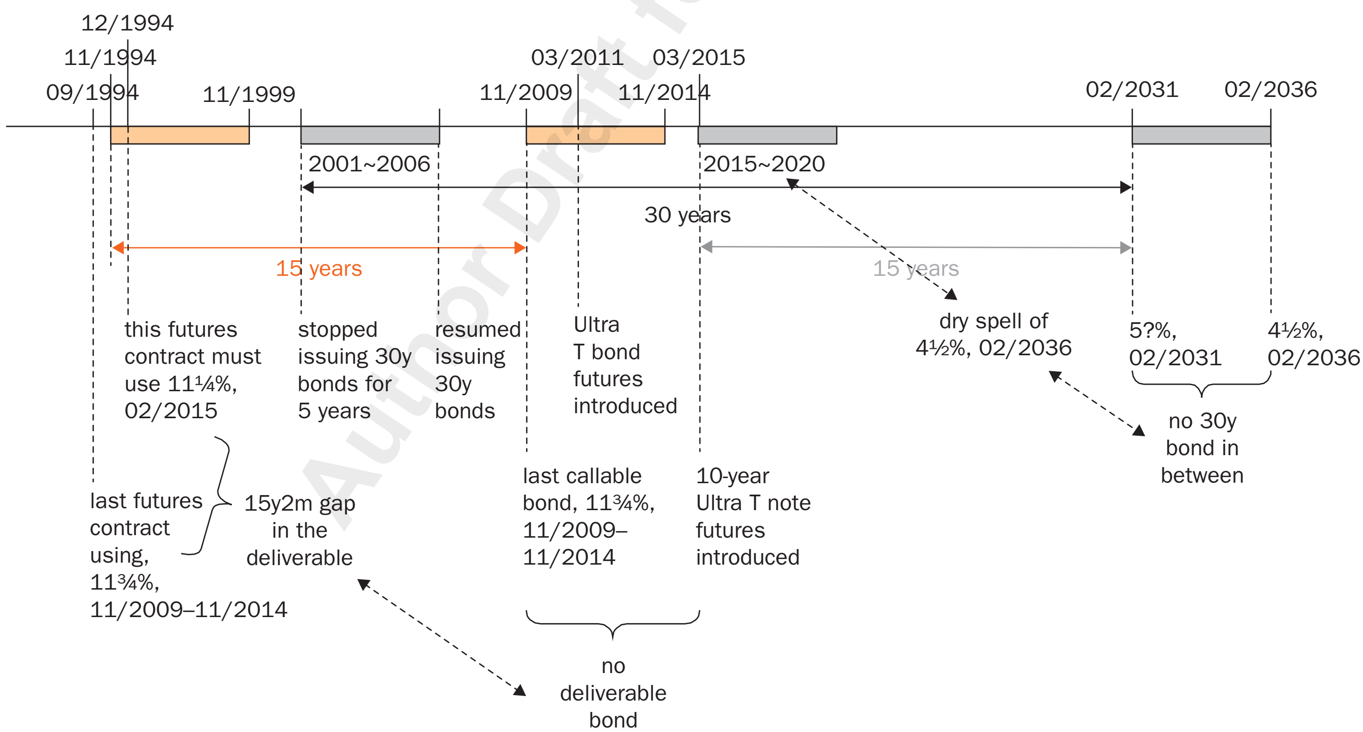
Take an example, on 3/25/2011, the futures contract that is settled in June has a futures price is $120.25 to be settled in June of 2011 (hence time to settlement is 68 days (or roughly $T_f - t = \frac{68}{365} = 0.1863$ which is used in the HL model). There are 12 deliverable bonds ranging from 15 years and 2 months to 24 years and 8 months. Hence,

t_1	0.04658
t_2	0.09315
t_3	0.13973
t_4	0.18630

There are 12 bonds that are eligible for delivery in June 2011. Their maturities range from 15.1667 years to 24.6667 years. As a result, the first bond has 31 coupon semi-annual payments $n(1) = 31$ and the last bond has 50 semi-annual coupon payments $n(12) = 50$. As a result, $T_{31}^{(1)} = 15.1667$ and $T_{50}^{(12)} = 24.1667$. In the above Ho-Lee table, $t^* = T_1^{(k)} \ldots T_{n(k)}^{(k)}$. As a result, $t^* = T_1^{(1)}, \ldots, T_{n(1)}^{(1)} = 0.1667, 0.6667, \ldots, 15.1667$ for the first bond and $t^* = T_1^{(12)}, \ldots, T_{n(12)}^{(12)} = 0.1667, 0.6667, \ldots, 24.1667$.

DRY SPELL EPISODES

The following diagram lays out the timeline of so-called the “dry spells”. There are two such episodes. One happens in 1994–1999 (highlighted as the orange period) due to the disappearance of callable Treasuries in 2009. As indicated in the diagram, in the end of 2009 the US Treasury department stopped issuing any more callable bonds (and the call period is 5 years prior to maturity)—leaving the 11¾%, November 2009–2014 the last callable in the US history. As a result, the last futures contract that can use 11¼%, November 2009–2014 for delivery is the September 1994 futures. The next contract, December 1994 futures, must move to the next available bond which is 11¼%, February 2015 and this bond will remain the CTD bond for the next five years – leaving a 5-year dry spell. The second episode happened in the 2015–2020 period (highlighted as the gray period). This time it is due to the budget surplus of the Clinton administration and the Treasury department stopped issuing the 30-year bond as a consequence in 2001 (until 2006). The gap between February 2031 (last 30-year bond prior to 2001 is 5⅜%, February 2031) and February 2036 (the first 30-year bond when issuance resumed is 4¼%, February 2036) causes another dry spell in 2015 and 2020 which is the period of this study.



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