SLOPES OF INTEGRAL LATTICES

Cris Poor, poor@fordham.edu
David S. Yuen, yuen@lfc.edu

Abstract. We use the dyadic trace to define the concept of slope for integral lattices. We present an introduction to the theory of the slope invariant. The main theorem states that a Siegel modular cusp form f of slope strictly less than the slope of an integral lattice with Gram matrix s satisfies $f(s\tau) = 0$ for all τ in the upper half plane. We compute the dyadic trace and the slope of each root lattice and we give applications to Siegel modular cusp forms.

§0. Introduction.

For a Siegel modular cusp form $f \in S_n^k$ with $f(\Omega) = \sum_T a_T e\,(\mathrm{tr}\,(\Omega T))$ as its Fourier expansion define $\mu(f) = \min\{m(T) : a_T \neq 0\}$ where $m(T) = \min_{x \in \mathbb{Z}^n \setminus \{0\}} x'Tx$ is Hermite's function. The integer μ is a measure of the order of vanishing of f on the boundary of moduli space and the ratio k/μ is called the slope of f. Cusp forms of small slope necessarily vanish on certain geometric loci of $\mathcal{A}_n$, the moduli space of principally polarized abelian varieties. For example all cusp forms of slope less than $8 + \frac{4}{n}$ vanish on the hyperelliptic locus inside $\mathcal{A}_n$. This follows from the work of Igusa [7] and may also be found in the work of Harris and Morrison [6]. Here one also finds a result on the trigonal locus and an elegant conjecture for the Jacobian locus. Cusp forms of slope less than

$$\frac{72(2n+3)(3^{2n+2}-n)}{(2n+3)(3^{2n+4}+2\cdot 3^{2n+2}-27)-(3^{2n+5}-27)}$$

vanish on the trigonal locus. As n increases this formula monotonically decreases to $72/11$. The conjecture of Harris and Morrison would imply that cusp forms of slope less than $6 + \frac{12}{n+1}$ vanish on the Jacobian locus and their conjecture has been verified for $n \le 6$.

These interesting results compute "critical slopes" for natural geometric loci inside of $\mathcal{A}_n$. In this paper we compute critical slopes for an extensive family of modular curves embedded in $\mathcal{A}_n$. In the process we exhibit homomorphisms that can profitably be used to study all Siegel modular cusp forms, see [15]. Let Λ be an integral lattice of rank n and let s be any Gram matrix for Λ. Let ℓ be the unique positive integer with ℓs^{-1} integral and primitive; ℓ is the exponent of the abelian group Λ^*/Λ. The map $\phi_s : \mathcal{H}_1 \to \mathcal{H}_n$ given by $\tau \mapsto s\tau$ descends to a map $\phi_\Lambda : X_0(\ell) \to \mathcal{A}_n$ that does not depend upon the choice of Gram matrix s. Our main result is the following: if a cusp form $f \in S_n^k$ has slope less than

$$(0.1)\qquad \frac{12}{n}\frac{1}{[\Gamma_1 : \Gamma_0(\ell)]} \sum_{\left[\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right] \in \Gamma_0(\ell)\backslash\Gamma_1} w\,(c\,\Lambda^* + \Lambda)$$

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then f vanishes on the modular curve $\phi_\Lambda\left(X_0(\ell)\right)$. We explain the notation: The Riemann surface $X_0(\ell)$ is constructed as $X_0(\ell) = \Gamma_0(\ell)\backslash\mathcal{H}_1$. Let $\mathcal{P}_n(\mathbb{R})$ be the cone of $n \times n$ symmetric positive definite real matrices. The dyadic trace [14] $w : \mathcal{P}_n(\mathbb{R}) \to \mathbb{R}_{\geq 0}$ is a class function defined by $w(s) = \inf_{Y\in\mathcal{P}_n(\mathbb{R})} \operatorname{tr}(sY)/m(Y)$ and so is dual to m in the sense of convexity theory [16]. The dyadic trace of a lattice Λ is given by $w\left(\Lambda\right) = w\left(\operatorname{Gram}(\Lambda)\right)$ for any choice of Gram matrix associated to Λ. Note that for $\alpha > 0$ we have $w(\alpha\Lambda) = w(\alpha^2 \operatorname{Gram}(\Lambda)) = \alpha^2 w(\Lambda)$. We refer to the rational number in 0.1 as the slope of Λ or of $\operatorname{Gram}(\Lambda)$ and we compute the slopes of the lattices A_n^r, D_n, E_i and their duals. For example, we have $\operatorname{slope}(D_n) = 8$ and so any Siegel modular cusp form of slope less than 8 must vanish on $\operatorname{Gram}(D_n)\tau$ for all $\tau \in \mathcal{H}_1$. As a consequence of this we show that for $n \geq 5$ the principally polarized abelian variety defined by $\mathbb{C}^n/(\operatorname{Gram}(D_n)\tau\mathbb{Z}^n + \mathbb{Z}^n)$ has a singular theta locus. Other examples and comments may be found in the Conclusion.

§1. Notation.

Let $M_{m\times n}(\mathbb{F})$ denote $m \times n$ matrices with coefficients in $\mathbb{F}$ for $\mathbb{F} = \mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$ or $\mathbb{Z}$. Let $M^{\text{sym}}_{n\times n}(\mathbb{F}) = \{M \in M_{n\times n}(\mathbb{F}) : M' = M\}$. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a typical element of $\operatorname{Sp}_n(\mathbb{F}) = \{M \in M_{2n\times 2n}(\mathbb{F}) : M'JM = J\}$ where A, B, C, $D \in M_{n\times n}(\mathbb{F})$ and $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Elements in $\operatorname{Sp}_n(\mathbb{Q})$ are called rational; elements in $\operatorname{Sp}_n(\mathbb{R}) \cap \mathbb{R}^+ M_{2n\times 2n}(\mathbb{Q})$ are called projective rational. Let $\Gamma_n = \operatorname{Sp}_n(\mathbb{Z})$ and for positive integers ℓ let $\Gamma_{n,0}(\ell) = \{M \in \Gamma_n : C \equiv 0 \mod \ell I_n\}$. Let $\Delta_n(\mathbb{F}) = \{M \in \operatorname{Sp}_n(\mathbb{F}) : C = 0\}$ and let $\Delta_n = \Delta_n(\mathbb{Z})$. For $n = 1$ we write $\Gamma_0(\ell) = \Gamma_{1,0}(\ell)$. For $\sigma \in \Gamma_1$, $\operatorname{width}_\ell(\sigma)$ is the number of cosets of $\Gamma_0(\ell)\backslash\Gamma_1$ contained in the double coset $\Gamma_0(\ell)\sigma\Delta_1$ and we have $\operatorname{width}_\ell(\sigma) = \ell/(\ell, c^2)$. We set $\gamma(\ell) = [\Gamma_1 : \Gamma_0(\ell)]$ and note that $\gamma(\ell) = \ell\prod(1 + \frac{1}{p})$ where the product is over the primes p dividing ℓ. For $U \in \operatorname{GL}_n(\mathbb{F})$ let U^* denote the transpose inverse and define a homomorphism $u : \operatorname{GL}_n(\mathbb{R}) \to \operatorname{Sp}_n(\mathbb{R})$ by $u(U) = \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix}$. Similarly let $t : M^{\text{sym}}_{n\times n}(\mathbb{R}) \to \operatorname{Sp}_n(\mathbb{R})$ be defined by $t(T) = \begin{pmatrix} I_n & T \\ 0 & I_n \end{pmatrix}$. For any $s \in \mathcal{P}_n(\mathbb{R})$ there is a homomorphism $\alpha_s : \operatorname{Sp}_1(\mathbb{R}) \to \operatorname{Sp}_n(\mathbb{R})$ given by: for $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{Sp}_1(\mathbb{R})$ define $\alpha_s(\sigma) = \begin{pmatrix} aI_n & bs \\ cs^{-1} & dI_n \end{pmatrix}$. Let $\mathcal{H}_n = \{\Omega \in M^{\text{sym}}_{n\times n}(\mathbb{C}) : \Im\Omega \in \mathcal{P}_n(\mathbb{R})\}$ be the Siegel upper half space. Let $M \in \operatorname{Sp}_n(\mathbb{R})$ act on $\Omega \in \mathcal{H}_n$ by linear fractional transformations: $M\langle\Omega\rangle = (A\Omega + B)(C\Omega + D)^{-1}$; we have $\phi_s \circ \sigma = \alpha_s(\sigma) \circ \phi_s$ as maps from $\mathcal{H}_1$ to $\mathcal{H}_n$, see [5, p. 301]. We set $\mathcal{A}_n = \Gamma_n\backslash\mathcal{H}_n$.

Fix $n, k \in \mathbb{Z}$, $n \geq 1, k \geq 0$. For a function $f : \mathcal{H}_n \to \mathbb{C}$ and $M \in \operatorname{Sp}_n(\mathbb{R})$ define $f\underset{k}{|}M : \mathcal{H}_n \to \mathbb{C}$ by $(f\underset{k}{|}M)(\Omega) = \det(C\Omega + D)^{-k} f(M\langle\Omega\rangle)$. We then have a right action of the group $\operatorname{Sp}_n(\mathbb{R})$ on functions from $\mathcal{H}_n$ to $\mathbb{C}$. For any $s \in \mathcal{P}_n(\mathbb{R})$, $\sigma \in \operatorname{Sp}_1(\mathbb{R})$ and function $f : \mathcal{H}_n \to \mathbb{C}$ the following equation is demonstrated in [5, pp. 300-301]

$$(\phi_s^* f) \underset{nk}{|} \sigma = \phi_s^* \left(f \underset{k}{|} \alpha_s(\sigma) \right). \tag{1.1}$$

Let $\Gamma \subseteq \Gamma_n$ be a subgroup of finite index. The $\mathbb{C}$-vector space $M_n^k(\Gamma)$ of Siegel modular forms of degree n and weight k for Γ is the set of holomorphic $f : \mathcal{H}_n \to \mathbb{C}$ such that for all $M \in \Gamma$ we have $f|M = f$ and for all projective rational $M \in \mathrm{Sp}_n(\mathbb{R})$ we have $f|M$ bounded on domains of type $\{\Omega \in \mathcal{H}_n : \Im\Omega > Y_0\}$. The $\mathbb{C}$-vector space $S_n^k(\Gamma)$ of Siegel modular cusp forms consists of the elememts of $M_n^k(\Gamma)$ which satisfy $\Phi_0(f|M) = 0$ for all projective rational M, where Φ_0 is the standard Siegel operator, see [5, p.45]. For $\mathbb{F} \subseteq \mathbb{R}$ let $\mathcal{P}_n^{\mathrm{semi}}(\mathbb{F}) = \{T \in M_{n\times n}^{\mathrm{sym}}(\mathbb{F}) : T \geq 0\}$. For $z \in \mathbb{C}$ let $e(z) = e^{2\pi i z}$. For A, $B \in M_{n\times n}^{\mathrm{sym}}(\mathbb{F})$ let $\langle A, B\rangle = \mathrm{tr}(AB)$. Any $f \in M_n^k(\Gamma)$ has a Fourier expansion $f(\Omega) = \sum_{T\in\mathcal{P}_n^{\mathrm{semi}}(\mathbb{Q})} a_T e(\langle T, \Omega\rangle)$. Let $\mathrm{supp}(f) = \{T \in \mathcal{P}_n^{\mathrm{semi}}(\mathbb{Q}) : a_T \neq 0\}$. We know that $2\ell\,\mathrm{supp}(f)$ are all even forms in $\mathcal{P}_n^{\mathrm{semi}}(\mathbb{Z})$ when we have $t\left(\ell M_{n\times n}^{\mathrm{sym}}(\mathbb{Z})\right) \subseteq \Gamma$. The function f is a cusp form precisely when $\mathrm{supp}(f) \subseteq \mathcal{P}_n(\mathbb{Q})$.

By a lattice Λ we mean a free $\mathbb{Z}$-module contained inside a Euclidean inner product space. If Λ has a $\mathbb{Z}$-basis of n elements we write $\mathrm{rank}(\Lambda) = n$. Write $\Lambda_{\mathbb{F}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{F}$; then $\Lambda_{\mathbb{Q}}$ is a $\mathbb{Q}$-vector space and $\mathrm{rank}(\Lambda) = \dim_{\mathbb{Q}}(\Lambda_{\mathbb{Q}})$. In this paper we always choose the Euclidean space containing a rank n lattice to be the column vectors in $\mathbb{R}^n$ with the standard dot product. If $M \in \mathrm{GL}_n(\mathbb{R})$ is a basis of column vectors for Λ then we have $\Lambda = M\mathbb{Z}^n$ and $s = M'M$ is called a Gram matrix for Λ. A Gram matrix s is not unique but the $\mathrm{GL}_n(\mathbb{Z})$ equivalence class $[s]$ is. The following are class properties of s and so apply to the lattice Λ as well: s is integral (rational) if its coefficients are in $\mathbb{Z}$ ($\mathbb{Q}$), an integral s is primitive if the gcd of its coefficients is 1, s is even if for all $v \in \mathbb{Z}^n$ we have $v'sv \in 2\mathbb{Z}$. The dual lattice $\Lambda^* = \{\xi \in \mathbb{R}^n : \forall x \in \Lambda, x\cdot\xi \in \mathbb{Z}\}$ also has rank n and is rational precisely when Λ is. We have that Λ is integral if and only if $\Lambda \subseteq \Lambda^*$. If $\Lambda = M\mathbb{Z}^n$ then $\Lambda^* = M^*\mathbb{Z}^n$. Given an abelian group $(G, +)$ we set $\exp(G) = \min\{j \in \mathbb{Z}^+ : \forall g \in G, jg = 0\}$. The minimal vectors of s are the $v \in \mathbb{Z}^n$ satisfying $v'sv = m(s)$. The minimal vectors of Λ are the $x \in \Lambda$ satisfying $x\cdot x = m(\Lambda)$.

Consider a Siegel modular form $f \in M_n^k$. For a rank n lattice we pick a basis $\Lambda = M\mathbb{Z}^n$ with Gram matrix $s = M'M$ and define $\phi_\Lambda^* f = \phi_s^* f = f \circ \phi_s$. Since f is a Siegel modular form, $\phi_\Lambda^* f$ is independent of the choice of M. Occasionally a more complicated construction is requisite: Let $X \in \mathrm{Ext}^{\mathrm{sym}}(\Lambda, \Lambda^*; \mathbb{R}) = \mathrm{Sym}\,(\Lambda^*\otimes\Lambda^*)_{\mathbb{R}} / \mathrm{Sym}\,(\Lambda^*\otimes\Lambda^*)_{\mathbb{Z}}$ be given along with Λ. Pick a representative $\bar{X} \in \mathrm{Sym}\,(\Lambda^*\otimes\Lambda^*)_{\mathbb{R}}$ so that $X = [\bar{X}]$. Let $j : \mathbb{R}^n \otimes \mathbb{R}^n \to M_{n\times n}(\mathbb{R})$ be the isomorphism of vector spaces given by $j(x\otimes y) = xy'$. We may define $\phi_{\Lambda,X}^* f$ by $\left(\phi_{\Lambda,X}^* f\right)(\tau) = f(M'(I_n\tau + j\bar{X})M)$; since f is a Siegel modular form $\phi_{\Lambda,X}^* f$ is independent of the choice of basis M and representative $\bar{X}$. We then have $\phi_{\Lambda,0}^* f = \phi_\Lambda^* f = \phi_s^* f$.

§2. The $\Box$ operation.

We now explain the operation of the cusps on lattices Λ with rational Gram matrices. Motivation for the definitions in this section may be found in the proof of Lemma 5.3.

Definition 2.1. *Let* $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$. *Let Λ be a rational lattice. Define the rational lattice* $\Lambda \Box \sigma$ *by* $\Lambda \Box \sigma = c\Lambda^* + a\Lambda$.

As examples of this operation we have: $\Lambda \Box I = \Lambda$, $\Lambda \Box J = \Lambda^*$ and $\Lambda \Box (J\sigma)^* \supseteq (\Lambda \Box \sigma)^*$.

There is another invariant that we can extract from Λ and σ, it is an element $X \in \mathrm{Ext}^{\mathrm{sym}}(\Lambda\,\square\sigma, (\Lambda\,\square\sigma)^*; \mathbb{Q})$. First we need the following two Lemmas. By treating the elements of $\Lambda \oplus \Lambda^*_{\mathbb{Q}}$ as 2×1 column vectors we may let Γ_1 act on $\Lambda \oplus \Lambda^*_{\mathbb{Q}}$ from the left. For a lattice Υ, we may also let $\bar{X} \in (\Upsilon^* \otimes \Upsilon^*)_{\mathbb{Q}}$ map Υ to $\Upsilon^*_{\mathbb{Q}}$ via $\bar{X}x = (j\bar{X})x$ for $x \in \Upsilon$. Lemma 2.2 recounts the standard properties of Ext and we omit the proof.

Lemma 2.2. *Let Υ be a lattice. The lattices $\mathcal{L}$ such that $0 \oplus \Upsilon^* \subseteq \mathcal{L} \subseteq \Upsilon \oplus \Upsilon^*_{\mathbb{Q}}$ which make the sequence $0 \hookrightarrow \Upsilon^* \xhookrightarrow{\iota_2} \mathcal{L} \xrightarrow{\pi_1} \Upsilon \longrightarrow 0$ exact are in one-to-one correspondence with the elements of $\mathrm{Ext}(\Upsilon, \Upsilon^*; \mathbb{Q}) = (\Upsilon^* \otimes \Upsilon^*)_{\mathbb{Q}} / (\Upsilon^* \otimes \Upsilon^*)_{\mathbb{Z}}$. The elements of $\mathrm{Ext}^{\mathrm{sym}}(\Upsilon, \Upsilon^*; \mathbb{Q}) = \mathrm{Sym}(\Upsilon^* \otimes \Upsilon^*)_{\mathbb{Q}} / \mathrm{Sym}(\Upsilon^* \otimes \Upsilon^*)_{\mathbb{Z}}$ are in one-to-one correspondence with the lattices $\mathcal{L}$ such that $\mathcal{L} = J\mathcal{L}^*$. The correspondence is as follows: For any $\bar{X} \in (\Upsilon^* \otimes \Upsilon^*)_{\mathbb{Q}}$ define $\mathcal{F}\bar{X} : \Upsilon \oplus \Upsilon^* \to \Upsilon \oplus \Upsilon^*_{\mathbb{Q}}$ by $\mathcal{F}\bar{X}(x,\xi) = (x, \bar{X}x + \xi)$. An element $[\bar{X}] \in \mathrm{Ext}(\Upsilon, \Upsilon^*; \mathbb{Q})$ determines the lattice $\mathcal{L} = \mathcal{F}\bar{X}(\Upsilon \oplus \Upsilon^*)$. For any lattice $\mathcal{L}$ choose a rectraction $r_1 : \Upsilon \to \mathcal{L}$; r_1 is a homomorphism such that $\pi_1 \circ r_1 = \mathrm{id}_{\Upsilon}$. Choose any basis $\{m_i\}$ for Υ and let $\{m_i^*\}$ be the corresponding dual basis for Υ^*, then we may select $\bar{X} = \sum_i (\pi_2 r_1 m_i) \otimes m_i^*$.*

Lemma 2.3. *Let $\sigma \in \Gamma_1$. Let Λ be a rational lattice. Let $\mathcal{L} = \sigma'(\Lambda \oplus \Lambda^*)$. We have $0 \oplus (\Lambda\,\square\sigma)^* \subseteq \mathcal{L} \subseteq (\Lambda\,\square\sigma) \oplus (\Lambda\,\square\sigma)^*_{\mathbb{Q}}$, the sequence $0 \hookrightarrow (\Lambda\,\square\sigma)^* \xhookrightarrow{\iota_2} \mathcal{L} \xrightarrow{\pi_1} (\Lambda\,\square\sigma) \longrightarrow 0$ is exact and $\mathcal{L} = J\mathcal{L}^*$.*

Proof. We have $\mathcal{L} = \sigma'(\Lambda \oplus \Lambda^*) = \{(ax + c\xi, bx + d\xi) : x \in \Lambda, \xi \in \Lambda^*\} \subseteq (a\Lambda + c\Lambda^*) \oplus (b\Lambda + d\Lambda^*) = (\Lambda\,\square\sigma) \oplus (\Lambda\,\square(J\sigma)^*) \subseteq (\Lambda\,\square\sigma) \oplus (\Lambda\,\square\sigma)_{\mathbb{Q}}$. To show that $0 \oplus (\Lambda\,\square\sigma)^* \subseteq \mathcal{L}$ so that the range of the injective ι_2 is as stated pick any $\theta \in (\Lambda\,\square\sigma)^*$. From $\theta \in (a\Lambda + c\Lambda^*)^*$ we have $c\theta \in \Lambda$ and $a\theta \in \Lambda^*$. Selecting $x = -c\theta$ and $\xi = a\theta$ we have $(0, \theta) \in \mathcal{L}$. To show that $\ker \pi_1 = \mathrm{Im}\,\iota_2$ suppose we have that $ax + c\xi = 0$ for some $x \in \Lambda, \xi \in \Lambda^*$. We must show that $bx + d\xi \in (\Lambda\,\square\sigma)^*$ or equivalently that $\langle bx + d\xi, ay + c\eta\rangle \in \mathbb{Z}$ for any $y \in \Lambda, \eta \in \Lambda^*$. Using $ax = -c\xi$ and $ad - bc = 1$ we have $\langle bx + d\xi, ay + c\eta\rangle = \langle \xi, y\rangle - \langle x, \eta\rangle \in \mathbb{Z}$. To show that $\mathcal{L} = J\mathcal{L}^*$ we simply calculate: $J\mathcal{L}^* = J\sigma^{-1}(\Lambda^* \oplus \Lambda) = \sigma' J(\Lambda^* \oplus \Lambda) = \sigma'(\Lambda \oplus \Lambda^*) = \mathcal{L}$. $\square$

Definition 2.4. *Let $\sigma \in \Gamma_1$. Let Λ be a rational lattice. Let $X[\Lambda, \sigma]$ be the element of $\mathrm{Ext}^{\mathrm{sym}}(\Lambda\,\square\sigma, (\Lambda\,\square\sigma)^*; \mathbb{Q})$ determined by $\mathcal{L} = \sigma'(\Lambda \oplus \Lambda^*)$ in the one-to-one correspondence of Lemma 2.2 with $\Upsilon = \Lambda\,\square\sigma$.*

Theorem 2.5. *Let $\sigma \in \Gamma_1$. Let Λ be a rational lattice of rank n. Choose any basis $M \in \mathrm{GL}_n(\mathbb{R})$ for Λ so that we have $\Lambda = M\mathbb{Z}^n$ and $s = M'M$ is the Gram matrix associated to the basis M. Let $M_1 \in \Gamma_n$ and $T = \begin{pmatrix} K^* & \beta \\ 0 & K \end{pmatrix} \in \Delta_n(\mathbb{Q})$ be any choices giving the factorization $\alpha_s(\sigma) = M_1 T$. We have $\Lambda\,\square\sigma = MK^{-1}\mathbb{Z}^n$ and $X[\Lambda, \sigma] = [j^{-1}\left(M^*\beta' K M^{-1}\right)]$.*

Proof. Write $\alpha_s(\sigma) = M_1 T$ as

$$\begin{pmatrix} aI_n & bs \\ cs^{-1} & dI_n \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} K^* & \beta \\ 0 & K \end{pmatrix}.$$

Transposing this equation and multiplying on the left by $u\,(M)$ we have

$$\begin{pmatrix} aM & cM^* \\ bM & dM^* \end{pmatrix} = \begin{pmatrix} MK^{-1} & 0 \\ M^*\beta' & M^*K' \end{pmatrix} \begin{pmatrix} A' & C' \\ B' & D' \end{pmatrix}.$$

Apply $\mathbb{Z}^{2n}$ on the right of the equation $(\,aM \quad cM^*\,) = (\,MK^{-1} \quad 0\,)(\,\text{unimodular}\,)$ and obtain $a\Lambda + c\Lambda^* = MK^{-1}\mathbb{Z}^n$ or $\Lambda\,\square\sigma = MK^{-1}\mathbb{Z}^n$.

Now let $\bar{X} = j^{-1}\left(M^*\beta' KM^{-1}\right)$ and rewrite our basic relation in the form:

$$\begin{pmatrix} aI_n & cI_n \\ bI_n & dI_n \end{pmatrix} \begin{pmatrix} M & 0 \\ 0 & M^* \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ j\bar{X} & I_n \end{pmatrix} \begin{pmatrix} (MK^{-1}) & 0 \\ 0 & (MK^{-1})^* \end{pmatrix} \begin{pmatrix} A' & C' \\ B' & D' \end{pmatrix}.$$

The fact that M_1' is unimodular gives $\sigma'\left(\Lambda \oplus \Lambda^*\right) = \mathcal{F}\bar{X}\left((\Lambda\,\square\sigma) \oplus (\Lambda\,\square\sigma)^*\right)$ so that $[\bar{X}] = [j^{-1}\left(M^*\beta' KM^{-1}\right)]$ gives the class of $X[\Lambda, \sigma]$ by Definition 2.4. $\square$

§3. Dyadic traces of Root Lattices.

Let C_n^* be the cone inside $\mathcal{P}_n^{\mathrm{semi}}(\mathbb{R})$ generated by all vv' for $v \in \mathbb{Z}^n$. C_n^* can be characterized as the elements of $\mathcal{P}_n^{\mathrm{semi}}(\mathbb{R})$ whose radical is defined over $\mathbb{Q}$ and C_n^* contains $\mathcal{P}_n(\mathbb{R})$ as the elements of radical zero. The dyadic trace w is a class function $w : C_n^* \to \mathbb{R}_{\geq 0}$ defined by $w(s) = \inf_{Y \in \mathcal{P}_n(\mathbb{R})} \frac{\langle s, Y\rangle}{m(Y)}$. The dyadic trace may also be characterized as a supremum. A dyadic representation of $s \in C_n^*$ is an equation $s = \sum \alpha_i v_i v_i'$ where $\alpha_i \geq 0$ and $v_i \in \mathbb{Z}^n \backslash \{0\}$. We have $w(s) = \sup \sum \alpha_i$ where the supremum is over all the dyadic representations of s. Both the supremum and infimum are attained. The inequality $\langle t, u\rangle \geq m(t)w(u)$ holds for all $u \in C_n^*$, $t \in \mathcal{P}_n(\mathbb{R})$ and equality holds precisely when u has a dyadic representation in the minimal vectors of t. If we let $\varpi(t)$ be the cone in C_n^* generated by the minimal vectors of t then equality in $\langle t, u\rangle \geq m(t)w(u)$ holds precisely when $u \in \varpi(t)$. A form $s \in \mathcal{P}_n(\mathbb{R})$ is called semieutactic when $s^{-1} \in \varpi(s)$ and eutactic when s^{-1} is in the relative interior of $\varpi(s)$. Since $\langle s, s^{-1}\rangle = n$ an $s \in \mathcal{P}_n(\mathbb{R})$ is semieutactic if and only if $n = m(s)w(s^{-1})$. According to Coexeter [2] all of the lattices $R = A_n^r$, D_n, E_n and their duals are eutactic so that we may compute $w(R^*) = n/m(R)$ and $w(R) = n/m(R^*)$.

Proposition 3.1. *The minimal norms of the irreducible root lattices and their dual lattices are as follows:*

(1) $m(A_n^*) = n/(n+1)$ *for* $n \geq 2$.
(2) $m(D_n^*) = 1$ *for* $n \geq 4$.
(3) $m(E_6^*) = \frac{4}{3}$, $m(E_7^*) = \frac{3}{2}$, *and* $m(E_8) = 2$.
(4) *In the above cases the minimal norm of the dual lattice is* 2.
(5) $m(A_n^r) = \min(2, \frac{r(q-1)}{q})$ *for* $rq = n+1$ *and for* $n, q > 1$.

Proof. See [1] and [2]. $\square$

Corollary 3.2. *The dyadic traces of the irreducible root lattices and their dual lattices are as follows:*

(1) $w(A_n) = n + 1$ *for* $n \geq 2$.
(2) $w(D_n) = n$ *for* $n \geq 4$.
(3) $w(E_6) = \frac{9}{2}$, $w(E_7) = \frac{14}{3}$, *and* $w(E_8) = 4$.
(4) *In the above cases the dyadic trace of the dual lattice is* $\frac{n}{2}$, *where* n *is the corresponding dimension.*
(5) $w(A_n^r) = \max(\frac{n}{2}, \frac{nq}{r(q-1)})$ *for* $rq = n + 1$ *and for* $n, q > 1$.

The dyadic traces just computed are rational and the dyadic trace has the following general rationality property.

Theorem 3.3. *Let* $K \subseteq \mathbb{R}$ *be a* $\mathbb{Q}$*-vector space. We have* $w : C_n^* \cap \mathcal{P}_n^{\mathrm{semi}}(K) \setminus \{0\} \to K^+$.

Proof. Let $s \in C_n^* \cap \mathcal{P}_n^{\mathrm{semi}}(K) \setminus \{0\}$; that is, the entries of s satisfy $s_{ij} \in K$. There exists a dyadic representation [14] $s = \sum_{i=1}^N \alpha_i \, v_i \, v_i'$, such that $\alpha_i \geq 0$, $v_i \in \mathbb{Z}^n \setminus \{0\}$, and $w(s) = \sum_i \alpha_i$. Now consider the region in $\mathbb{R}^N$ defined by $D = \{x \in \mathbb{R}^N : s = \sum_{i=1}^N x_i \, v_i v_i'$, and $x_j \geq 0$ each $j = 1, \dots, N\}$. It is a region defined by a finite system of linear equations and linear inequalities. Note that D is bounded since it is in the first "quadrant" and every $x \in D$ satisfies $x_1 + \cdots + x_N \leq w(s)$. Note also that D is convex since if $x, x' \in D$ then $px + qx' \in D$, for $p, q \geq 0$ with $p + q = 1$. Note also that D is nonempty since $\alpha \in D$, where α has components α_i. Then the linear function $f(x) = \sum_{i=1}^N x_i$ must attain its maximum at some vertex of the region D. Since a vertex $y \in D$ is the unique solution to a system of linear equations given by $s = \sum_{i=1}^N x_i \, v_i v_i'$ and $x_j = 0$ for j belonging to a subset of $\{1, \dots, N\}$, Cramer's Rule shows that the coordinates of y must be rational expressions in the coefficients of this system of equations. Since the coefficients of the matrix being inverted are all integers and since K is a $\mathbb{Q}$-vector space, then the coordinates of y are also in K. So we have $f(y) = y_1 + \cdots + y_N \in K$. Since we know a priori that this maximum value is $w(s)$, then $w(s) = f(y) \in K$. Since $w(s) > 0$ automatically for $s \neq 0$, we have $w(s) \in K^+$. $\square$

§4. Slopes of Root Lattices.

When Λ is integral and $\sigma \in \Gamma_1$ we have $\Lambda \,\square \sigma = c\,\Lambda^* + a\,\Lambda = c\,\Lambda^* + \Lambda$ because $\Lambda \subseteq \Lambda^*$ and $(a, c) = 1$. Similarly, for Λ with $\ell\Lambda^* \subseteq \Lambda$ we have $c\,\Lambda^* + \Lambda = (\ell, c)\Lambda^* + \Lambda$. This shows that $\Lambda \,\square \sigma$ depends only upon Λ and upon the double coset $\Gamma_0(\ell)\sigma\Delta_1$ because (ℓ, c) has the same value for every element of $\Gamma_0(\ell)\sigma\Delta_1$. We use this fact to make the following definition.

Definition 4.1. *Let* Λ *be an integral lattice of rank* n *with* $\ell = \exp(\Lambda^*/\Lambda)$. *Define:*

$$\mathrm{slope}(\Lambda) = \frac{12}{n} \frac{1}{[\Gamma_1 : \Gamma_0(\ell)]} \sum_{[\sigma] \in \Gamma_0(\ell) \backslash \Gamma_1 / \Delta_1} \mathrm{width}_\ell(\sigma) \, w\left(\Lambda \,\square \sigma\right).$$

Note that $\mathrm{slope}(\Lambda)$ is also $12/n$ times the average of $w\left(\Lambda \,\square \sigma\right)$ over $[\sigma] \in \Gamma_0(\ell) \backslash \Gamma_1$. The next Proposition is an immediate consequence of Theorem 3.3.

Proposition 4.2. *If Λ is an integral lattice then* $\mathrm{slope}(\Lambda) \in \mathbb{Q}^+$.

By a good choice of coset representatives for $\Gamma_0(\ell)\backslash\Gamma_1/\Delta_1$ we can rewrite the formula for the slope of an integral lattice as a number theoretic sum over the divisors of ℓ. Here ϕ is Euler's function $\phi(\ell) = \mathrm{card}\,(\mathbb{Z}/\ell\mathbb{Z})^\times$.

Lemma 4.3. *Let Λ be an integral lattice of rank n with $\ell = \exp(\Lambda^*/\Lambda)$. We have*

$$\mathrm{slope}(\Lambda) = \frac{12}{n}\frac{1}{[\Gamma_1 : \Gamma_0(\ell)]}\sum_{q,r\in\mathbb{Z}^+:\, qr=\ell}\frac{r}{(q,r)}\,\phi\left((q,r)\right) w\left(q\,\Lambda^* + \Lambda\right).$$

Proof. We use coset representatives for $\Gamma_0(\ell)\backslash\Gamma_1/\Delta_1$ that are given by $\sigma_{u,q} = \begin{pmatrix} u & * \\ q & * \end{pmatrix}$ of width $\ell/(\ell,q^2)$ for each $q|\ell$ and where for each class $[u] \in (\mathbb{Z}/(q,\ell/q)\mathbb{Z})^*$ we pick a representative u with $(u,q) = 1$, see [9, pp. 35-37]. Then Definition 4.1 reads

$$\mathrm{slope}(\Lambda) = \frac{12}{n}\frac{1}{[\Gamma_1 : \Gamma_0(\ell)]}\sum_{q:q|\ell}\left(\sum_{u:1\le u\le(q,\frac{\ell}{q})\text{ and }(u,(q,\frac{\ell}{q}))=1}\frac{\ell}{(\ell,q^2)}\,w\left(q\,\Lambda^* + u\,\Lambda\right)\right).$$

Since $\Lambda \subseteq \Lambda^*$ and $(u,q) = 1$ we have $q\,\Lambda^* + u\,\Lambda = q\,\Lambda^* + \Lambda$ so that the summand is independent of u. Setting $r = \frac{\ell}{q}$ this renders the double sum as

$$\sum_{q,r\in\mathbb{Z}^+:\, qr=\ell}\phi\left((q,r)\right)\frac{r}{(r,q)}\,w\left(q\,\Lambda^* + \Lambda\right). \quad \square$$

Proposition 4.4. *We have* $\mathrm{slope}(E_6) = \frac{27}{4}$, $\mathrm{slope}(E_7) = \frac{20}{3}$ *and* $\mathrm{slope}(E_8) = 6$.

Proof. When $\ell = 1$ we have $\mathrm{slope}(\Lambda) = \frac{12}{n}w(\Lambda)$ and hence $\mathrm{slope}(E_8) = \frac{12}{8}\cdot 4 = 6$. When ℓ is prime we have $\mathrm{slope}(\Lambda) = \frac{12}{n}\frac{1}{\ell+1}\left(w(\Lambda) + \ell w(\Lambda^*)\right)$ since the only two cusps are $[I]$ and $[J]$. We have $\mathrm{slope}(E_7) = \frac{12}{7}\frac{1}{2+1}\left(\frac{14}{3} + 2\cdot\frac{7}{2}\right) = \frac{20}{3}$ and $\mathrm{slope}(E_6) = \frac{12}{6}\frac{1}{3+1}\left(\frac{9}{2} + 3\cdot\frac{6}{2}\right) = \frac{27}{4}$. $\square$

Proposition 4.5. *We have* $\mathrm{slope}(D_n) = 8$ *for* $n \ge 4$.

Proof. When n is even $\exp(D_n^*/D_n)$ is the prime $\ell = 2$ so that we have

$$\mathrm{slope}(D_n) = \frac{12}{n}\frac{1}{[\Gamma_1 : \Gamma_0(2)]}\left(w(D_n) + 2w(D_n^*)\right) = \frac{12}{n}\frac{1}{3}\left(n + 2\cdot\frac{n}{2}\right) = 8.$$

When n is odd $\ell = \exp(D_n^*/D_n) = 4$ and $\Gamma_0(4)\backslash\Gamma_1/\Delta_1$ has 3 cusps represented by: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ of widths 1, 1, 4 respectively. When the rank n is odd we have $D_n \,\square \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = 2D_n^* + D_n = I_n$ and therefore we have

$$\mathrm{slope}(D_n) = \frac{12}{n}\frac{1}{[\Gamma_1 : \Gamma_0(4)]}\left(w(D_n) + w(I_n) + 4w(D_n^*)\right) = \frac{12}{n}\frac{1}{6}\left(n + n + 4\cdot\frac{n}{2}\right) = 8. \quad \square$$

Proposition 4.6. *Let $n \geq 2$ and $\beta = 4/(4, n+1)$. The slope of A_n is*

$$\mathrm{slope}(A_n) = \begin{cases} \frac{77}{10} & \text{if } n = 5 \\ 6 + \frac{6}{\gamma(n+1)}\left(\frac{n+2}{n}\right) & \text{if } n \text{ is even} \\ 6 + \frac{6}{\gamma(n+1)}\left(\frac{n^2+2\beta n+n-2}{n(n-1)}\right) & \text{if } n \text{ is odd, } n \neq 5. \end{cases}$$

Proof. We have $\ell = \exp\left(A_n^*/A_n\right) = n+1$. Set $\gamma = \gamma(\ell)$ for simplicity. Since we have $qA_n^* + A_n = qA_n[1] + A_n = A_n[q] = A_n^r$ the formula from Lemma 4.3 for the slope of A_n reads:

$$\mathrm{slope}(A_n) = \frac{12}{n}\frac{1}{\gamma}\sum_{qr=n+1}\frac{r\,\phi((q,r))}{(q,r)}\,w\left(A_n^r\right).$$

Most of the $w(A_n^r)$ will be $\frac{n}{2}$ as can be seen if we rewrite Corollary 3.2 (5) as follows:

$$w(A_n^r) = \begin{cases} n+1 & \text{if } r = 1 \\ \frac{n(n+1)}{2(n-1)} & \text{if } r = 2 \\ \frac{10}{3} & \text{if } (r,q) = (3,2) \\ \frac{n}{2} & \text{otherwise} \end{cases}$$

We need to consider the three possible cases.

Case $n = 5$. Then we have $n+1 = 6$ and this is the case where $(r,q) = (3,2)$ can happen. The enumeration of (r,q) is $(6,1), (3,2), (2,3), (1,6)$. Thus we have

$$\begin{aligned} w(A_5) &= \frac{12}{5}\frac{1}{12}\left(\frac{6}{1}w(A_5^6) + \frac{3}{1}w(A_5^3) + \frac{2}{1}w(A_5^2) + \frac{1}{1}w(A_5)\right) \\ &= \frac{12}{5}\frac{1}{12}\left(\frac{6}{1}\cdot\frac{5}{2} + \frac{3}{1}\cdot\frac{10}{3} + \frac{2}{1}\cdot\frac{15}{4} + \frac{1}{1}\cdot 6\right) = \frac{77}{10}. \end{aligned}$$

Case n is even. Then $n+1$ is odd. Then for all $r > 1$ with $r|(n+1)$, we have $w(A_n^r) = \frac{n}{2}$. The only dyadic trace that is not $\frac{n}{2}$ is when $(r,q) = (1, n+1)$, which has $\frac{r\,\phi((r,q))}{(r,q)} = 1$. Thus we have

$$w(A_n) = \frac{12}{n}\frac{1}{\gamma}\left(1w(A_n) + (\gamma-1)\frac{n}{2}\right) = \frac{12}{n}\frac{1}{\gamma}\left(1(n+1) + (\gamma-1)\frac{n}{2}\right) = 6 + \frac{6}{\gamma}\left(\frac{n+2}{n}\right).$$

Case n is odd, $n \neq 5$. Then $n+1$ is even. Then for all $r > 1$ with $r|(n+1)$, we have $w(A_n^r) = \frac{n}{2}$, unless $r = 2$ and $q = \frac{n+1}{2}$. So the only dyadic traces that are not $\frac{n}{2}$ are when $(r,q) = (1, n+1)$, which has $\frac{r\,\phi((r,q))}{(r,q)} = 1$, and when $(r,q) = (2, \frac{n+1}{2})$, which has $\frac{r\,\phi((r,q))}{(r,q)} = \frac{2}{(2,\frac{n+1}{2})} = \frac{4}{(4,n+1)} = \beta$. Thus we have

$$\begin{aligned} w(A_n) &= \frac{12}{n}\frac{1}{\gamma}\left(1w(A_n) + \beta w(A_n^2) + (\gamma-1-\beta)\frac{n}{2}\right) \\ &= \frac{12}{n}\frac{1}{\gamma}\left(1(n+1) + \beta\frac{n(n+1)}{2(n-1)} + (\gamma-1-\beta)\frac{n}{2}\right) = 6 + \frac{6}{\gamma}\left(\frac{n^2+2\beta n+n-2}{n(n-1)}\right) \end{aligned}$$

□

Here is a table of the slopes of the first few A_n, along with decimal approximations truncated to two decimal places.

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15
slope(A_n)	9	$\frac{26}{3}$	$\frac{15}{2}$	$\frac{77}{10}$	7	$\frac{143}{21}$	$\frac{53}{8}$	$\frac{355}{54}$	$\frac{33}{5}$	$\frac{349}{55}$	$\frac{13}{2}$	$\frac{497}{78}$	$\frac{44}{7}$	$\frac{1327}{210}$
approx	9	8.66	7.5	7.7	7	6.80	6.62	6.57	6.6	6.34	6.5	6.37	6.28	6.31

A similar computation to that for A_n will give the slope of A_n^r.

Proposition 4.7. *Let $rq = n+1$, $\bar{r} = r/(r,q)$ and $\ell = (n+1)/(r,q)^2$. We have*

$$\mathrm{slope}(\sqrt{\bar{r}}A_n^r) = \begin{cases} \mathrm{slope}(A_n) & \text{if } (r,q) = 1 \\ 6 + \frac{12}{(n-1)\gamma(\ell)} & \text{if } (r,q) = 2 \\ 6 & \text{if } (r,q) \geq 3. \end{cases}$$

We have casually referred to the slope of lattices such as A_n^r which are rational but not integral. The following Proposition shows that we may define the slope of a rational lattice to be the slope of any integral rescaling.

Proposition 4.8. *Let Λ be an integral lattice. Let α be a positive real number. If $\alpha\Lambda$ is integral then we have* $\mathrm{slope}(\alpha\Lambda) = \mathrm{slope}(\Lambda)$.

Proof. Since both $\alpha\Lambda$ and Λ are multiples of the same primitive lattice we may reduce to the case where $\alpha = \sqrt{N}$ for some integer N. By induction, we only need to prove that $\mathrm{slope}(\sqrt{p}\Lambda) = \mathrm{slope}(\Lambda)$ where p is prime. If we denote $\ell = \exp(\Lambda^*/\Lambda)$ then we have $\exp((\sqrt{p}\Lambda)^*/(\sqrt{p}\Lambda)) = p\ell$. We first consider the case $p \not| \ell$. Then $\gamma(p\ell) = (p+1)\gamma(\ell)$ and we have

$$\begin{aligned}
\mathrm{slope}(\sqrt{p}\Lambda) &= \frac{12}{n}\frac{1}{\gamma(p\ell)}\sum_{qr=p\ell}\frac{\phi((q,r))r}{(q,r)}w(\frac{q}{\sqrt{p}}\Lambda^* + \sqrt{p}\Lambda) \\
&= \frac{12}{n}\frac{1}{(p+1)\gamma(\ell)}\sum_{qr=p\ell}\frac{\phi((q,r))r}{(q,r)}\frac{1}{p}w(q\Lambda^* + p\Lambda) \\
&= \frac{12}{n}\frac{1}{(p+1)\gamma(\ell)}\sum_{uv=\ell}\left(\frac{\phi((pu,v))v}{(pu,v)}\frac{1}{p}w(pu\Lambda^* + p\Lambda) + \frac{\phi((u,pv))pv}{(u,pv)}\frac{1}{p}w(u\Lambda^* + p\Lambda)\right) \\
&= \frac{12}{n}\frac{1}{(p+1)\gamma(\ell)}\sum_{uv=\ell}\left(\frac{\phi((u,v))v}{(u,v)}\frac{1}{p}p^2w(u\Lambda^* + \Lambda) + \frac{\phi((u,v))pv}{(u,v)}\frac{1}{p}w(u\Lambda^* + \Lambda)\right) \\
&= \frac{12}{n}\frac{1}{\gamma(\ell)}\sum_{uv=\ell}\frac{\phi((u,v))v}{(u,v)}w(u\Lambda^* + \Lambda) = \mathrm{slope}(\Lambda).
\end{aligned}$$

We then consider the case $p|\ell$. Then $\gamma(p\ell) = p\gamma(\ell)$. Let $\ell = p^t m$ with $(p,m) = 1$. We first write out the summation for $\mathrm{slope}(\Lambda)$. Instead of summing over q and r such that

$qr = p^t m$, we let $q = p^j u$, $r = p^{t-j} v$ and sum over j, u and v such that $0 \le j \le t$ and $uv = m$. Thus we get

$$\begin{aligned}
\mathrm{slope}(\Lambda) &= \frac{12}{n}\frac{1}{\gamma(\ell)} \sum_{uv=m} \sum_{j=0}^{t} \phi((p^j u, p^{t-j} v)) \frac{p^{t-j} v}{(p^j u, p^{t-j} v)} w(p^j u \Lambda^* + \Lambda) \\
&= \frac{12}{n}\frac{1}{\gamma(\ell)} \sum_{uv=m} \frac{v\phi((u,v))}{(u,v)} \sum_{j=0}^{t} \frac{p^{t-j}\phi((p^j, p^{t-j}))}{(p^j, p^{t-j})} w(p^j u \Lambda^* + \Lambda) \\
&= \frac{12}{n}\frac{1}{\gamma(\ell)} \sum_{uv=m} \frac{v\phi((u,v))}{(u,v)} \sum_{j=0}^{t} \frac{p^{t-j}\phi(p^{\min(j,t-j)})}{p^{\min(j,t-j)}} w(p^j u \Lambda^* + \Lambda).
\end{aligned}$$

Here we have used the fact that $p \not| u$ and $p \not| v$, and the multiplicative property of ϕ. Using that $\frac{\phi(p^i)}{p^i} = \begin{cases} 1 & \text{if } i = 0 \\ \frac{p-1}{p} & \text{if } i > 0 \end{cases}$ we have

$$\text{(4.9)} \quad \mathrm{slope}(\Lambda) = \frac{12}{n}\frac{1}{\gamma(\ell)} \sum_{uv=m} \frac{v\phi((u,v))}{(u,v)} \sum_{j=0}^{t} p^{t-j} \left\{ \begin{matrix} 1 & \text{if } j = 0, t \\ \frac{p-1}{p} & \text{otherwise} \end{matrix} \right\} w(p^j u \Lambda^* + \Lambda).$$

Similarly, we have $\mathrm{slope}(\sqrt{p}\Lambda) =$

$$\begin{aligned}
&\frac{12}{n}\frac{1}{p\gamma(\ell)} \sum_{uv=m} \frac{v\phi((u,v))}{(u,v)} \sum_{j=0}^{t+1} p^{t+1-j} \left\{ \begin{matrix} 1 & \text{if } j = 0, t+1 \\ \frac{p-1}{p} & \text{otherwise} \end{matrix} \right\} w(\frac{p^j u}{\sqrt{p}} \Lambda^* + \sqrt{p}\Lambda). \\
&= \frac{12}{n}\frac{1}{p\gamma(\ell)} \sum_{uv=m} \frac{v\phi((u,v))}{(u,v)} \sum_{j=0}^{t+1} p^{t+1-j} \left\{ \begin{matrix} 1 & \text{if } j = 0, t+1 \\ \frac{p-1}{p} & \text{otherwise} \end{matrix} \right\} p w(p^{j-1} u \Lambda^* + \Lambda). \\
&= \frac{12}{n}\frac{1}{\gamma(\ell)} \sum_{uv=m} \frac{v\phi((u,v))}{(u,v)} \sum_{j=-1}^{t} p^{t-j} \left\{ \begin{matrix} 1 & \text{if } j = -1, t \\ \frac{p-1}{p} & \text{otherwise} \end{matrix} \right\} w(p^j u \Lambda^* + \Lambda).
\end{aligned}$$

In the inner sum, when $j = -1$, we have $p^{t+1} \cdot w(\frac{1}{p} u \Lambda^* + \Lambda) = p^{t-1} w(u\Lambda^* + p\Lambda) = p^{t-1} w(u\Lambda^* + \Lambda)$ because $(u, p) = 1$. Thus in the inner sum, combining $j = -1, 0$ yields $p^{t-1} w(u\Lambda^* + \Lambda) + p^t \frac{p-1}{p} w(u\Lambda^* + \Lambda) = p^t w(u\Lambda^* + \Lambda)$. This is the $j = 0$ term in the inner sum of equation 4.9 and so by comparison with equation 4.9, we are done. $\square$

Theorem 4.10. *Let Λ be an integral lattice with $\ell = \exp(\Lambda^*/\Lambda)$. We have the equality:* $\mathrm{slope}(\sqrt{\ell}\,\Lambda^*) = \mathrm{slope}(\Lambda)$.

Proof. We first reduce the Theorem to the case where Λ is primitive. Any integral lattice $\tilde{\Lambda}$ may be written as $\tilde{\Lambda} = \sqrt{N}\Lambda$ where Λ is primitive. If Λ has $\exp(\Lambda^*/\Lambda) = \ell$ then $\tilde{\Lambda}$ has $\exp(\tilde{\Lambda}^*/\tilde{\Lambda}) = \tilde{\ell} = N\ell$. Assuming the Theorem for primitive lattices we have $\mathrm{slope}(\sqrt{\tilde{\ell}}\tilde{\Lambda}^*) = \mathrm{slope}(\sqrt{N\ell}\frac{1}{\sqrt{N}}\Lambda^*) = \mathrm{slope}(\sqrt{\ell}\Lambda^*) = \mathrm{slope}(\Lambda)$. On the other hand using Proposition 4.8 we also have $\mathrm{slope}(\tilde{\Lambda}) = \mathrm{slope}(\sqrt{N}\Lambda) = \mathrm{slope}(\Lambda)$.

For a primitive Λ both Λ and $\sqrt{\ell}\Lambda^*$ have glue groups of exponent ℓ. From Lemma 4.3 we have

$$\operatorname{slope}(\sqrt{\ell}\Lambda^*) = \frac{12}{n}\frac{1}{[\Gamma_1 : \Gamma_0(\ell)]}\sum_{q,r\in\mathbb{Z}^+ :\, qr=\ell}\frac{r}{(q,r)}\,\phi\left((q,r)\right) w\left(q\,\left(\sqrt{\ell}\Lambda^*\right)^* + \sqrt{\ell}\Lambda^*\right).$$

Now we have $w\left(\frac{q}{\sqrt{\ell}}\Lambda + \sqrt{\ell}\Lambda^*\right) = \frac{q^2}{\ell} w\left(\Lambda + \frac{\ell}{q}\Lambda^*\right) = \frac{q}{r} w\left(r\,\Lambda^* + \Lambda\right)$. Hence, if we switch q and r, the above sum is the formula of Lemma 4.3 for slope(Λ). $\square$

With this Theorem we complete the computation of the slopes of the irreducible root lattices and their duals. The slopes of reducible lattices may be computed using the following Proposition whose proof we omit.

Proposition 4.11. *Let Λ_1, Λ_2 be integral lattices with ranks n_1, n_2, respectively. We have the equaltity:* $\operatorname{slope}(\Lambda_1 \oplus \Lambda_2) = \frac{n_1}{n_1+n_2}\operatorname{slope}(\Lambda_1) + \frac{n_2}{n_1+n_2}\operatorname{slope}(\Lambda_2)$.

§5. Main Theorem.

Our goal in this section is to prove that a Siegel modular cusp form f vanishes on all $s\tau$ for $\tau \in \mathcal{H}_1$ when we have slope$(f) <$ slope(s). We first introduce homomorphisms from rings of Siegel modular forms to elliptic modular forms.

Theorem 5.1. *Let ℓ, n, $N \in \mathbb{Z}^+$. Let Λ be an integral lattice of rank n with $\ell = \exp(\Lambda^*/\Lambda)$. The map ϕ_Λ^* is a graded ring homomorphism*

$$\phi_\Lambda^* : M_n\left(\Gamma_{n,0}(N)\right) \to M_1\left(\Gamma_{1,0}(N\ell)\right)$$

that multiplies weights by n and takes cusp forms to cusp forms.

Proof. Let $f \in M_n^k\left(\Gamma_{n,0}(N)\right)$ for integral weight k. Let s be a Gram matrix for Λ. For $\tau \in \mathcal{H}_1$ we have $\left(\phi_\Lambda^* f\right)(\tau) = \left(\phi_s^* f\right)(\tau) = f(s\tau)$ so that $\phi_s^* f$ is a composition of holomorphic functions. For $\sigma \in \Gamma_{1,0}(N\ell)$ we have $\left(\phi_s^* f\right) \underset{nk}{|} \sigma = \phi_s^*\left(f \underset{k}{|} \alpha_s(\sigma)\right)$ by equation 1.1 and $\phi_s^*\left(f \underset{k}{|} \alpha_s(\sigma)\right) = \phi_s^*\left(f\right)$ because $\alpha_s(\sigma) \in \Gamma_{n,0}(N)$. For projective rational $\sigma \in \mathrm{Sp}_1(\mathbb{R})$ the boundedness of $\left(\phi_s^* f\right)|\sigma = \phi_s^*\left(f|\alpha_s(\sigma)\right)$ on domains of type $\{\tau \in \mathcal{H}_1 : \Im\tau > y_0\}$ follows from the boundedness of $f|\alpha_s(\sigma)$ on domains of type $\{\Omega \in \mathcal{H}_n : \Im\Omega > y_0 s\}$, $\alpha_s(\sigma)$ being projective rational when σ is. These conditions show $\phi_s^* f \in M_1^{nk}\left(\Gamma_{1,0}(N\ell)\right)$ so that ϕ_s^* multiplies weights by n; substitution always defines a homomorphism.

To show that ϕ_s^* takes cusp forms to cusp forms let $f \in S_n^k\left(\Gamma_{n,0}(N)\right)$ and let σ be projective rational. Using equation 1.1 we have

$$\Phi_0\left(\left(\phi_s^* f\right)|\sigma\right) = \Phi_0\left(\phi_s^*\left(f|\alpha_s(\sigma)\right)\right) = \lim_{\lambda\to+\infty}\phi_s^*\left(f|\alpha_s(\sigma)\right)(i\lambda) = \lim_{\lambda\to+\infty}\left(f|\alpha_s(\sigma)\right)(i\lambda s).$$

The eigenvalues of λs go to infinity so this equals $\Phi_0^n\left(f|\alpha_s(\sigma)\right) = 0$, compare [10, p54]. $\square$

Let $\nu_\infty(f)$ be the order of the Fourier series of f in powers of $e\left(\tau\right)$.

Lemma 5.2. *Let Λ be an integral lattice of rank n and let $X \in \operatorname{Ext}^{\mathrm{sym}}(\Lambda, \Lambda^*; \mathbb{R})$. Let $f \in M_n^k$. We have $\nu_\infty\left(\phi_{\Lambda,X}^* f\right) \geq \mu(f)\, w(\Lambda)$.*

Proof. We have $\left(\phi_{\Lambda,X}^* f\right)(\tau) = f(s\tau + R)$ where for some basis M with $\Lambda = M\mathbb{Z}^n$ we have $s = M'M$ and $R = M'j\bar{X}M$. If $f(\Omega) = \sum_{T \in \mathrm{supp}(f)} a_T e(\langle T, \Omega\rangle)$ then $f(s\tau + R) = \sum_T a_T e(\langle T, R\rangle)\, e(\tau)^{\langle T, s\rangle}$ so that $\nu_\infty\left(\phi_{\Lambda,X}^* f\right) \geq \min_{T \in \mathrm{supp}(f)} \langle T, s\rangle$. Since we have $\langle T, s\rangle \geq m(T)w(s)$ we also have $\min_{T \in \mathrm{supp}(f)} \langle T, s\rangle \geq \min_{T \in \mathrm{supp}(f)} m(T)w(s) = \mu(f)w(s)$. $\square$

The next Lemma allows us to compute the Fourier expansion of $\phi_\Lambda^* f$ at each cusp of $\hat{X}_0(\ell)$ in terms of the Fourier expansion of f.

Lemma 5.3. *Let Λ be an integral lattice of rank n. Let $f \in M_n^k$ and $\sigma \in \Gamma_1$. We have*

$$(\phi_\Lambda^* f)\,|\sigma = \frac{1}{[\Lambda \,\square\sigma : \Lambda]^k}\ \ \phi^*_{\Lambda\,\square\sigma, X[\Lambda,\sigma]} f.$$

Proof. Write $\Lambda = M\mathbb{Z}^n$ and $s = M'M$. We have $(\phi_\Lambda^* f)\,|\sigma = (\phi_s^* f)\,|\sigma = \phi_s^*[f|\alpha_s(\sigma)]$. As in [5, p. 125] factor $\alpha_s(\sigma) = M_1 T$ where $M_1 \in \Gamma_n$ and $T = \begin{pmatrix} K^* & \beta \\ 0 & K \end{pmatrix} \in \Delta_n(\mathbb{Q})$. Then we have $((\phi_\Lambda^* f)|\sigma)(\tau) = (\phi_s^*[f|T])(\tau) = [f|T](s\tau) = \det(K)^{-k} f\left((K^* s\tau + \beta)K^{-1}\right) = \det(K)^{-k} f\left(K^* s K^{-1}\tau + \beta K^{-1}\right)$.

On the other hand, using the basis $\Lambda\,\square\sigma = MK^{-1}\mathbb{Z}^n$ and the representative $j\bar{X}[\Lambda,\sigma] = M^*\beta' K M^{-1}$ from Theorem 2.5 we also have

$$\begin{aligned}
\left(\phi^*_{\Lambda\,\square\sigma, X[\Lambda,\sigma]} f\right)(\tau) &= f\left((MK^{-1})'\tau(MK^{-1}) + (MK^{-1})' j\bar{X}[\Lambda,\sigma](MK^{-1})\right) \\
&= f\left(K^* M' M K^{-1}\tau + K^* M' M^* \beta' K M^{-1} M K^{-1}\right) = f\left(K^* s K^{-1}\tau + K^*\beta'\right) \\
&= f\left(K^* s K^{-1}\tau + \beta K^{-1}\right).
\end{aligned}$$

We have $K^*\beta' = \beta K^{-1}$ since T is symplectic. Clearly we have $[\Lambda\,\square\sigma : \Lambda] = \det(K)$. $\square$

Let Γ be any Fuchsian group of the first kind contained in Γ_1. We recall [18] the construction of the compact Riemann surface $\hat{X}(\Gamma)$. Let $\mathcal{H}_1$ have the standard topology and let $\hat{\mathcal{H}}_1 = \mathcal{H}_1 \cup \mathbb{P}^1(\mathbb{Q})$ have a basis of deleted neighborhoods about each point $x \in \mathbb{P}^1(\mathbb{Q})$ given by the open horodisks in $\mathcal{H}_1$ tangent to $\mathbb{R}$ at x. We define $\hat{X}(\Gamma) = \Gamma\backslash\hat{\mathcal{H}}_1$ and choose the minimal topology on $\hat{X}(\Gamma)$ such that the orbit map $\pi : \hat{\mathcal{H}}_1 \to \hat{X}(\Gamma)$ is open. With this topology $\hat{X}(\Gamma)$ is compact. The complex manifold structure on $\hat{X}(\Gamma)$ is given by the following charts: For generic points $\pi(\tau_0)$ with $\mathrm{Iso}_\Gamma(\tau_0) = e$ and any neighborhood N of τ_0 with the $\{\gamma N : \gamma \in \Gamma\}$ all disjoint a chart ψ_{N,τ_0} is given by the bijection: $\pi(N) \longleftrightarrow N$. For exceptional points $\pi(\tau_0)$ with $\mathrm{Iso}_\Gamma(\tau_0)$ cyclic of order $m > 1$ and any neighborhood N of τ_0 stable under $\mathrm{Iso}_\Gamma(\tau_0)$ with the $\{\gamma N : [\gamma] \in \Gamma/\,\mathrm{Iso}_\Gamma(\tau_0)\}$ all disjoint choose $\sigma \in \mathrm{Sp}_1(\mathbb{C})$ with $\sigma(\tau_0) = 0$ and $\sigma(\bar{\tau}_0) = \infty$; the chart ψ_{N,σ,τ_0} is given by the bijection: $\pi(N) \longleftrightarrow \{\sigma(\tau)^m : \tau \in N\}$. For cusps $\pi(x)$ choose $\sigma \in \mathrm{Sp}_1(\mathbb{Z})$ with $\sigma(\infty) = x$ and note that $\mathrm{width}_\ell(\sigma) = [\mathrm{Iso}_{\Gamma_1}(x) : \mathrm{Iso}_\Gamma(x)]$. For any neighborhood N of x stable under $\mathrm{Iso}_{\Gamma_1}(x)$

with the $\{\gamma N : [\gamma] \in \Gamma/\operatorname{Iso}_\Gamma(x)\}$ all disjoint the chart $\psi_{N,\sigma,x}$ is given by the bijection: $\pi(x) \cup \pi(N^*) \longleftrightarrow \{0\} \cup \{e\left(\sigma^{-1}(\tau)/\operatorname{width}_\ell(\sigma)\right) : \tau \in N^*\}$; here $N^* = N \setminus \{x\}$. One checks that the images of the charts are open in $\mathbb{C}$ and that the transition functions form an analytic atlas. Hence $\hat{X}(\Gamma) = \Gamma\backslash\hat{\mathcal{H}}_1$ is a compact Riemann surface and $X(\Gamma) = \Gamma\backslash\mathcal{H}_1$ is obtained from $\hat{X}(\Gamma)$ by removing the finite number of cusps $\Gamma\backslash\mathbb{P}^1(\mathbb{Q})$. The set $\Gamma\backslash\Gamma_1/\Delta_1$ is in bijection with the set of cusps $\Gamma\backslash\mathbb{P}^1(\mathbb{Q})$; we send $\Gamma\sigma\Delta_1$ to $\Gamma\sigma(\infty)$.

For every $p \in \hat{X}(\Gamma)$ define [18, p. 38] valuations $\nu_p : M_1^{\text{even}}(\Gamma) \to \mathbb{Q}$ as follows: For generic $\pi(\tau_0)$, $\nu_{\pi(\tau_0)}(f)$ is the order of the Taylor series of f in powers of $(\tau - \tau_0)$. For exceptional $\pi(\tau_0)$ with $\operatorname{Iso}_\Gamma(\tau_0)$ cyclic of order m, $\nu_{\pi(\tau_0)}(f)$ is $\frac{1}{m}$ times the order of the Taylor series of f in powers of $(\tau - \tau_0)$. For cusps $\pi(x)$ and $\sigma \in \operatorname{Sp}_1(\mathbb{Z})$ with $\sigma(\infty) = x$, $\nu_{\pi(x)}(f)$ is the order of the Fourier series of $f|\sigma$ in powers of $e\left(\tau/\operatorname{width}_\ell(\sigma)\right)$. This notation almost coincides with our earlier notation when $x = \infty$ because $\nu_{\pi(\infty)}(f) = \nu_\infty(f)$ is the order of the Fourier series of f in powers of $e\left(\tau\right)$. The importance of these valuations lies in the following bound on the total vanishing order of a nontrivial $g \in M_1^k(\Gamma)$ of even weight k (combine [18, p. 39] Proposition 2.16 and [18, p. 23] Proposition 1.40):

$$[\Gamma_1 : \Gamma]\frac{k}{12} = \sum_{p \in \hat{X}(\Gamma)} \nu_p(g). \tag{5.4}$$

Theorem 5.5 (Main Result). *Let $f \in S_n^k$ be a Siegel modular cusp form and let Λ be an integral lattice of rank n. If we have* $\operatorname{slope}(f) < \operatorname{slope}(\Lambda)$ *then we have $\phi_\Lambda^* f = 0$.*

Proof. By restricting the summation in equation 5.4 to the cusps of $\hat{X}(\Gamma_0(\ell))$ we have for nontrivial $g \in M_1^k(\Gamma_0(\ell))$ and for even k:

$$[\Gamma_1 : \Gamma_0(\ell)]\frac{k}{12} \geq \sum_{[\sigma] \in \Gamma_0(\ell)\backslash\Gamma_1/\Delta_1} \nu_{\pi(\sigma(\infty))}(g).$$

We apply this inequality to $\phi_\Lambda^* f \in M_1^{nk}(\Gamma_0(\ell))$ noting that nk is even for nontrivial f:

$$[\Gamma_1 : \Gamma_0(\ell)]\frac{nk}{12} \geq \sum_{[\sigma] \in \Gamma_0(\ell)\backslash\Gamma_1/\Delta_1} \nu_{\pi(\sigma(\infty))}(\phi_\Lambda^* f).$$

We now apply $\nu_{\pi(\sigma(\infty))}(\phi_\Lambda^* f) = \operatorname{width}_\ell(\sigma)\nu_\infty\left(\phi_\Lambda^* f|\sigma\right) = \operatorname{width}_\ell(\sigma)\nu_\infty\left(\phi^*_{\Lambda\,\Box\sigma, X[\Lambda,\sigma]} f\right) \geq \operatorname{width}_\ell(\sigma)\mu\left(f\right)w\left(\Lambda\,\Box\sigma\right)$. The first equality is the definition of $\nu_{\pi(\sigma(\infty))}$, the second equality is given by Lemma 5.3 and the inequality is given by Lemma 5.2.

Therefore we have

$$[\Gamma_1 : \Gamma_0(\ell)]\frac{nk}{12} \geq \sum_{[\sigma] \in \Gamma_0(\ell)\backslash\Gamma_1/\Delta_1} \operatorname{width}_\ell(\sigma)\mu(f)w(\Lambda\,\Box\sigma)$$

and

$$\frac{k}{\mu(f)} \geq \frac{12}{n}\frac{1}{[\Gamma_1 : \Gamma_0(\ell)]} \sum_{[\sigma] \in \Gamma_0(\ell)\backslash\Gamma_1/\Delta_1} \operatorname{width}_\ell(\sigma)w(\Lambda\,\Box\sigma)$$

or $\operatorname{slope}(f) \geq \operatorname{slope}(\Lambda)$ for nontrivial $\phi_\Lambda^* f$. $\square$

§6. Conclusion.

To see that our results can be sharp consider S^8_4. The Schottky modular form J spans S^8_4, see [17][13][3], and has slope 8. The lattice D_4 also has slope 8 and from the following definition [8] of J (write $\frac{1}{2} = x$):

$$J(\Omega) = r_{00}^2 + r_{0x}^2 + r_{x0}^2 - 2\left(r_{00}r_{0x} + r_{00}r_{x0} + r_{0x}r_{x0}\right),$$
$$r_{\mu\nu} = \prod_{\alpha,\beta,\gamma\in\{0,x\}}^{8} \theta\left[\begin{matrix}\mu & 0 & 0 & 0\\ \nu & \alpha & \beta & \gamma\end{matrix}\right](0,\Omega) \quad \text{for } \mu,\nu \in \{0,x\}$$

one can show that

$$\left(\phi^*_{D_4} J\right)(\tau) = 2^{-16}\theta\left[\begin{matrix}0\\0\end{matrix}\right](\tau)^{16}\theta\left[\begin{matrix}0\\x\end{matrix}\right](\tau)^{16}\theta\left[\begin{matrix}x\\0\end{matrix}\right](\tau)^{32}$$

and so $\phi^*_{D_4} J \in S^{32}_1(\Gamma_0(2))$ is not zero. Since $\phi^*_{D_4} J$ vanishes only at the cusps this incidentally shows that no $\mathrm{Gram}(D_4)\tau$ is a Jacobian.

To give an application to the determination of spaces of cusp forms consider S^6_5. Duke and Imamoḡlu [3] proved that $S^6_n = 0$ for all n by using L-functions and explicit formulae. We give a second proof that $S^6_5 = 0$. According to [14, p. 218] an $f \in S^k_n$ is determined by the subset of Fourier coefficients a_T that satisfy $w(T) \le n\frac{2}{\sqrt{3}}\frac{k}{4\pi}$. We have $5\frac{2}{\sqrt{3}}\frac{6}{4\pi} \approx 2.76$ and the only semi-integral class $[T]$ with $w(T) < 3$ is represented by $\frac{1}{2}\,\mathrm{Gram}(D_5)$. Hence the map from S^6_5 to $\mathbb{C}$ given by $f \mapsto a_{\frac{1}{2}\mathrm{Gram}(D_5)}(f)$ is an isomorphism. By Propositions 4.5 and 4.10 we have $\mathrm{slope}(2D^*_5) = \mathrm{slope}(D_5) = 8$ whereas an $f \in S^6_5$ has $\mathrm{slope}(f) \le 6$ so that by Theorem 5.5 we have $\phi^*_{2D^*_5} f = 0$. The Fourier expansion of $\phi^*_{2D^*_5} f$, however, is given by $\left(\phi^*_{2D^*_5} f\right)(\tau) = a_{\frac{1}{2}\mathrm{Gram}(D_5)} q^{10} + O(q^{11})$ where $q = e(\tau)$ so that $a_{\frac{1}{2}\mathrm{Gram}(D_5)}$ and hence f itself vanish. This shows that $S^6_5 = 0$. The above Fourier expansion is an immediate consequence of the arithmetic-geometric inequality $\langle 4\,\mathrm{Gram}(D_5)^{-1}, T\rangle \ge 5\delta(4\,\mathrm{Gram}(D_5)^{-1})\delta(T)$ where $\delta(s) = \det(s)^{1/n}$, equality holding only when T is a multiple of $\mathrm{Gram}(D_5)$. Our results are not as deep as the results mentioned in the Introduction on the hyperelliptic and trigonal loci because their function fields are not so simple as that of $\hat{X}_0(\ell)$. However, it is more serviceable to have many simple homomorphisms ϕ^*_Λ than to have a few complicated ones. For example, although slope considerations show that any $f \in S^6_5$ must vanish on the hyperelliptic locus in $\mathcal{A}_5$ it is not elementary to relate the Fourier coefficients of f to the image of f in the ring $S(2,12)$ of binary invariants.

Finally consider the lattice D_n of slope 8. There are interesting Siegel modular cusp forms of slope less than 8 when $n \ge 5$. Following Mumford [11] let N_0 denote the divisor on $\mathcal{A}_n$ containing principally polarized abelian varieties possessing a singular theta locus, N^*_0 denote the divisor on $\mathcal{A}_n$ containing principally polarized abelian varieties that have some nontwo-torsion in the singular part of the theta locus and Θ_{null} the divisor on $\mathcal{A}_n$ containing principally polarized abelian varieties that have two-torsion in the singular part of the theta locus, i.e., a vanishing even thetanull. In the Picard group $\mathrm{Pic}(\bar{\mathcal{A}}^{(1),0}_n)$ of a partial compactification $\bar{\mathcal{A}}^{(1),0}_n$ of $\mathcal{A}_n$ the divisor classes are related by $[\bar{N}_0] = [\Theta_{\mathrm{null}}] + 2[\bar{N}^*_0]$.

The support of each of these divisors is the zero set of some Siegel modular form by a result of Freitag [4] and the slope k/μ of a Siegel modular cusp form in our sense equals the "slope" of its associated divisor class in the sense of Mumford, see comments in [11]. The slope of the divisor class $\bar{N}_0^*$ is [11, p. 368]

$$6\frac{1+\frac{2}{n+1}-\frac{2^{n-1}(2^n+1)}{(n+1)!}}{1-2^{2n-6}\frac{24}{(n+1)!}},$$

asymptotic to 6 and less than 8 for $n \geq 5$. By Theorem 5.5 we have $[\mathrm{Gram}(D_n)\tau] \in \bar{N}_0^*$. Thus for $n \geq 5$ the principally polarized abelian variety $\mathbb{C}^n/\left(\mathbb{Z}^n + \mathrm{Gram}(D_n)\tau\mathbb{Z}^n\right)$ has a singular theta locus and every deleted neighborhood of $\mathbb{C}^n/\left(\mathbb{Z}^n + \mathrm{Gram}(D_n)\tau\mathbb{Z}^n\right)$ in $\mathcal{A}_n$ contains principally polarized abelian varieties with nontwo-torsion in the singular part of the theta locus. This result, achieved through numerical criterion, seems difficult to approach in any other way. Actually it seems period matrices like $\mathrm{Gram}(D_n)\tau$ are quite special. In $n = 5$ we can show that $\mathrm{Gram}(D_5)\tau$ has 60 vanishing even thetanulls and so $[\mathrm{Gram}(D_5)\tau] \in \bar{N}_0^* \cap \Theta_{\mathrm{null}}$.

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Department of Mathematics, Fordham University, Bronx, NY 10458
Email: poor@fordham.edu

Math/CS Department, Lake Forest College, 555 N. Sheridan Rd., Lake Forest, IL 60045
Email: yuen@lfc.edu