LINEAR DEPENDENCE AMONG SIEGEL MODULAR FORMS

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Abstract. Theorems are given which describe when high enough vanishing at the cusps implies that a Siegel modular cusp form is zero. Formerly impractical computations become practical and examples are given in degree four. Vanishing order is described by kernels, a type of polyhedral convex hull.

§0. Introduction.

This paper extends to Siegel modular forms certain practical computational techniques available for modular forms on the upper half plane. Two modular forms are equal when enough of their Fourier coefficients agree; more generally, a linear dependence relation holds among modular forms when it holds among enough of their Fourier coefficients. For example, in [30] Schiemann shows that the theta series for two distinct classes of 4×4 integral positive definite quadratic forms are equal by showing that their first 375 Fourier coefficients agree. The type of theorem one requires is that a cusp form is zero if it vanishes to a sufficiently large order; in the above example the cusp form in question is given by the difference of the theta series. In the case of Siegel modular forms, Siegel provided a version of the following result for the full modular group.

Theorem (Siegel). *Let $f \in S_n^k$ have the Fourier expansion $f(\Omega) = \sum_{s>0} a_s e(\mathrm{tr}(s\Omega))$. The following conditions are equivalent.*

(1) $f = 0$.
(2) *For all s such that $\mathrm{tr}(s) \le \kappa_n \frac{k}{4\pi}$, we have $a_s = 0$.*
(3) *For all s such that $\mathrm{tr}(s) \le n\mu_n^n \frac{2}{\sqrt{3}} \frac{k}{4\pi}$, we have $a_s = 0$.*
(4) *For all s such that $\det(s)^{1/n} \le \mu_n^n \frac{2}{\sqrt{3}} \frac{k}{4\pi}$, we have $a_s = 0$.*

Here S_n^k is the $\mathbb{C}$-vector space of Siegel modular cusp forms of weight k on the Siegel upper half space, $\mathcal{H}_n$, we denote $e^{2\pi i z}$ by $e(z)$ and the positive constant κ_n is defined by $\kappa_n = \sup \mathrm{tr}((\mathrm{Im}\,\Omega)^{-1})$ where the supremum is taken over $\Omega \in \mathcal{F}_n$, Siegel's fundamental domain. This theorem shows that the vanishing of a finite number of Fourier coefficients a_s, those for which $\mathrm{tr}(s) \le \kappa_n \frac{k}{4\pi}$, implies that $f \equiv 0$. This theorem however is very impractical for $n > 1$. First of all, the vanishing of a Fourier coefficient a_s depends only upon the $\mathrm{GL}_n(\mathbb{Z})$ equivalence class of s but when $n > 1$ the trace is not a class function. Secondly, the upper bounds known for κ_n when $n > 1$ are probably much larger than κ_n.

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This means that many unnecessary Fourier coefficients must be computed in applications of Siegel's Theorem. A result of this paper which remedies both these ills replaces the trace $\mathrm{tr}(s)$ with the dyadic trace $w(s)$.

Theorem 2.9. *Let $f \in S_n^k$ have the Fourier expansion $f(\Omega) = \sum_{s>0} a_s e(\mathrm{tr}(s\Omega))$. The following conditions are equivalent.*

1. *$f = 0$.*
2. *For all s such that $w(s) \le w_n \frac{k}{4\pi}$, we have $a_s = 0$.*
3. *For all s such that $w(s) \le n \frac{2}{\sqrt{3}} \frac{k}{4\pi}$, we have $a_s = 0$.*
4. *For all s such that $\det(s)^{1/n} \le \mu_n \frac{2}{\sqrt{3}} \frac{k}{4\pi}$, we have $a_s = 0$.*

The dyadic trace w is defined and studied in detail in §3, here we just mention the following characterization:

$$w(s) = \inf_{Y>0} \frac{\mathrm{tr}(sY)}{m(Y)}$$

where m is Hermite's function defined by $m(Y) = \min_{x \in \mathbb{Z}^n \setminus 0} {}^t xYx$. The optimal constant μ_n in $m(s) \le \mu_n \det(s)^{1/n}$ has been the object of much study [34][4].

Unlike the trace, w is a class function and $w_n = \sup_{\Omega \in \mathcal{H}_n} \inf_{\sigma \in \Gamma_n} w\left(\{\mathrm{Im}(\sigma\Omega)\}^{-1}\right)$ has good known bounds, $n \le w_n \le \frac{2}{\sqrt{3}} n$. Theorem 2.9 is indeed more practical than Siegel's theorem for $n > 1$ as can be seen in many examples. To determine the linear span in S_4^{12} of the theta series attached to the Niemeier lattices using Theorem 2.9 requires 23 Fourier coefficients to be computed for each Niemeier lattice whereas the use of Siegel's Theorem requires over $48,000$ apiece. See §5 for this example and further comparisons.

Theorems analogous to Theorem 2.9 and Siegel's Theorem can be obtained for a broad class of functions ϕ which we call type two, see Definition 2.3. Besides $\mathrm{tr}(s)$ and $w(s)$, type two functions include $m(s)$ and $\det(s)^{1/n}$. The computation of relevant constants like $m_n = \sup_{\Omega \in \mathcal{H}_n} \inf_{\sigma \in \Gamma_n} m\left(\{\mathrm{Im}(\sigma\Omega)\}^{-1}\right)$ and $\det_n = \sup_{\Omega \in \mathcal{H}_n} \inf_{\sigma \in \Gamma_n} \det\left(\{\mathrm{Im}(\sigma\Omega)\}^{-1}\right)^{\frac{1}{n}}$ is an interesting question in the symplectic geometry of numbers. For reasons not understood, however, no type two ϕ seems to give better results than Theorem 2.9 corresponding to the choice of $\phi(s) = w(s)$. Theorem 2.9 can be generalized to apply to half-integral weights, characters, and subgroups of finite index. The extension to subgroups is the most interesting. Instead of a vanishing condition on one Fourier expansion we need an average vanishing condition on the expansions corresponding to each cusp, see Theorem 2.6.

There is an essential reason, of independent interest, why theorems like Theorem 2.9 and Siegel's Theorem exist for any type two function. Let $f \in S_n^k$ be a nontrivial cusp form with Fourier expansion $f(\Omega) = \sum a_s e(\mathrm{tr}(s\Omega))$. Let $\mathrm{supp}(f) = \{s : a_s \ne 0\}$ and let $\nu(f)$ be the closure of the convex hull of $\mathbb{R}_{\ge 1}\,\mathrm{supp}(f)$ inside $\mathcal{P}_n^{\mathrm{semi}}(\mathbb{R})$, the cone of semidefinite, symmetric, $n \times n$ matrices over $\mathbb{R}$. In Lemma 1.1 we see that $\nu(f)$ is a *kernel* in the sense of [1, p.120]. The function $\phi_f(\Omega) = \det(\mathrm{Im}\,\Omega)^{k/2} |f(\Omega)|$ is known to attain a maximum at some point, say $\Omega_0 = X_0 + iY_0 \in \mathcal{H}_n$. The essential new result is:

$$\frac{k}{4\pi} Y_0^{-1} \in \nu(f) = \text{closure of the convex hull of } \mathbb{R}_{\ge 1}\,\mathrm{supp}(f) \tag{0.1}$$

This is the Semihull Theorem 1.2. By applying a type two ϕ to both sides of 0.1 one obtains Theorems 2.5 and 2.6. The type two functions ϕ are thus merely an expedient to enhance computations. One can avoid type two functions and use 0.1 directly to yield a Theorem in which high vanishing implies that a cusp form is zero, see Theorem 1.6. For

$n > 1$ Theorem 1.6 is not very practical but a description is still worthwhile: Let Γ be a subgroup of finite index in Γ_n and $f \in S_n^k(\Gamma)$. The function $\phi_f(\Omega) = \det(Y)^{k/2}|f(\Omega)|$ has a maximum which cannot be attained in some deleted neighborhood of each cusp because f is a cusp form. So any point $\Omega_0 = X_0 + iY_0$ where ϕ_f attains its maximum cannot be in these neighborhoods. Theorem 1.6 gives an explicit description, in terms of the vanishing of the Fourier expansion corresponding to each cusp, of the deleted neighborhoods of the cusps forbidden to Ω_0. *When the vanishing of f at the cusps is high enough that these forbidden neighborhoods cover a fundamental domain for Γ then no nontrivial cusp forms with this vanishing can exist.* In general these coverings pose delicate questions but when $n = 1$ this description can be worked out directly because the forbidden neighborhoods are horocircles.

Here is an outline of the paper. In §1 we prove our main result, Theorem 1.3, generalizing 0.1 above to each cusp. We interpret our main result in terms of coverings in Theorem 1.6 which gives an explicit description of the neighborhoods of the cusps forbidden to the extreme points Ω_0. In §2 we use the main result 1.3 of §1 to produce theorems like the theorem of Siegel mentioned in the Introduction for any type two function. We describe four such theorems for the type two functions: the trace, Hermite's function, the dyadic trace, and the reduced determinant. The dyadic trace version is the most efficient. The version for Hermite's function has a relation to the theory of toroidal compactifications of moduli space. In §3 we describe the properties of the dyadic trace from a computational point of view. These properties include: the domain of definition, class invariance, characterizations as both an infimum and a supremum which are attained, an inequality with the reduced determinant, and rationality. In §4 we prove some formulae for the dyadic trace $w(s)$ when $n = 2, 3$ and also make tables of quadratic forms with low dyadic trace in $n = 3, 4$. In §5 we discuss examples of explicit computations with Siegel modular forms.

We now fix notations and list elementary results. We let $V_n(\mathbb{F}) = M_{n\times n}^{\mathrm{sym}}(\mathbb{F})$ for $\mathbb{F} = \mathbb{C}, \mathbb{R}, \mathbb{Q}$, or $\mathbb{Z}$. We let $\mathcal{P}_n(\mathbb{F})$ and $\mathcal{P}_n^{\mathrm{semi}}(\mathbb{F})$ denote the definite and semidefinite matrices in $V_n(\mathbb{F})$. We let $e_{ij} \in M_{n\times n}(\mathbb{Z})$ be the standard basis. For $A, B \in V_n(\mathbb{C})$, we set $\langle A, B\rangle = \mathrm{tr}(AB)$. A **cone** in $V_n(\mathbb{R})$ is an $\mathbb{R}_{\geq 0}$-semigroup. The inclusion reversing duality operation relevant to cones is:

$$C^\vee = \{x \in V_n(\mathbb{R}) : \text{ for all } y \in C, \langle x, y\rangle \geq 0\}.$$

For any nonempty set C, the set $C^\vee$ is always a cone and the set $C^{\vee\vee}$ is the smallest closed cone containing C. So if C is a nonempty closed cone then $C^{\vee\vee} = C$. A cone C contains an open set if and only if $C^\vee$ does not contain a line. Examples of cones are $\mathcal{P}_n(\mathbb{R})$ and $\mathcal{P}_n^{\mathrm{semi}}(\mathbb{R})$. We have $\mathcal{P}_n^{\mathrm{semi}}(\mathbb{R})^\vee = \mathcal{P}_n^{\mathrm{semi}}(\mathbb{R})$ and that $\mathcal{P}_n^{\mathrm{semi}}(\mathbb{R})$ is the closure of $\mathcal{P}_n(\mathbb{R})$.

We define the concept of **kernel** only for sets which are closed and convex in $\mathcal{P}_n^{\mathrm{semi}}(\mathbb{R})$, for these sets our definition coincides with that in [1, p.120]. A convex, closed set $K \subseteq \mathcal{P}_n^{\mathrm{semi}}(\mathbb{R})$ is a kernel if:

(1) $\mathbb{R}_{\geq 1}K = K$,
(2) $0 \notin K$,
(3) $\mathbb{R}_{>0}K \supseteq \mathcal{P}_n(\mathbb{R})$.

The inclusion reversing duality operation relevant to kernels is:

$$K^\sqcup = \{x \in V_n(\mathbb{R}) : \text{ for all } y \in K, \langle x, y\rangle \geq 1\}.$$

If K is a kernel then $K^\sqcup$ is a kernel and $K^{\sqcup\sqcup} = K$. Kernels are $\mathbb{R}_{\geq 1}$-semigroups. The set $K_{\mathbb{Q}} = K \cap \mathcal{P}_n^{\mathrm{semi}}(\mathbb{Q})$ is dense in K and $(K_{\mathbb{Q}})^\sqcup = K^\sqcup$. We have $K^{\vee\vee} = \mathbb{R}_{\geq 0}K$ because each is the smallest closed cone containing K.

For any function f on $\mathcal{H}_n$ and $M \in \mathrm{Sp}_n(\mathbb{R})$ we set $(f|M)(\Omega) = f(M\Omega)\det(C\Omega + D)^{-k}$. A modular form f in $M_n^k(\Gamma, \chi)$ is a holomorphic function on $\mathcal{H}_n$ which satisfies $f|M = \chi(M)f$ for all M in Γ. Here χ is some character, a homomorphism $\chi : \Gamma \to e(\mathbb{Q})$. We also require $f|M$ to be bounded in regions of the type $\{\Omega : \mathrm{Im}\,\Omega > Y_0\}$ for all $M \in \Gamma_n$. A cusp form f in $S_n^k(\Gamma, \chi)$ is an $f \in M_n^k(\Gamma, \chi)$ which satisfies $\Phi(f|M) = 0$ for all M in Γ_n, where Φ is the standard Siegel operator, see [11, p.45]. These definitions make sense when k is an integer. When $k \in \frac{1}{2}\mathbb{Z}$ the usual transformation condition [21, p.200] is that f transforms like $\theta(0, \Omega)$ under Γ. The theorems in this paper hold in both cases.

The notation for special subgroups of $\Gamma_n = \mathrm{Sp}_n(\mathbb{Z})$ is: $\Gamma_n(\ell) = \{\sigma \in \Gamma_n : \sigma \equiv I_{2n} \bmod \ell\}$, $\Delta_n = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n : C = 0\}$, $S\Delta_n = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n : C = 0, \det(A) = 1\}$, $\Delta_n(\ell) = \Delta_n \cap \Gamma_n(\ell)$, $S\Delta_n(\ell) = S\Delta_n \cap \Gamma_n(\ell)$. We note that the $\Gamma_n(\ell)$ are finitely generated and that, for $n \geq 2$, any Γ of finite index in Γ_n contains some $\Gamma_n(\ell)$, [20]. For any $\Gamma \subseteq \Gamma_n$ we define $u(\Gamma) \subseteq \mathrm{GL}_n(\mathbb{Z})$ by $u(\Gamma) = \{A \in \mathrm{GL}_n(\mathbb{Z}) : \exists \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \in \Gamma\}$; in the same way we define $t(\Gamma) \subseteq V_n(\mathbb{Z})$ by $t(\Gamma) = \{B \in V_n(\mathbb{Z}) : \exists \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \in \Gamma\}$. These operations u and t cannot increase group index, [11, p.128].

In this paper we especially study holomorphic functions f invariant under $S\Delta_n(\ell)$ for some $\ell \in \mathbb{Z}^+$. For example, $\theta(0, \Omega)$ is invariant under $\Delta_n(2)$. For $n \geq 2$, the Koecher Principle [11, p.175] provides a Fourier expansion of the form

$$f(\Omega) = \sum_{s \in \mathcal{P}_n^{\mathrm{semi}}(\mathbb{Q})} a_s e(\langle s, \Omega \rangle).$$

In the case $n = 1$ we must use the boundedness hypothesis to obtain the same result. We let $\mathrm{supp}(f) = \{s \in \mathcal{P}_n^{\mathrm{semi}}(\mathbb{Q}) : a_s \neq 0\}$; we note that the above Fourier series need only be summed over $s \in \mathrm{supp}(f)$ and that the elements of $2\ell\,\mathrm{supp}(f)$ are even quadratic forms in $\mathcal{P}_n^{\mathrm{semi}}(\mathbb{Z})$. For any $S \subseteq \mathcal{P}_n^{\mathrm{semi}}(\mathbb{R})$ we define $\mathrm{Semihull}(S) = \mathrm{closure}\{$ convex hull$(R_{\geq 1} S)\}$. Applying this concept to $S = \mathrm{supp}(f)$ we define $\nu(f) = \mathrm{Semihull}[\mathrm{supp}(f)]$. This notion $\nu(f)$ measures the order of vanishing of f at the cusp at infinity. When f and g are nontrivial cusp forms we have $\nu(fg) = \nu(f) + \nu(g)$. Our first lemma says that $\nu(f)$ is a *kernel* when f has no constant term in a nontrivial Fourier expansion.

§1. The Semihull Theorem.

1.1 Lemma. (Kernel Lemma) *Let $f : \mathcal{H}_n \to \mathbb{C}$ be holomorphic, invariant under $S\Delta_n(\ell)$ for some $\ell \in \mathbb{Z}^+$, and have $0 \notin \mathrm{supp}(f)$. If $n = 1$, we further assume that $0 < \mathrm{supp}(f)$. Then either $\nu(f)$ is a kernel or f is identically zero.*

Proof. For $n \geq 2$, the Koecher principle assures us that $\mathrm{supp}(f) \subseteq \mathcal{P}_n^{\mathrm{semi}}(\mathbb{Q})$, in the case $n = 1$ we must use our assumption that $0 \leq \mathrm{supp}(f)$ to obtain the same conclusion. Let $K = \nu(f)$. We have $K \subseteq \overline{\mathcal{P}_n^{\mathrm{semi}}(\mathbb{R})} = \mathcal{P}_n^{\mathrm{semi}}(\mathbb{R})$ and that K is closed and convex; K is a kernel if and only if we have (1) $\mathbb{R}_{\geq 1} K = K$, (2) $0 \notin K$, (3) $\mathbb{R}_{>0} K \supseteq \mathcal{P}_n(\mathbb{R})$. Item (1) holds automatically because K is the semihull of a set. Item (2) holds because f is periodic with respect to the translation lattice $t(S\Delta_n(\ell))$ so that the elements of $2\ell\,\mathrm{supp}(f) \subseteq \mathcal{P}_n^{\mathrm{semi}}(\mathbb{Z}) \backslash \{0\}$ are all even quadratic forms. Here we have used the hypothesis $0 \notin \mathrm{supp}(f)$. This implies that for all $s \in \mathrm{supp}(f)$ we have $\mathrm{tr}(s) \geq \frac{1}{\ell}$ and hence that for all $x \in K$ we have $\mathrm{tr}(x) \geq \frac{1}{\ell}$. Therefore we have item (2). The main condition to be proven is item (3).

If we are in the case $n = 1$ then using our assumption $0 < \mathrm{supp}(f)$ we have either $K = [\frac{m}{\ell}, \infty)$ for some $m \in \mathbb{Z}^+$ or $K = \emptyset$. If $K = [\frac{m}{\ell}, \infty)$ then item (3) holds because $\mathbb{R}_{>0}K = (0, \infty) = \mathcal{P}_1(\mathbb{R})$; if $K = \emptyset$ then $f \equiv 0$. ¿From now on we assume that $n \geq 2$. If it is true that $K^\vee \subseteq \mathcal{P}_n^{\mathrm{semi}}(\mathbb{R})$ then we have $\mathbb{R}_{\geq 0}K = K^{\vee\vee} \supseteq \left(\mathcal{P}_n^{\mathrm{semi}}(\mathbb{R})\right)^\vee = \mathcal{P}_n^{\mathrm{semi}}(\mathbb{R})$ so that $\mathbb{R}_{>0}K \supseteq \mathcal{P}_n^{\mathrm{semi}}(\mathbb{R}) \setminus \{0\} \supset \mathcal{P}_n(\mathbb{R})$ and item (3) holds.

If it is not true that $K^\vee \subseteq \mathcal{P}_n^{\mathrm{semi}}(\mathbb{R})$ then there is a $T \in K^\vee$ that is not semidefinite and we will show that $f \equiv 0$. Select an $A \in \mathrm{SL}_n(\mathbb{Z})$: such that $P = A^{-1} T\, {}^t\!A^{-1}$ has $P_{11} < 0$ [18, p.45]. Let $\gamma \in S\Delta_n$ be defined by $\gamma(\Omega) = A\Omega\, {}^t\!A$; then $f|\gamma$ is also $S\Delta_n(\ell)$ invariant because $S\Delta_n(\ell)$ is normal in Δ_n. We have $\mathrm{supp}(f|\gamma) = {}^t\!A\, \mathrm{supp}(f) A$, $\nu(f|\gamma) = {}^t\!A \nu(f) A = {}^t\!A K A$, $\nu(f|\gamma)^\vee = A^{-1} K^\vee\, {}^t\!A^{-1}$, and $P = A^{-1} T\, {}^t\!A^{-1} \in A^{-1} K^\vee\, {}^t\!A^{-1} = \nu(f|\gamma)^\vee$.

If it is true that $\mathrm{supp}(f|\gamma) \subseteq \mathbb{R}_{\geq 0} e_{11}$ then we can show that $f \equiv 0$. Let $E = I_n + \ell e_{11} - \ell e_{22} + \ell e_{12} - \ell e_{21} \in u(S\Delta_n(\ell))$ and let $h \in S\Delta_n(\ell)$ be defined by $h(\Omega) = E\Omega\, {}^t\!E$. The invariance $f|\gamma|h = f|\gamma$ implies that $\mathrm{supp}(f|\gamma|h) = \mathrm{supp}(f|\gamma)$ or ${}^t\!E\, \mathrm{supp}(f|\gamma) E = \mathrm{supp}(f|\gamma)$. However, for $s = \lambda e_{11} \in \mathbb{R}_{\geq 0} e_{11}$ we have

$$ {}^t\!E s E = \lambda\{(1 + 2\ell + \ell^2) e_{11} + (\ell + \ell^2)(e_{12} + e_{21}) + \ell^2 e_{22}\} $$

so that we must have $\lambda = 0$ if ${}^t\!EsE \in \mathbb{R}_{\geq 0} e_{11}$. Since $0 \notin \mathrm{supp}(f|\gamma)$ we have $\mathrm{supp}(f|\gamma) = \emptyset$, $f|\gamma \equiv 0$, and $f \equiv 0$.

Let us suppose by contradiction that $\mathrm{supp}(f|\gamma) \not\subseteq \mathbb{R}_{\geq 0} e_{11}$. Then there is a $\sigma \in \mathrm{supp}(f|\gamma)$ with $\sigma_{mm} > 0$ for $m > 1$. Let $E_k = I_n + k\ell e_{1m} \in u(S\Delta_n(\ell))$ and let $\delta_k = E_k \sigma\, {}^t\!E_k$. The δ_k are all in $\mathrm{supp}(f|\gamma)$ and are in fact distinct for all sufficiently large k because

$$ \delta_k = (I_n + k\ell e_{1m})\, \sigma\, (I_n + k\ell e_{m1}) = O(k) + k^2 \ell^2 \sigma_{mm} e_{11} $$

so that the $(1,1)$-entry is eventually increasing. Let N be such that the δ_k are distinct for $k > N$. Recall that we have a $P \in \nu(f|\gamma)^\vee$ with $P_{11} < 0$. For any $\zeta \in \mathcal{H}_1$, we can let $\Omega_\zeta = iI_n + \zeta P \in V_n(\mathbb{C})$. The Fourier series of $f|\gamma$ converges absolutely at Ω_ζ in view of $\langle P, \mathrm{supp}(f|\gamma)\rangle \geq 0$ and the estimate:

$$ \sum_{s \in \mathrm{supp}(f|\gamma)} |a_s e(\langle s, \Omega_\zeta \rangle)| \leq \sum_{s \in \mathrm{supp}(f|\gamma)} |a_s| |e(\langle s, iI_n \rangle)|. $$

We will show the divergence of the subseries

$$ \sum_{k > N} |a_{\delta_k}| |e(\langle \delta_k, \Omega_\zeta \rangle)| = |a_\sigma| \sum_{k > N} |e(\langle \delta_k, \Omega_\zeta \rangle)|. $$

We compute $\mathrm{Im}\langle \delta_k, \Omega_\zeta \rangle = \mathrm{Im}\, \mathrm{tr}\, (\delta_k(iI_n + \zeta P)) = O(k) + k^2 \ell^2 \sigma_{mm}(1 + P_{11}\, \mathrm{Im}\, \zeta)$. Since $P_{11} < 0$ if we select $\mathrm{Im}\, \zeta$ large enough then $1 + (\mathrm{Im}\, \zeta) P_{11} < 0$ and the subseries diverges. $\square$

Kernels provide the correct point of view for our theorems.

1.2 Theorem. (Semihull Theorem) *Let $f : \mathcal{H}_n \to \mathbb{C}$ be holomorphic, not identically zero, and invariant under $S\Delta_n(\ell)$ for some $\ell \in \mathbb{Z}^+$. Assume $\phi_f(\Omega) = \det(Y)^{k/2} |f(\Omega)|$ attains a maximum at $\Omega_0 = X_0 + iY_0 \in \mathcal{H}_n$. Then we have $\frac{k}{4\pi} Y_0^{-1} \in \nu(f)$.*

Proof. In order to apply Lemma 1.1 we need $0 \notin \mathrm{supp}(f)$ for $n \geq 2$. We cannot have $0 \in \mathrm{supp}(f)$ since then $\lim_{\Omega \to +\infty iI} |\phi_f(\Omega)| = +\infty$ implies that ϕ_f does not attain a maximum in $\mathcal{H}_n$. For the case $n = 1$ we need the condition $0 < \mathrm{supp}(f)$; this holds because any nonpositive indices in the Fourier series of f would imply that ϕ_f is unbounded in a deleted neighborhood of $i\infty$. Let $K = \nu(f)$, by Lemma 1.1 K is a kernel. We will prove that

$$\left(\frac{k}{4\pi}Y_0^{-1}\right)^{\sqcup} \supseteq (K^{\sqcup})_{\mathbb{Q}}$$

and it then follows that

$$\frac{k}{4\pi}Y_0^{-1} \in \left(\frac{k}{4\pi}Y_0^{-1}\right)^{\sqcup\sqcup} \subseteq \left((K^{\sqcup})_{\mathbb{Q}}\right)^{\sqcup} = K^{\sqcup\sqcup} = K$$

since $K^{\sqcup}$ and K are kernels. We need to show that $\langle \frac{k}{4\pi}Y_0^{-1}, T\rangle \geq 1$ for any $T = \frac{P}{q} \in (K^{\sqcup})_{\mathbb{Q}}$ where $P \in \mathcal{P}_n^{\mathrm{semi}}(\mathbb{Z})$, $q \in \mathbb{Z}^+$.

Let $\Omega_\zeta = \Omega_0 + \zeta\ell P$ be an analytic map for ζ with $\operatorname{Im}\zeta \geq -\epsilon$ where $\epsilon > 0$ is sufficiently small to ensure that $\Omega_\zeta \in \mathcal{H}_n$. Because P is integral the function $f(\Omega_\zeta)$ is a holomorphic function of $z = e(\zeta)$ for $0 < |z| \leq e^{2\pi\epsilon}$. Note that the Laurent expansion about $z = 0$,

$$f(\Omega_\zeta) = \sum_{s \in \mathrm{supp}(f)} a_s e(\langle s, \Omega_\zeta\rangle) = \sum_{s \in \mathrm{supp}(f)} a_s e(\langle s, \Omega_0\rangle) e(\zeta\langle s, \ell P\rangle)$$

has order at least $\min_{s\in\mathrm{supp}(f)}\langle s, \ell P\rangle = \ell q \min_{s\in\mathrm{supp}(f)}\langle s, \frac{P}{q}\rangle \geq \ell q$ since $\langle s, T\rangle \geq 1$. Therefore the function $\dfrac{f(\Omega_\zeta)}{e(\ell q\zeta)}$ extends holomorphically to $z = 0$ and must attain its maximum modulus on $|z| = e^{2\pi\epsilon}$, or equivalently for some ζ' with $\operatorname{Im}\zeta' = -\epsilon$. This maximum must be greater than or equal to the modulus at $z = 1$ so that we must have the inequality

$$\left|\frac{f(\Omega_0)}{e(\ell q \cdot 0)}\right| \leq \left|\frac{f(\Omega_{\zeta'})}{e(\ell q\zeta')}\right|.$$

Use of the inequality $\det(Y)^{k/2}|f(\Omega)| \leq \phi_f(\Omega_0) = \det(Y_0)^{k/2}|f(\Omega_0)|$ renders the above inequality as

$$\frac{k}{2}\ln\det(I - \epsilon\ell Y_0^{-1}P) \leq -2\pi\ell q\epsilon$$

since f is not identically zero. Expanding in powers of ϵ, dividing by ϵ, and letting $\epsilon \to 0^+$ we conclude that $\frac{k}{2}\langle Y_0^{-1}, \ell P\rangle \geq 2\pi\ell q$. This is $\langle \frac{k}{4\pi}Y_0^{-1}, \frac{P}{q}\rangle \geq 1$ as was to be shown. □

This theorem is our main technical result. This analysis closely parallels that in Freitag [11, p.48-50] and Eichler [7] [8]. Their arguments, however, are restricted to specific choices of $T \in (K^{\sqcup})_{\mathbb{Z}}$ and hence their various conclusions are simple corollaries of the Semihull theorem. This theorem is the real reason underlying all the various types of special estimates prescribing which Fourier coefficients must vanish in order to imply that a cusp form is identically zero.

1.3 Theorem. (Main Result) *Let $f \in S_n^k(\Gamma, \chi)$ be nontrivial with Γ of finite index in Γ_n. Let $\phi_f(\Omega) = \det(Y)^{k/2}|f(\Omega)|$ attain its maximum at $\Omega_0 = X_0 + iY_0 \in \mathcal{H}_n$. Then for all $M \in \Gamma_n$ we have $\frac{k}{4\pi}\left(\operatorname{Im}\{M^{-1}\Omega_0\}\right)^{-1} \in \nu(f|M)$.*

Proof. Recall that f being a cusp form ensures that ϕ_f attains a maximum [11, pg. 129]. In the case of half-integral weight, ϕ_f has a maximum because $(\phi_f)^2 = \phi_{f^2}$ and f^2 is a modular form of integral weight. Recall we assume that $\operatorname{Im}\chi \subseteq e(\mathbb{Q})$. We will first show that $\forall M \in \Gamma_n$, $\exists \ell \in \mathbb{Z}^+ : f|M$ is invariant under $S\Delta_n(\ell)$. The function $f|M$ transforms by χ^M $(\chi^M(g) = \chi(MgM^{-1}))$ under $\Gamma^M = M^{-1}\Gamma M$ which is also a subgroup of finite index. For $n \geq 2$ Γ^M is finitely generated so that $\operatorname{Im}\chi^M$ is finite. Since $\ker(\chi^M)$ is of finite index in Γ there is an ℓ such that we have $\ker(\chi^M) \supseteq \Gamma_n(\ell)$. Thus $f|M$ is invariant under $S\Delta_n(\ell)$. For $n = 1$, $t(\Gamma^M) = \ell_1\mathbb{Z}$ for some $\ell_1 \in \mathbb{Z}^+$ since $t(\Gamma^M)$ is of finite index in $t(\Gamma_1) = \mathbb{Z}$. Therefore $\ker_{S\Delta_1}(\chi^M) \cong \ell_1\ell_2\mathbb{Z}$ for some ℓ_2 and $f|M$ is invariant under $S\Delta_1(\ell_1\ell_2)$. The function $\phi_{f|M}$ attains its maximum at $M^{-1}\Omega_0$ since $\phi_{f|M}(\Omega) = \phi_f(M\Omega)$. Now we use Theorem 1.2. $\square$

Discussion of the Main Result. It is often expedient to introduce an auxiliary function ϕ as a height function on $\mathcal{H}_n$ or as a means to linearly order the support of a cusp form f; however, the natural way to measure both support and height is via kernels. A maximum point Ω_0 of ϕ_f is forbidden from some neighborhood of each cusp and Theorem 1.3 explicitly gives such a neighborhood in terms of the kernel $\nu(f|M)$. Should the vanishing requirements placed on f be sufficiently demanding that these forbidden neighborhoods cover a fundamental domain for Γ then f must itself vanish. In the next section we fall back upon vanishing theorems stated in terms of type two functions ϕ because for $n > 1$ we cannot handle the relation of the forbidden regions to the fundamental domain for Γ but we conclude this section with a vanishing theorem in terms of coverings.

Given a cusp form $f \in S_n^k(\Gamma)$ we are interested in all of the kernels $\nu(f|M)$ for $M \in \Gamma_n$. These coincide whenever M falls in the same coset of $\Gamma\backslash\Gamma_n$ so there are actually only a finite number of distinct kernels. The action of Δ_n on the right is particularly simple and further restricts the number of kernels one must consider. We will call a collection of kernels which transforms like $\nu(f|M)$ under the left action of Γ and the right action of Δ_n a Γ-admissible collection of kernels.

1.4 Definition. A collection of kernels $\{K_M\}_{M\in\Gamma_n}$ is called <u>Γ-admissible</u> if

$$\forall\, g \in \Gamma, \forall\, u = \begin{pmatrix} {}^tA & S \\ 0 & A^{-1} \end{pmatrix} \in \Delta_n, \text{ we have } K_{gMu} = AK_M\,{}^tA.$$

1.5 Lemma. *Let $f \in S_n^k(\Gamma, \chi)$ be nontrivial. Then $\{\nu(f|M)\}_{M\in\Gamma_n}$ is a Γ-admissible collection of kernels. Let $\{K_M\}$ be any Γ-admissible collection of kernels. Then the following two conditions are equivalent:*

(1) *For all M in Γ_n, we have the containment $\nu(f|M) \subseteq K_M$.*
(2) *For all M from a set of double coset representatives for $\Gamma\backslash\Gamma_n/\Delta_n$ we have the containment $\nu(f|M) \subseteq K_M$.*

Proof. Left to the reader. $\square$

1.6 Theorem. (Vanishing Theorem) *Let $f \in S_n^k(\Gamma, \chi)$ with Γ of finite index in Γ_n. Let $\phi_f(\Omega) = \det(Y)^{k/2}|f(\Omega)|$ attain a maximum at $\Omega_0 = X_0 + iY_0 \in \mathcal{H}_n$. Let $\{K_M\}$ be a Γ-admissible collection of kernels. Assume we have* $\operatorname{supp}(f|M) \subseteq K_M$ *for all M from a set of representatives for $\Gamma\backslash\Gamma_n/\Delta_n$.*

If we have $f \not\equiv 0$ then we have that Ω_0 is not in the union:

$$\bigcup_{M\in\Gamma_n} M\{\Omega \in \mathcal{H}_n : \frac{k}{4\pi}(\operatorname{Im}\Omega)^{-1} \notin K_M\}.$$

If this union contains a fundamental domain $\mathcal{F}_n(\Gamma)$ then we have $f \equiv 0$.

Proof. Assume that f is not identically zero. Since $\phi_f(\Omega)$ is Γ-invariant we can choose a maximum point $\Omega_0 \in \mathcal{F}_n(\Gamma)$. Apply the Main Result 1.3 to f and conclude that

$\frac{k}{4\pi}\left(\operatorname{Im}\{M^{-1}\Omega_0\}\right)^{-1} \in \nu(f|M)$ for all $M \in \Gamma_n$. Since $\{K_M\}$ is a Γ-admissible collection of kernels, Lemma 1.5 tells us that $\nu(f|M) \subseteq K_M$ for all $M \in \Gamma_n$ so that we have $\frac{k}{4\pi}\left(\operatorname{Im}\{M^{-1}\Omega_0\}\right)^{-1} \in K_M$ for all $M \in \Gamma_n$. Therefore we have

$$\begin{aligned} M^{-1}\Omega_0 \in &\{\Omega \in \mathcal{H}_n : \frac{k}{4\pi}(\operatorname{Im}\Omega)^{-1} \in K_M\}, \\ \Omega_0 \notin M&\{\Omega \in \mathcal{H}_n : \frac{k}{4\pi}(\operatorname{Im}\Omega)^{-1} \notin K_M\}, \\ \Omega_0 \notin \bigcup_{M \in \Gamma_n} M&\{\Omega \in \mathcal{H}_n : \frac{k}{4\pi}(\operatorname{Im}\Omega)^{-1} \notin K_M\}. \end{aligned}$$

However if the last union contains $\mathcal{F}_n(\Gamma)$ and we have $\Omega_0 \in \mathcal{F}_n(\Gamma)$ then this contradicts that $f \not\equiv 0$. $\square$

§2. Estimation with type two functions

In this section we define type one and type two functions and use type two functions in the statement of vanishing theorems.

Definition 2.1. A function $\phi : \operatorname{Dom}\phi \to \mathbb{R}_{\geq 0}$ with $\mathcal{P}_n(\mathbb{R}) \subseteq \operatorname{Dom}\phi \subseteq \mathcal{P}_n^{\text{semi}}(\mathbb{R})$ is called type one if

(1) $\phi(s) > 0$ for all $s \in \mathcal{P}_n(\mathbb{R})$,
(2) $\phi(\lambda s) = \lambda\phi(s)$ for all $\lambda \in \mathbb{R}_{\geq 0}$ and $s \in \operatorname{Dom}\phi$,
(3) $\phi(s_1 + s_2) \geq \phi(s_1) + \phi(s_2)$ for all $s_1, s_2 \in \operatorname{Dom}\phi$.

2.2 Proposition. *A type one function ϕ is continuous on $\mathcal{P}_n(\mathbb{R})$.*

Proof. Pick $\epsilon > 0$. We will define a neighborhood N of a fixed $t \in \mathcal{P}_n(\mathbb{R})$ such that $|\phi(s) - \phi(t)| < \epsilon$ for all $s \in N$.

Pick $q \in \mathbb{R}$ such that $0 < q < \frac{\epsilon}{\phi(t)}$ and $0 < q < 1$. Since $qt > 0$, we can choose a neighborhood N of t such that $qt \pm (t - s) > 0$ for all $s \in N$. Then for $s \in N$, we have $\phi(t)+\epsilon > \phi(t)+q\phi(t) = (1+q)\phi(t) = \phi((1+q)t) = \phi(s+qt+(t-s)) \geq \phi(s)+\phi(qt+(t-s)) > \phi(s)$. Also, we have $\phi(s) = \phi(t-qt+qt+s-t) \geq \phi((1-q)t)+\phi(qt+s-t) > \phi((1-q)t) = (1-q)\phi(t) = \phi(t) - q\phi(t) > \phi(t) - \epsilon$. Therefore $\epsilon > \phi(s) - \phi(t) > -\epsilon$ for $s \in N$. $\square$

Definition 2.3. A type one function ϕ is called type two if $\phi(\mathcal{P}_n(\mathbb{Z}))$ is discrete in $\mathbb{R}$.

Examples. The following are some useful type two functions: the trace $\operatorname{tr}(s)$, the reduced determinant $\det(s)^{1/n}$, Hermite's function $m(s)$, and the dyadic trace $w(s)$. Among these only the trace is not a class function. The smallest eigenvalue of s, $\lambda_1(s)$ is type one but not type two.

2.4 Lemma. *Let ϕ be type one. Let $S \subseteq \frac{1}{\ell}\mathcal{P}_n(\mathbb{Z})$ for some $\ell \in \mathbb{Z}^+$. Then we have* $\inf\phi\left(\operatorname{Semihull}(S) \cap \mathcal{P}_n(\mathbb{R})\right) = \inf\phi(S)$. *If ϕ is type two we also have* $\inf\phi(S) = \min\phi(S)$.

Proof. The inequality $\inf\phi(S) \geq \inf\phi\left(\operatorname{Semihull}(S) \cap \mathcal{P}_n(\mathbb{R})\right)$ follows from the inclusion $S \subseteq \operatorname{Semihull}(S) \cap \mathcal{P}_n(\mathbb{R})$. On the other hand take any $x \in \operatorname{Semihull}(S) \cap \mathcal{P}_n(\mathbb{R})$. There are choices of $s_i \in S$ and $a_i \geq 0$ satisfying $\sum_i a_i \geq 1$ such that $\sum_i a_i s_i$ is arbitrarily close to x. The continuity of ϕ on $\mathcal{P}_n(\mathbb{R})$ implies that $\phi(\sum_i a_i s_i)$ is arbitrarily close to $\phi(x)$. Since ϕ is type one we also have $\phi(\sum_i a_i s_i) \geq \sum_i a_i \phi(s_i) \geq \sum_i a_i \inf\phi(S) \geq \inf\phi(S)$; therefore we conclude $\phi(x) \geq \inf\phi(S)$. If ϕ is type two we know that $\phi(S)$ is discrete in $\mathbb{R}$ and so we have $\inf\phi(S) = \min\phi(S)$. $\square$

The uniform vanishing hypothesis in the next theorem in too restrictive and will be weakened to an average vanishing hypothesis in the theorem that follows it.

2.5 Theorem. (Uniform Estimation) *Let $f \in S_n^k(\Gamma, \chi)$ with Γ of finite index in Γ_n. Let ϕ be type two. If f has the following uniform ϕ-vanishing*

$$\forall\, [M] \in \Gamma \backslash \Gamma_n, \quad \min \phi\left(\operatorname{supp}(f|M)\right) > \frac{k}{4\pi} \sup_{\Omega \in \mathcal{H}_n} \inf_{\sigma \in \Gamma_n} \phi\left(\operatorname{Im}(\sigma\Omega)^{-1}\right)$$

then we have $f \equiv 0$.

Proof. Note that since $\Gamma\backslash\Gamma_n$ is a finite set, we can rewrite the hypothesis as $\exists \delta > 0$, $\forall M \in \Gamma_n$, $\min \phi\left(\operatorname{supp}(f|M)\right) > \frac{k}{4\pi} \sup_{\Omega \in \mathcal{H}_n} \inf_{\sigma \in \Gamma_n} \phi\left(\operatorname{Im}(\sigma\Omega)^{-1}\right) + \delta$. We assume $f \not\equiv 0$ and obtain a contradiction. Apply the Main Result 1.3 to conclude that

$$\frac{k}{4\pi}\left(\operatorname{Im}\{M^{-1}\Omega_0\}\right)^{-1} \in \operatorname{Semihull}[\operatorname{supp}(f|M)]$$

for some $\Omega_0 \in \mathcal{H}_n$ and for all $M \in \Gamma_n$. Then we have

$$\begin{aligned}\frac{k}{4\pi}\phi\left(\{\operatorname{Im}(M^{-1}\Omega_0)\}^{-1}\right) \in &\phi\left(\operatorname{Semihull}[\operatorname{supp}(f|M)] \cap \mathcal{P}_n(\mathbb{R})\right), \\ =&\mathbb{R}_{\geq 1} \min \phi\left(\operatorname{supp}(f|M)\right) \text{ by Lemma 2.4 with } S = \operatorname{supp}(f|M).\end{aligned}$$

Combined with the hypothesis, this yields

$$\frac{k}{4\pi}\phi\left(\{\operatorname{Im}(M^{-1}\Omega_0)\}^{-1}\right) > \frac{k}{4\pi} \sup_{\Omega \in \mathcal{H}_n} \inf_{\sigma \in \Gamma_n} \phi\left(\operatorname{Im}(\sigma\Omega)^{-1}\right) + \delta.$$

Take any $\epsilon > 0$, there exists a $\sigma_0 \in \Gamma_n$ such that

$$\inf_{\sigma \in \Gamma_n} \phi\left(\operatorname{Im}(\sigma\Omega_0)^{-1}\right) + \epsilon \geq \phi\left(\operatorname{Im}(\sigma_0\Omega_0)^{-1}\right).$$

Combined with the previous inequality in the instance where $M = \sigma_0^{-1}$, we obtain

$$\inf_{\sigma \in \Gamma_n} \phi\left(\operatorname{Im}(\sigma\Omega_0)^{-1}\right) + \epsilon > \sup_{\Omega \in \mathcal{H}_n} \inf_{\sigma \in \Gamma_n} \phi\left(\operatorname{Im}(\sigma\Omega)^{-1}\right) + \delta.$$

Taking $\epsilon \to 0^+$ yields the contradiction

$$\inf_{\sigma \in \Gamma_n} \phi\left(\operatorname{Im}(\sigma\Omega_0)^{-1}\right) \geq \sup_{\Omega \in \mathcal{H}_n} \inf_{\sigma \in \Gamma_n} \phi\left(\operatorname{Im}(\sigma\Omega)^{-1}\right) + \delta. \quad \square$$

2.6 Theorem. (Average Estimation) *Let $f \in S_n^k(\Gamma, \chi)$ with Γ of finite index I in Γ_n. Let ϕ be type two. Let $M_1, \ldots, M_I$ be a set of representatives for $\Gamma\backslash\Gamma_n$. If f has the following average ϕ-vanishing*

$$\frac{1}{I}\sum_{i=1}^{I} \min \phi\left(\operatorname{supp}(f|M_i)\right) > \frac{k}{4\pi} \sup_{\Omega \in \mathcal{H}_n} \inf_{\sigma \in \Gamma_n} \phi\left(\operatorname{Im}(\sigma\Omega)^{-1}\right)$$

then we have $f \equiv 0$.

Proof. We assume $f \not\equiv 0$ and obtain a contradiction. Let $\sigma_i = \min \phi\left(\operatorname{supp}(f|M_i)\right)$. Let the Fourier series of $f|M_i$ be written as

$$(f|M_i)(\Omega) = \sum_{s_i \in \operatorname{supp}(f|M_i)} a_{s_i}^i e(\langle s_i, \Omega \rangle).$$

Consider the norm F of f given by $F(\Omega) = \prod_{i=1}^{I}(f|M_i)(\Omega) \in S_n^{kI}(\Gamma_n)$ of weight kI modular with respect to the full group Γ_n. We will apply the previous Theorem 2.5 to F to show that $F \equiv 0$ by verifying the following condition:

$$\min \phi\left(\operatorname{supp}(F)\right) > \frac{kI}{4\pi} \sup_{\Omega \in \mathcal{H}_n} \inf_{\sigma \in \Gamma_n} \phi\left(\operatorname{Im}(\sigma\Omega)^{-1}\right). \tag{2.7}$$

In order to express the condition 2.7 in terms of the σ_i we expand F in a Fourier series

$$F(\Omega) = \sum_{s_1} \cdots \sum_{s_I} \left(\prod_{i=1}^{I} a_{s_i}^{i} \right) e(\langle s_1 + \cdots + s_I, \Omega \rangle).$$

We have $s \in \operatorname{supp}(F)$ only if there exist $s_i \in \operatorname{supp}(f|M_i)$ such that $s = s_1 + \cdots + s_I$. This implies that $\phi(s) \geq \sum_i \phi(s_i) \geq \sum_i \sigma_i$ for all $s \in \operatorname{supp}(F)$ so that we have $\min \phi(\operatorname{supp}(F)) \geq \sum_i \sigma_i$. By the hypothesis $\sum \sigma_i > \frac{kI}{4\pi} \sup_{\Omega \in \mathcal{H}_n} \inf_{\sigma \in \Gamma_n} \phi\left(\operatorname{Im}(\sigma\Omega)^{-1}\right)$, we see that condition 2.7 is true. Then we have $F \equiv 0$, whence $f \equiv 0$. $\square$

In order to apply the previous two theorems, it becomes necessary to compute $\phi_n = \sup_{\Omega \in \mathcal{H}_n} \inf_{\sigma \in \Gamma_n} \phi\left(\{\operatorname{Im}(\sigma\Omega)\}^{-1}\right)$ for various type two functions ϕ, or at least to compute upper bounds. Currently, we know of no better upper bound for ϕ_n other than $\phi_n \leq \sup_{\Omega \in \mathcal{F}_n} \phi(Y^{-1})$. The quality of the upper bound is important in making the previous two theorems of practical value. Here are the best upper bounds that we know of and their relation to the mathematical literature.

For $n = 1$ we have $\operatorname{tr}(s) = \det(s)^{1/n} = m(s) = w(s)$ and $\sup_{\mathcal{F}_1} \phi(y^{-1}) = \frac{2}{\sqrt{3}}$ for these ϕ from the well known construction of $\mathcal{F}_1$, $\sup \phi(y^{-1}) = \phi(1) \sup_{\tau} y^{-1} = \phi(1)\frac{2}{\sqrt{3}}$.

For the type two function, the trace, let $\kappa_n = \sup_{\mathcal{F}_n} \operatorname{tr}(Y^{-1})$. We have $\operatorname{tr}_n \leq \kappa_n$ and the best known upper bound for κ_n is $\kappa_n \leq n\mu_n^n \frac{2}{\sqrt{3}}$, see [14, p.197], [5]. If we use the trace as the type two function in Theorem 2.5 and restrict ourselves to the full modular group Γ_n we obtain the result of Siegel (whose statement was given in the Introduction), see [14, p.200].

To compute an upper bound for $\dim S_n^k$ or to show that a particular cusp form is zero using Siegel's Theorem, one lists the semi-integral *classes* $[s]$ such that a representative s exists satisfying $\operatorname{tr}(s) \leq \kappa_n \frac{k}{4\pi}$. The number of such classes $[s]$ is then an upper bound for $\dim S_n^k$ and is the number of Fourier coefficients a_s of f that must be computed in an application. It suffices to count classes because the vanishing of a_s is a class function. Notice that the trace is not a class function so that representatives from a table must be checked to ensure that they have minimal trace, if they in fact do. In practice, of course, one must use the upper bound of item (3). Since tables order forms by determinant it is also helpful to express the condition in terms of the determinant as in item (4) to use as a preliminary sorting method. This theorem, the most commonly known, is actually the most inefficient among those we discuss.

For the type two Hermite's function we have $m_n \leq \sup_{\mathcal{F}_n} m(Y^{-1})$. The best known upper bound for $\sup_{\mathcal{F}_n} m(Y^{-1})$ occurs in [7]: $m(Y^{-1}) \leq \mu_n \det(Y^{-1})^{1/n} = \mu_n / \det(Y)^{1/n} \leq \mu_n/(m(Y)/\mu_n) = \mu_n^2/m(Y) \leq \mu_n^2 \frac{2}{\sqrt{3}}$. The equivalence of items (1) and (3) in the following theorem is due to Eichler, see [7]. The equivalence of items (1) and (2) is likely an improvement in that m_n may be smaller than $\sup_{\mathcal{F}_n} m(Y^{-1})$.

2.8 Theorem. *Let $f \in S_n^k$ have Fourier expansion $f(\Omega) = \sum_{s>0} a_s e^{2\pi i \operatorname{tr}(s\Omega)}$. The following conditions are equivalent.*

(1) $f = 0$.

(2) *For all s such that* $m(s) \le m_n \frac{k}{4\pi}$, *we have* $a_s = 0$.
(3) *For all s such that* $m(s) \le \mu_n^2 \frac{2}{\sqrt{3}} \frac{k}{4\pi}$, *we have* $a_s = 0$.

This Theorem is not a finiteness theorem in the usual sense because the condition (2) may hold for infinitely many classes $[s]$. A similar theorem for Fourier-Jacobi expansions may be found in [7][28]. When we have $m_n \frac{k}{4\pi} < 1$ then condition (2) holds and so $S_n^k = 0$. Eichler used condition (3) to show that $S_3^k = 0$ for $k \le 6$; $S_4^k = 0$ for $k \le 5$; $S_5^k = 0$ for $k \le 5$.

The constant m_n also has significance in the theory of toroidal compactifications of $\mathcal{H}_n/\Gamma_n$. Let $A_n^{(1)}$ be the coarse moduli space of dimension n principally polarized abelian varieties and rank one degenerations used in [23]. Let $A_n^{(1),0}$ be the elements of $A_n^{(1)}$ with minimal automorphism group. Each $f \in S_n^k$ defines a divisor $\mathrm{div}(f) \subseteq A_n = \mathcal{H}_n/\Gamma_n$ which under certain conditions extends to a divisor on $A_n^{(1)}$. The divisor class of $\mathrm{div}(f)$ in $\mathrm{Pic}(A_n^{(1),0}) \otimes \mathbb{Q}$ is given by $[\mathrm{div}(f)] = k\lambda - \mu\delta$ where λ is the class of the Hodge bundle, k is the weight of f, δ is the divisor class of the boundary $[A_n^{(1)} \setminus A_n]$ and $\mu = \min m(\mathrm{supp}(f))$. The "slope" of a divisor refers to k/μ. The existence of effective divisors with slope less than the slope of the canonical bundle was used in [32][23] to show that A_n is of general type for $n \ge 7$. This theorem shows that an effective divisor has slope which is bounded below by:

$$\text{slope } = \frac{k}{\mu} \ge \frac{4\pi}{m_n} \ge \frac{2\pi\sqrt{3}}{\mu_n^2}.$$

The asymptotic growth of this lower bound for the slope is $\frac{2\pi\sqrt{3}}{\mu_n^2} \ge \mathrm{const}/n^2$ [4, p.20]. A more careful study of the new constant m_n defined here may reveal a slower growth than $O(n^2)$. This gives the only known lower bound on the slope for large n.

For the type two function the dyadic trace, $w_n \le \sup_{\mathcal{F}_n} w(Y^{-1})$ gives the best known upper bound for w_n: $\sup_{\mathcal{F}_n} w(Y^{-1}) \le n\frac{2}{\sqrt{3}}$. From section 3, use $Y^{-1} > 0$ and Lemma 3.4 to obtain $\langle Y, s\rangle \ge m(Y)w(s)$ so that $w(Y^{-1}) \le \frac{\langle Y, Y^{-1}\rangle}{m(Y)} = \frac{n}{m(Y)} \le \frac{n}{\sqrt{3}/2} = \frac{2n}{\sqrt{3}}$.

2.9 Theorem. *Let $f \in S_n^k$ have Fourier expansion $f(\Omega) = \sum_{s>0} a_s e^{2\pi i\,\mathrm{tr}(s\Omega)}$. The following conditions are equivalent.*

(1) $f = 0$.
(2) *For all s such that* $w(s) \le w_n \frac{k}{4\pi}$, *we have* $a_s = 0$.
(3) *For all s such that* $w(s) \le n\frac{2}{\sqrt{3}}\frac{k}{4\pi}$, *we have* $a_s = 0$.
(4) *For all s such that* $\det(s)^{1/n} \le \mu_n \frac{2}{\sqrt{3}}\frac{k}{4\pi}$, *we have* $a_s = 0$.

This estimate seems to give the best results, see the examples in §5 for comparisons. Corollary 3.8 of [28, p. 340] could be used to prove items (3) and (4) above.

For the type two function the reduced determinant we have the following equality $\det_n = \sup_{\Omega \in \mathcal{H}_n} \inf_{\sigma \in \Gamma_n} \det\left(\{\mathrm{Im}(\sigma\Omega)\}^{-1}\right)^{\frac{1}{n}} = \sup_{\mathcal{F}_n} \det(Y^{-1})^{1/n}$ from the construction of Siegel's fundamental domain $\mathcal{F}_n$. The best known upper bound for $\det_n$ is $\det_n \le \mu_n \frac{2}{\sqrt{3}}$ which follows from Hermite's Inequality.

2.10 Theorem. *Let $f \in S_n^k$ have Fourier expansion $f(\Omega) = \sum_{s>0} a_s e^{2\pi i\,\mathrm{tr}(s\Omega)}$. The following conditions are equivalent.*

(1) $f = 0$.
(2) *For all s such that* $\det(s)^{1/n} \le \det_n \frac{k}{4\pi}$, *we have* $a_s = 0$.

(3) *For all s such that* $\det(s)^{1/n} \le \mu_n \frac{2}{\sqrt{3}}\frac{k}{4\pi}$, *we have* $a_s = 0$.

If $h(\Omega) = \det(Y)$ is taken as a height function on $\mathcal{H}_n$ then computing $\det_n$ amounts to finding a *lowest* point in $\mathcal{F}_n$. Siegel gave upper bounds for this in [31, p.64–65]. Lower bounds on $\det_n$ can be computed from the existence of nontrivial cusp forms. Just as the covering issue, these are interesting problems in the symplectic geometry of numbers.

§3. Dyadic Trace

In this section we develop the theory of the dyadic trace, a particular type two class function. This theory follows from certain facts about the perfect cone decomposition [1, pp.144–150] but we present a more elementary and self-contained account aimed at computational use. We let $C_n^* = \mathbb{R}_{\ge 0}\langle v\,{}^t v\rangle_{v \in \mathbb{Z}^n \setminus 0}$ and later characterize C_n^* as the elements of $\mathcal{P}_n^{\mathrm{semi}}(\mathbb{R})$ whose radical is defined over $\mathbb{Q}$. A subspace W of $\mathbb{R}^m$ is said to be defined over $\mathbb{Q}$ if it is spanned by vectors from $W \cap \mathbb{Q}^m$. Recall $\mathrm{rad}(s) = \{v \in \mathbb{R}^n : {}^t v s v = 0\}$ and note that for $s \ge 0$ we have $\mathrm{rad}(s) = \mathrm{Null}(s) = \{v \in \mathbb{R}^n : sv = 0\}$. For $s \in \mathcal{P}_n(\mathbb{R})$ we use the notation $\mathrm{MinVec}(s) = \{\, x \in \mathbb{Z}^n : {}^t x s x = m(s)\,\}$. We extend the notation, in a consistent but perhaps nonstandard way, to singular $s \in \mathcal{P}_n^{\mathrm{semi}}(\mathbb{R}) \backslash \mathcal{P}_n(\mathbb{R})$ by: $\mathrm{MinVec}(s) = \{\, x \in \mathbb{Z}^n : {}^t x s x = 0\,\}$. For a singular s, the $\mathbb{R}$-span of $\mathrm{MinVec}(s)$ is $\mathrm{rad}(s)$ precisely when $s \in C_n^*$. We let e_i for $i = 1, \dots, n$ denote the standard basis for $\mathbb{Z}^n$.

3.1 Definition. *The matrix $s \in V_n(\mathbb{R})$ has a* **dyadic representation** *if there exist $\alpha_i \in \mathbb{R}_{\ge 0}$ and $v_i \in \mathbb{Z}^n \setminus \{0\}$ such that $s = \sum \alpha_i v_i {}^t v_i$.*

For example, $s = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$ has a dyadic representation

$$s = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix}.$$

A dyadic representation is termed *strict* when we have $\alpha_i > 0$ for all i. A matrix s with a dyadic representation is semidefinite but not all semidefinite s have dyadic representations; for example, $s = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix}$ does not have a dyadic representation. To see this assume that we have a dyadic representation $s = \sum \alpha_i v_i {}^t v_i$, which we may assume strict, and note that

$$0 = \begin{pmatrix} \sqrt{2} & -1 \end{pmatrix} s \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} = \sum \alpha_i \left({}^t v_i \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} \right)^2$$

implies that ${}^t v_i \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} = 0$ for all i contradicting $v_i \in \mathbb{Z}^2 \setminus \{0\}$ and $\sqrt{2} \notin \mathbb{Q}$. This example illustrates the general case. The cone inside $\mathcal{P}_n^{\mathrm{semi}}(\mathbb{R})$ consisting of elements s which possess a dyadic representation is clearly $C_n^* = \mathbb{R}_{\ge 0}\langle v\,{}^t v\rangle_{v \in \mathbb{Z}^n \setminus 0}$; we may characterize C_n^* as the elements of $\mathcal{P}_n^{\mathrm{semi}}(\mathbb{R})$ whose radical is defined over $\mathbb{Q}$.

3.2 Proposition. $C_n^* = \{s \in \mathcal{P}_n^{\mathrm{semi}}(\mathbb{R}) : \mathrm{rad}(s)$ *is defined over* $\mathbb{Q}\}$

Proof. The case $n = 1$ is trivial because both sides of the equation are equal to $\mathbb{R}_{\ge 0}$. We clearly have $C_n^* \subseteq \{s \in \mathcal{P}_n^{\mathrm{semi}}(\mathbb{R}) : \mathrm{rad}(s)$ is defined over $\mathbb{Q}\}$ because if $s = \sum \alpha_i v_i {}^t v_i$ is a strict dyadic representation then ${}^t x s x = \sum \alpha_i \left({}^t x v_i\right)^2$ and so $\mathrm{rad}(s) = \{x \in \mathbb{R}^n : {}^t x v_i = 0$ for all $i\}$ is defined over $\mathbb{Q}$. The main step needed to prove the other inclusion is that $C_n^* \supseteq \mathcal{P}_n(\mathbb{R})$; granting this we can prove $C_n^* \supseteq \{s \in \mathcal{P}_n^{\mathrm{semi}}(\mathbb{R}) : \mathrm{rad}(s)$ is defined over $\mathbb{Q}\}$ by induction on n. Suppose that $s \geq 0$ and that $\mathrm{rad}(s) \neq 0$ is defined over $\mathbb{Q}$, then there exists a $u \in \mathrm{SL}_n(\mathbb{Z})$ such that ${}^t\!usu = \begin{pmatrix} 0 & 0 \\ 0 & s_{n-1} \end{pmatrix}$. This shows that $s_{n-1} \geq 0$ has $\mathrm{rad}(s_{n-1})$ defined over $\mathbb{Q}$, thus the induction hypothesis gives a dyadic representation $s_{n-1} = \sum \alpha_i v_i {}^t v_i$ and hence ${}^t\!usu = \sum \alpha_i \begin{pmatrix} 0 \\ v_i \end{pmatrix} {}^t \begin{pmatrix} 0 \\ v_i \end{pmatrix}$. Possessing a dyadic representation is a class property so that ${}^t\!usu \in C_n^*$ implies that $s \in C_n^*$. This completes the induction.

In order to handle the case $\mathrm{rad}(s) = 0$ and show that $\mathcal{P}_n(\mathbb{R}) \subseteq C_n^*$ we first show that $\mathcal{P}_n(\mathbb{Q}) \subseteq C_n^*$. Let $s \in \mathcal{P}_n(\mathbb{Q})$. By completing the square successively we have ${}^t\!xsx = \sum \alpha_i \left({}^t\!xv_i\right)^2$ for some $v_i \in \mathbb{Q}^n$ and $\alpha_i \in \mathbb{Q}_{\geq 0}$. Choose a $q \in \mathbb{Z}^+$ such that $qv_i \in \mathbb{Z}^n$ for all i, then we have $s = \sum \frac{\alpha_i}{q^2}(qv_i)^t(qv_i)$. Omitting the terms with $v_i = 0$ gives a dyadic representation of s. Next we show that any $s \in V_n(\mathbb{R})$ which is near-diagonal is in C_n^*. A matrix s is *near-diagonal* when for all i we have $s_{ii} \geq \sum_{j:j\neq i} |s_{ij}|$. A near-diagonal s has the immediate dyadic representation

$$s = \sum_{i,j:i<j} |s_{ij}|(e_i + \mathrm{sgn}(s_{ij})e_j)\ {}^t(e_i + \mathrm{sgn}(s_{ij})e_j) + \sum_i \left(s_{ii} - \sum_{j:j\neq i} |s_{ij}| \right) e_i\, {}^t e_i.$$

This shows that $s \in C_n^*$. We can now demonstrate the general case $\mathcal{P}_n(\mathbb{R}) \subseteq C_n^*$ by combining the two previous special cases. Let $s \in \mathcal{P}_n(\mathbb{R})$; choose $\eta \in \mathbb{R}^+$ so that $s - \eta I \in \mathcal{P}_n(\mathbb{R})$, and choose $\tilde{s} \in \mathcal{P}_n(\mathbb{Q})$ so that $E = (s - \eta I) - \tilde{s}$ has all its entries less than $\frac{\eta}{n}$ in absolute value. Then $s = \tilde{s} + (\eta I + E)$ has a dyadic representation because both the rational $\tilde{s}$ and the near-diagonal $\eta I + E$ do. This demonstrates that $\mathcal{P}_n(\mathbb{R}) \subseteq C_n^*$. □

3.3 Definition. Define the dyadic trace $w : C_n^* \to \mathbb{R}_{\geq 0}$ for $s \in C_n^*$ by

$$w(s) = \sup \left(\sum_i \alpha_i \right)$$

where the supremum is over all dyadic representations of $s = \sum \alpha_i v_i {}^t v_i$.

3.4 Lemma. *The dyadic trace w is a type one class function $w : C_n^* \to \mathbb{R}_{\geq 0}$ satisfying for all $s \in C_n^*$:*

(1) $\forall\, Y \in \mathcal{P}_n^{\mathrm{semi}}(\mathbb{R}), \langle Y, s\rangle \geq w(s)m(Y)$
(2) $w(s) = 0 \iff s = 0$

Proof. Let $s = \sum \alpha_i v_i {}^t v_i$ be any dyadic representation. Then $sY = \sum \alpha_i v_i {}^t v_i Y$ and $\langle Y, s\rangle = \sum \alpha_i {}^t v_i Y v_i \geq \sum \alpha_i m(Y)$. Taking the supremum over all dyadic representations gives $\langle Y, s\rangle \geq w(s)m(Y)$. This shows that $w(s)$ is finite by choosing, say, $Y = I$. Since $w(s) \geq \sum \alpha_i$ for any dyadic representation of s we have $w(s) = 0 \iff s = 0$. Hence note that w maps from $\mathcal{P}_n(\mathbb{R})$ to $\mathbb{R}_{>0}$. The remainder of the proof is left to the reader. □

Lemma 3.4 has an improved version in Proposition 3.12. We will now work towards showing that w also has a characterization as an infimum:

$$\text{for } s \in C_n^*, \quad w(s) = \inf_{Y \in \mathcal{P}_n(\mathbb{R})} \frac{\langle s, Y\rangle}{m(Y)}.$$

Once this is proven, the two characterizations as supremum and infimum allow one to quickly bound $w(s)$ in computations. The first step toward the infimum characterization is to show that the infimum is attained by some $Y \in \mathcal{P}_n(\mathbb{R})$ and for this we need some facts about $m(Y)$.

3.5 Lemma. *The type two function m is continuous on $\mathcal{P}_n^{\mathrm{semi}}(\mathbb{R})$ and vanishes on the singular set $\mathcal{P}_n^{\mathrm{semi}}(\mathbb{R}) \setminus \mathcal{P}_n(\mathbb{R})$.*

Proof. This is a consequence of the fact that Hermite's inequality $m(Y) \le \mu_n \det(Y)^{1/n}$ holds for all $Y \in \mathcal{P}_n^{\mathrm{semi}}(\mathbb{R})$. At nonsingular Y the continuity of m follows from Proposition 2.2 because m is type one. $\square$

3.6 Lemma. *Let ϕ be a type one function continuous on $\mathcal{P}_n^{\mathrm{semi}}(\mathbb{R})$ and vanishing on the singular set $\mathcal{P}_n^{\mathrm{semi}}(\mathbb{R}) \setminus \mathcal{P}_n(\mathbb{R})$. For all $s \in \mathcal{P}_n(\mathbb{R})$ the infimum*

$$\hat{\phi}(s) = \inf_{Y \in \mathcal{P}_n(\mathbb{R})} \frac{\langle s, Y\rangle}{\phi(Y)}$$

is attained at some $Y_0 \in \mathcal{P}_n(\mathbb{R})$.

Proof. Let D be the set $D = \{\, Y \in \mathcal{P}_n(\mathbb{R}) : \phi(Y) = 1 \,\}$. Note that we have $\hat{\phi}(s) = \inf_{Y \in D} \langle s, Y\rangle$ since we may replace Y by $\frac{Y}{\phi(Y)}$. Next, let E be the subset of D defined by $E = \left\{ Y \in D : \mathrm{tr}(Y) \le \frac{\mathrm{tr}(s)}{\lambda_1(s)\phi(I)} \right\}$. Note that $E \ne \emptyset$ because $\frac{I}{\phi(I)} \in E$. Now consider any $Y \in D \setminus E$, so that we have $\mathrm{tr}(s) < \mathrm{tr}(Y)\lambda_1(s)\phi(I)$. For all such Y we have $\langle s, \frac{I}{\phi(I)}\rangle = \frac{\mathrm{tr}(s)}{\phi(I)} < \lambda_1(s)\,\mathrm{tr}(Y) \le \langle s, Y\rangle$; together with $\frac{I}{\phi(I)} \in E$ this implies that $\inf_{Y\in D} \langle s, Y\rangle = \inf_{Y \in E} \langle s, Y\rangle$. The set E is bounded due to the bound on $\mathrm{tr}(Y)$. The set E is closed in $\mathcal{P}_n^{\mathrm{semi}}(\mathbb{R})$ because if $Y \in \mathcal{P}_n^{\mathrm{semi}}(\mathbb{R})$ is a cluster point of E, then $\phi(Y) = 1$ forces $Y \in \mathcal{P}_n(\mathbb{R})$, which forces $Y \in E$ since E is closed in $\mathcal{P}_n(\mathbb{R})$. Therefore, E is closed in $V_n(\mathbb{R})$ since $\mathcal{P}_n^{\mathrm{semi}}(\mathbb{R})$ is closed in $V_n(\mathbb{R})$. Thus E is compact. Since $\langle s, Y\rangle$ is a continuous function of Y, then $\inf_{Y\in E} \langle s, Y\rangle$ is attained at some $Y_0 \in E$. $\square$

3.7 Proposition. *For all $s \in C_n^*$ the infimum*

$$\hat{m}(s) = \inf_{Y \in \mathcal{P}_n(\mathbb{R})} \frac{\langle s, Y\rangle}{m(Y)}$$

is attained at some $Y_0 \in \mathcal{P}_n(\mathbb{R})$.

Proof. For $s \in \mathcal{P}_n(\mathbb{R})$ this follows from Lemmas 3.5 and 3.6. If $s \in C_n^*$ has $\mathrm{rad}(s) \ne 0$ then there exists a $u \in \mathrm{SL}_n(\mathbb{Z})$: ${}^t usu = \begin{pmatrix} 0 & 0 \\ 0 & s' \end{pmatrix}$ for $s' \in C_{n-1}^*$ and we finish the proof by induction. $\square$

Now that we know that the above infimum is attained we want to show that it is equal to $w(s)$. This requires some information about the variation of minimal vectors.

3.8 Lemma. (Variation of Minimal Vectors) *Let $Y_0 \in \mathcal{P}_n(\mathbb{R})$. There exists a neighborhood $N \subset \mathcal{P}_n(\mathbb{R})$ of Y_0 such that we have*

$$Y \in N \implies \mathrm{MinVec}(Y) \subseteq \mathrm{MinVec}(Y_0).$$

Proof. This follows from the continuity of m. Let $\eta > 0$ be such that $0 < Y_0 - \eta I$. Let $N_1 = \{Y : Y_0 - \eta I < Y\}$ be a neighborhood of Y_0. Select a second neighborhood $N_2 \subseteq N_1$ of Y_0 such that $Y \in N_2$ implies that $|m(Y) - m(Y_0)| < 1$. Setting $W = \{x \in \mathbb{Z}^n \setminus 0 : (Y_0 - \eta I)[x] < m(Y_0) + 1\}$ we have $\mathrm{MinVec}(Y) \subseteq W$ for $Y \in N_2$. This follows for a minimal vector x from the inequality $(Y_0 - \eta I)[x] < Y[x] = m(Y) < m(Y_0) + 1$. Note that W is a finite set because $Y_0 - \eta I > 0$.

Now assume that the desired neighborhood N does not exist and proceed by contradiction. We choose sequences $Y_k \in N_2$, $v_k \in \mathbb{Z}^n \setminus 0$ such that $Y_k \to Y_0$ and $Y_k[v_k] = m(Y_k)$ but $v_k \notin \mathrm{MinVec}(Y_0)$. We have $v_k \in \mathrm{MinVec}(Y_k) \subseteq W$ and since W is a finite set we can assume the sequence v_k to be constant, $v_k = v$, by choosing a subsequence if necessary. We have that $Y_0[v]$ is the limit of $Y_k[v]$ and that $Y_k[v] = Y_k[v_k] = m(Y_k)$ converges to $m(Y_0)$ by the continuity of m. Hence we have $Y_0[v] = m(Y_0)$ and $v \in \mathrm{MinVec}(Y_0)$ is the desired contradiction. □

The following theorem shows that the supremum defining the dyadic trace as well as the infimum characterizing it are both attained.

3.9 Theorem. *For any $s \in C_n^*$ there exists a $Y_0 \in \mathcal{P}_n(\mathbb{R})$ such that we have $w(s) = \inf_{Y\in\mathcal{P}_n(\mathbb{R})} \frac{\langle s,Y\rangle}{m(Y)} = \frac{\langle s,Y_0\rangle}{m(Y_0)}$ and for any such Y_0 we have*

(1) *s has a dyadic representation in the minimal vectors of Y_0, that is, there exist $v_i \in \mathrm{MinVec}(Y_0)$ and $\alpha_i \geq 0$ such that we have $s = \sum \alpha_i v_i {}^t v_i$,*
(2) *$\frac{\langle s,Y_0\rangle}{m(Y_0)} = w(s) = \sum \alpha_i$.*

Proof. For the existence of Y_0, see Proposition 3.7. Now, for any such Y_0,

$$\langle s, Y_0\rangle m(Y) \leq m(Y_0)\langle s, Y\rangle \quad \text{for all } Y \in \mathcal{P}_n(\mathbb{R}). \tag{3.10}$$

Let N be a neighborhood of Y_0 as in lemma 3.8. For any B sufficiently small in $V_n(\mathbb{R})$ we have $Y_B = Y_0 + B \in N$. Let $v_B \in \mathrm{MinVec}(Y_0)$ such that $m(Y_B) = Y_B[v_B]$. Then equation 3.10 yields $\langle s, Y_0\rangle\langle Y_0 + B, v_B\, {}^t v_B\rangle \leq m(Y_0)\langle s, Y_0 + B\rangle$ or

$$\langle s, Y_0\rangle\langle B, v_B\, {}^t v_B\rangle \leq m(Y_0)\langle s, B\rangle \quad \text{for } B \in N - Y_0. \tag{3.11}$$

Let $T \in V_n(\mathbb{R})$ such that $T[v] \geq 0$ for all $v \in \mathrm{MinVec}(Y_0)$. Set $B = \lambda T$ with $\lambda > 0$ so that $B \in N - Y_0$. By equation 3.11 $0 \leq \lambda\langle s, Y_0\rangle\langle T, v_B\, {}^t v_B\rangle \leq \lambda m(Y_0)\langle s, T\rangle$ and therefore $s \in \left(\{v\, {}^t v\}_{v\in\mathrm{MinVec}(Y_0)}\right)^{\vee\vee} = \mathbb{R}_{\geq 0}\langle v\, {}^t v\rangle_{v\in\mathrm{MinVec}(Y_0)}$.

For (2), $s = \sum \alpha_i v_i {}^t v_i$ be a dyadic representation in the minimal vectors of Y_0. Then $\langle s, Y_0\rangle = \sum \alpha_i m(Y_0)$. Now use Lemma 3.4 to get $\frac{\langle s,Y_0\rangle}{m(Y_0)} \geq w(s) \geq \sum \alpha_i$. □

Tables of quadratic forms are usually listed in order of increasing determinant and the next proposition allows us to compare the reduced determinant with the dyadic trace. This is the promised improvement of Lemma 3.4.

3.12 Proposition. (Summary) *The dyadic trace w is a type two class function $w : C_n^* \to \mathbb{R}_{\geq 0}$ satisfying for all $s \in C_n^*$:*

(1)
$$s = 0 \iff w(s) = 0,$$
(2)
$$\forall\, Y \in \mathcal{P}_n^{\mathrm{semi}}(\mathbb{R}), \quad \langle Y, s\rangle \geq w(s)m(Y).$$
Equality is attained if and only if s has a dyadic representation in the minimal vectors of Y.

(3)
$$w(s) \geq \frac{n}{\mu_n}\det(s)^{1/n}.$$
Equality is attained if and only if $s = 0$ or s^{-1} exists and attains equality in Hermite's inequality.

Proof. We prove item (2). By Lemma 3.4, $\langle Y, s\rangle \ge w(s)m(Y)$ holds for all $Y \in \mathcal{P}_n^{\mathrm{semi}}(\mathbb{R})$. Equality clearly holds if s has a dyadic representation in the minimal vectors of Y, and by Theorem 3.9 (1) this is the case for nonsingular Y whenever the equality $\frac{\langle Y,s\rangle}{m(Y)} = w(s)$ holds. For singular Y, the equality $\langle Y, s\rangle = 0$ implies that $0 = \langle Y, s\rangle = \sum \alpha_i Y[v_i]$ for any dyadic representation $s = \sum \alpha_i v_i\,{}^t v_i$ of s. The case $s = 0$ is trivial. For $s \ne 0$ we may assume the dyadic representation to be strict and thereby conclude: $Y[v_i] = 0$ and hence $v_i \in \mathrm{MinVec}(Y)$.

We demonstrate item (3), which by item (1) need only be proven for nonsingular s, by combining the arithmetic–geometric inequality with Hermite's inequality.

$$\forall\, Y \in \mathcal{P}_n(\mathbb{R}), \quad \frac{\langle s, Y\rangle}{m(Y)} \ge \frac{n\det(s)^{1/n}\det(Y)^{1/n}}{m(Y)} \ge \frac{n}{\mu_n}\det(s)^{1/n}.$$

Taking the infimum over $Y \in \mathcal{P}_n(\mathbb{R})$ gives $w(s) \ge \frac{n}{\mu_n}\det(s)^{1/n}$ by Theorem 3.9. On the other hand, equality holds above if and only if equality holds in both the arithmetic–geometric inequality and in Hermite's inequality. That is, if and only if $sY = \lambda I$ for some $\lambda \in \mathbb{R}^+$ and Y optimizes Hermite's inequality. By Theorem 3.9 there is a Y such that $w(s) = \frac{\langle s,Y\rangle}{m(Y)}$ and so the equality $w(s) = \frac{n}{\mu_n}\det(s)^{1/n}$ implies that s^{-1} optimizes Hermite's inequality.

The inequality $w(s) \ge \frac{n}{\mu_n}\det(s)^{1/n}$ shows that given any $B \in \mathbb{R}^+$ there are only a finite number of integral classes s satisfying $w(s) \le B$. Hence $w(\mathcal{P}_n(\mathbb{Z}))$ is discrete in $\mathbb{R}$ and w is type two. $\square$

§4. Calculations in low degrees

In this section, we prove some formulae for $w(s)$ for $n = 1, 2, 3$. We make tables of quadratic forms with low dyadic traces for $n = 3, 4$. We also make tables comparing the number of Fourier coefficients one needs to calculate for low weights using the old trace method versus using the new dyadic trace method for $n = 2, 3, 4$.

For $n = 1$ we have $w(s) = s$. For $n = 2$ we have a formula for $w(s)$ if $s = \begin{pmatrix} a & b \\ b & c\end{pmatrix} \in \mathcal{P}_2(\mathbb{R})$ is Minkowski reduced. This reduction condition is $2|b| \le a \le c$, see [4, p.396–397].

4.1 Proposition (Dyadic trace for $n = 2$). *Let $s = \begin{pmatrix} a & b \\ b & c\end{pmatrix} \in \mathcal{P}_2(\mathbb{R})$ be Minkowski reduced. Then we have $w(s) = a + c - |b| \ge \frac{3}{4}\operatorname{tr}(s)$.*

Proof. Consider $Y = \begin{pmatrix} 2 & \pm1 \\ \pm1 & 2\end{pmatrix} \in \mathcal{P}_2(\mathbb{Z})$ and note that $m(Y) = 2$ in either case. Then $w(s) \le \frac{\operatorname{tr}(sY)}{m(Y)} = a + c \pm b$ so that we have $w(s) \le a + c - |b|$. On the other hand, if we have $|b| \le a$ and $|b| \le c$, then s has the dyadic representation:

$$\begin{aligned} s = \begin{pmatrix} a & b \\ b & c\end{pmatrix} &= |b|\begin{pmatrix} 1 & \operatorname{sgn}(b) \\ \operatorname{sgn}(b) & 1\end{pmatrix} + (a-|b|)\begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix} + (c-|b|)\begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix} \\ &= |b|\begin{pmatrix} 1 \\ \operatorname{sgn}(b)\end{pmatrix}(1\,\operatorname{sgn}(b)) + (a-|b|)\begin{pmatrix} 1 \\ 0\end{pmatrix}(1\,0) + (c-|b|)\begin{pmatrix} 0 \\ 1\end{pmatrix}(0\,1), \end{aligned}$$

so that $w(s) \ge |b| + (a - |b|) + (c - |b|) = a + c - |b|$. Using the reduction conditions, we have: $w(s) = a+c-|b| \ge a+c-\frac{a}{2} = \frac{a+c}{2} + \frac{c}{2} = \frac{1}{2}\operatorname{tr}(s) + \frac{c}{2} \ge \frac{1}{2}\operatorname{tr}(s) + \frac{\frac{1}{2}\operatorname{tr}(s)}{2} = \frac{3}{4}\operatorname{tr}(s)$. $\square$

For $n = 3$ we also have a formula for $w(s)$ if $s = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix} \in \mathcal{P}_3(\mathbb{R})$ is in Minkowski reduced form. The reduction conditions are: $a \le b \le c$; $2|d|, 2|e| \le a$; $2|f| \le b$; and $2|d \pm e \pm f| \le a + b$, [4, p.397].

4.2 Proposition (Dyadic trace for $n = 3$). *Let $s = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix} \in \mathcal{P}_3(\mathbb{R})$ be Minkowski reduced. Then we have $w(s) \ge \frac{2}{3}\operatorname{tr}(s)$ and we have:*

Case I. If $def \le 0$, then $w(s) = a + b + c - (|d| + |e| + |f|)$;

Case II. If $def > 0$, then $w(s) = a + b + c - (|d| + |e| + |f|) + \min(|d|, |e|, |f|)$.

Proof. Consider $Y = \begin{bmatrix} 2 & \alpha & \beta \\ \alpha & 2 & \gamma \\ \beta & \gamma & 2 \end{bmatrix} \in V_3(\mathbb{Z})$ where $\alpha, \beta, \gamma \in \{0, \pm 1\}$. We compute $\det(Y) = 8 + 2\alpha\beta\gamma - 2(\alpha^2 + \beta^2 + \gamma^2)$ so that $Y \in \mathcal{P}_3(\mathbb{Z})$ *unless* $\alpha\beta\gamma = -1$. We have $m(Y) = 2$ when $\alpha\beta\gamma \ne -1$. Therefore $w(s) \le \frac{\operatorname{tr}(sY)}{m(Y)} = a + b + c + \min_{\substack{\alpha,\beta,\gamma \in \{0,1,-1\} \\ \alpha\beta\gamma \ne -1}} (\alpha d + \beta e + \gamma f)$. When $def \le 0$, we have $\min(\alpha d + \beta e + \gamma f) = -(|d| + |e| + |f|)$. When $def > 0$, we have $\min(\alpha d + \beta e + \gamma f) = -(|d| + |e| + |f|) + \min(|d|, |e|, |f|)$.

On the other hand, we can use the Minkowski reduction conditions to produce the dyadic representation: $s = |d| v_1 {}^t v_1 + |e| v_2 {}^t v_2 + |f| v_3 {}^t v_3 + (a - |d| - |e|) v_4 {}^t v_4 + (b - |d| - |f|) v_5 {}^t v_5 + (c - |e| - |f|) v_6 {}^t v_6$ where ${}^t v_1 = (1, \operatorname{sgn}(d), 0)$, ${}^t v_2 = (1, 0, \operatorname{sgn}(e))$, ${}^t v_3 = (0, 1, \operatorname{sgn}(f))$, ${}^t v_4 = (1,0,0)$, ${}^t v_5 = (0,1,0)$, ${}^t v_6 = (0,0,1)$. Therefore we have,

$$\begin{aligned} w(s) &\ge |d| + |e| + |f| + (a - |d| - |e|) + (b - |d| - |f|) + (c - |e| - |f|) \\ &= a + b + c - (|d| + |e| + |f|). \end{aligned}$$

Because of the previous upper bound, we see that $w(s) = a + b + c - (|d| + |e| + |f|)$ in Case I. In Case II we may assume that $d, e, f > 0$ by changing to an equivalent reduced s. Let $m = \min(|d|, |e|, |f|)$. We display the dyadic representation: $s = m u_1 {}^t u_1 + (d - m) u_2 {}^t u_2 + (e - m) u_3 {}^t u_3 + (f - m) u_4 {}^t u_4 + (a - d - e + m) u_5 {}^t u_5 + (b - d - f + m) u_6 {}^t u_6 + (c - e - f + m) u_7 {}^t u_7$ where ${}^t u_1 = (1,1,1)$, ${}^t u_2 = (1,1,0)$, ${}^t u_3 = (1,0,1)$, ${}^t u_4 = (0,1,1)$, ${}^t u_5 = (1,0,0)$, ${}^t u_6 = (0,1,0)$, ${}^t u_7 = (0,0,1)$, and conclude that

$$\begin{aligned} w(s) &\ge m + (d - m) + (e - m) + (f - m) + (a - d - e + m) + (b - d - f + m) + (c - e - f + m) \\ &= a + b + c - d - e - f + m. \end{aligned}$$

We then have $w(s) = a + b + c - (|d| + |e| + |f|) + m$ from the previous upper bound in Case II. We have shown the equality $w(s) = a + b + c + \min_{\substack{\alpha,\beta,\gamma \in \{0,1,-1\} \\ \alpha\beta\gamma \ne -1}} (\alpha d + \beta e + \gamma f)$ for reduced s. Further application of the reduction conditions implies that we have:

$$\begin{aligned} w(s) &\ge \operatorname{tr}(s) - \max|\alpha d + \beta e + \gamma f| \ge \operatorname{tr}(s) - \frac{a+b}{2} \\ &= \frac{a+b+c}{2} + \frac{c}{2} = \tfrac{1}{2}\operatorname{tr}(s) + \frac{c}{2} \ge \tfrac{1}{2}\operatorname{tr}(s) + \frac{\frac{1}{3}\operatorname{tr}(s)}{2} = \frac{2}{3}\operatorname{tr}(s). \quad \square \end{aligned}$$

For $n = 4$, we do not have a general rule such as Proposition 4.1 or 4.2 but we have worked out many instances. In $n = 4$, we computer search for upper bounds on $w(s)$ using

$w(s) \leq \frac{\langle s,Y\rangle}{m(Y)}$ and for lower bounds using $w(s) \geq \sum \alpha_i$ over dyadic representations of s until the upper and lower bounds coincide for some Y and some $s = \sum \alpha_i v_i\, {}^t v_i$. In this case we have $v_i \in \mathrm{MinVec}(Y)$ by Theorem 3.9. For $n = 4$, we use the tables of G. Nipp, "Quaternary Quadratic Forms: Computer Generated Tables" [24].

In Table 1, we list the first 10 even quaternary forms, [24], listed in order of increasing determinant. For $s = \begin{bmatrix} a & e & f & h \\ e & b & g & i \\ f & g & c & j \\ h & i & j & d \end{bmatrix}$ we write: $a \quad b \quad c \quad d \quad 2e \quad 2f \quad 2g \quad 2h \quad 2i \quad 2j$ and as is traditional we let $D = 16\det(s)$, the discriminant.

Table 1. (Even Quaternary Forms)

D	$w(s)$	s
4	2	1 1 1 1 0 0 0 1 1 1
5	2.5	1 1 1 1 1 0 0 1 0 1
8	3	1 1 1 1 0 0 0 1 1 0
9	3	1 1 1 1 1 0 0 0 0 1
12	3	1 1 1 2 1 1 0 1 0 0
12	3.5	1 1 1 1 0 0 0 1 0 0
13	3.5	1 1 1 2 1 1 0 0 1 0
16	4	1 1 1 1 0 0 0 0 0 0
16	4	1 1 1 2 1 1 0 0 0 0
17	3.5	1 1 1 2 1 0 0 1 0 1

In Tables 2, 3, and 4, we consider an $f \in S_n^k$ with Fourier expansion $\sum a_s e(\langle s, \Omega\rangle)$. The first column lists the weight k of the vector space S_n^k. The second column lists a number t_0 such that the condition that $a_s = 0$ for all $s : \mathrm{tr}(s) \leq t_0$ implies $f \equiv 0$. The given t_0 is the greatest integer less than or equal to $\frac{2}{\sqrt{3}} n \mu_n^n \frac{k}{4\pi}$. The third column lists a number w_0 such that the condition that $a_s = 0$ for all $s : w(s) \leq w_0$ implies $f \equiv 0$. The given w_0 is the greatest half-integer less than or equal to $\frac{2}{\sqrt{3}} n \frac{k}{4\pi}$. The fourth column gives the number T of integral-valued classes $[s]$ such that $\mathrm{tr}(s) \leq t_0$ for some s. The fifth column gives the number W of integral-valued classes $[s]$ such that $w(s) \leq w_0$. The sixth column gives $\dim S_n^k$, if known. The numbers T and W in the fourth and fifth columns are the number of Fourier coefficients of f one must compute to show that $f \equiv 0$ using the old and new methods. We always have $\dim S_n^k \leq W \leq T$ and the difference $T - W$ measures the superiority of the new method over the old one.

Table 2. (n=2)

k weight	t_0 trace	w_0 dyadic	T old estimate	W new estimate	$\dim S_n^k$ true dim.
0	0	0	0	0	0
2	0	0	0	0	0
4	0	0.5	0	0	0
6	1	1	0	0	0
8	1	1	0	0	0
10	2	1.5	2	1	1
12	2	2	2	2	1
14	3	2.5	4	3	1
16	3	2.5	4	3	2
18	4	3	9	5	2
20	4	3.5	9	7	3
22	5	4	14	10	4
24	5	4	14	10	5
26	6	4.5	23	13	5
28	6	5	23	17	7
30	7	5.5	32	21	8

Table 3. (n=3)

k weight	t_0 trace	w_0 dyadic	T old estimate	W new estimate	$\dim S_n^k$ true dim.
0	0	0	0	0	0
2	1	0.5	0	0	0
4	2	1	0	0	0
6	3	1.5	3	0	0
8	4	2	8	1	0
10	5	2.5	20	2	0
12	6	3	44	5	1
14	7	3.5	85	8	1
16	8	4	152	16	3
18	9	4.5	263	24	4
20	11	5.5	674	58	6

Table 4. (n=4)

k weight	t_0 trace	w_0 dyadic	T old estimate	W new estimate	$\dim S_n^k$ true dim.
0	0	0	0	0	0
1	1	0	0	0	0
2	2	0.5	0	0	0
3	4	1	6	0	0
4	5	1	17	0	0
5	7	1.5	131	0	0
6	8	2	334	1	0
7	10	2.5	1611	2	0
8	11	2.5	3285	2	1
9	13	3	12517+	5	0
10	14	3.5	22635+	10	1+
11	16	4	42014+	23	0
12	17	4	48800+	23	2
13	19	4.5	56977+	42	0

The numbers with a "+" in the "old estimate" column of Table 4 are due to the size limitations of Nipp's tables; the forms are listed there only up to discriminant 1732. The result of $\dim S_4^{11}$ is from an unpublished preprint of the authors. The results for $\dim S_4^9$, $\dim S_4^{10}$ and $\dim S_4^{13}$ are from [26]. The results for $\dim S_4^6$ and $\dim S_4^8$ may be found in [29][6][27] and for $\dim S_4^7$ in [29][6].

§5. Examples and Discussion.

Let us first consider Tables 2, 3, and 4. Table 2 shows that $S_2^k = 0$ for $k \leq 8$, a result that follows from the old trace method as well as from the new dyadic trace method. In weight 10, the new method correctly bounds the one dimensional space S_2^{10}.

For $n = 3$, Table 3 shows that $S_3^k = 0$ for $k \leq 6$, bettering the older method. The first nonzero S_3^k, however, is S_3^{12} and so the estimate appears to be far from the mark. This is not really the case, however, as there is a "missing" cusp form of weight 9 in S_3: the cusp form $\chi_{18} \in S_3^{18}$ defining the hyperelliptic locus inside $\mathcal{A}_3$ satisfies $\min m\,(\mathrm{supp}(\chi_{18})) = 2$ and although $\sqrt{\chi_{18}}$ does not exist over $\mathcal{H}_3$ as a Siegel modular form it does exist over Teichmüller space as a Teichmüller modular form [12]. A defining expression for χ_{18} can be taken as

$$\chi_{18}(\Omega) = \prod_{\text{even } \zeta}^{36} \theta[\zeta](0, \Omega)$$

and calculation shows that

$$\chi_{18}(\Omega) = -2^{28} e\left(\langle \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \Omega\rangle\right) + 2^{29} e\left(\langle \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & x \\ 1 & x & 2 \end{bmatrix}, \Omega\rangle\right) + \dots$$

plus equivalent terms and terms with classes of higher dyadic trace. Since a cusp form of weight 18 exists with $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ in its support, the linearity of our method of estimation means that it cannot rule out the existence of a cusp form of weight $9 = \frac{1}{2}(18)$ with $\begin{bmatrix} 1 & x & x \\ x & 1 & 0 \\ x & 0 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ in its support.

For $n = 4$, Table 4 shows that $S_4^k = 0$ for $k \leq 5$, and the contrast between the old and new methods is dramatic. Formerly intractable calculations become tractable. The first nonzero S_4^k, is $S_4^8 = \mathbb{C}J$, see [29][27][6], where J is Schottky's modular form vanishing on the Jacobian locus in $\mathcal{A}_4$. The following definition [15] of J can be given (write $\frac{1}{2} = x$):

$$J = r_{00}^2 + r_{0x}^2 + r_{x0}^2 - 2\left(r_{00}r_{0x} + r_{00}r_{x0} + r_{0x}r_{x0}\right),$$

$$r_{\mu\nu} = \prod_{\alpha,\beta,\gamma \in \{0,x\}}^{8} \theta\begin{bmatrix} \mu & 0 & 0 & 0 \\ \nu & \alpha & \beta & \gamma \end{bmatrix}(0, \Omega) \qquad \text{for } \mu, \nu \in \{0, x\}.$$

Our method, however, allows the possibility that S_4^6 is nonzero. We do not attribute, as in $n = 3$, this poor showing to the existence of cusp forms f of higher weight with $\min m\,\{\,\mathrm{supp}(f)\,\} > 1$ but suggest that the difference between the true value of $w_4 = \sup_{\Omega \in \mathcal{H}_4} \inf_{\sigma \in \Gamma_4} w\left((\mathrm{Im}\,\sigma\Omega)^{-1}\right)$ and the available upper bound $\frac{2}{\sqrt{3}}n = \frac{8}{\sqrt{3}}$ is affecting the sharpness of our estimates. Further progress in estimating the constants w_n, m_n, or even $\det_n$, will correspondingly improve the estimates given here.

As intimated in the Introduction, finding linear relations among the theta series attached to Type II lattices is an obvious application for our results. Recall that a lattice Λ in a M dimensional Euclidean space is called Type II [4] if it is self dual and if the norm of any element from Λ is an even integer. The corresponding class of M-by-M quadratic forms $[Q]$ is obtained from the inner product by choosing any basis for the lattice. The associated *theta series* is defined by $\vartheta^Q(\Omega) = \theta(0, \Omega \otimes Q)$ where we make use in this definition of the map $\Omega \mapsto \Omega \otimes Q$ from $\mathcal{H}_n$ to $\mathcal{H}_{Mn}$ [21, p. 217]. The significance of Type II lattices is that $\vartheta_\Lambda \in S_n^{\frac{M}{2}}$ for each n, see [11, p. 17]. Useful formal properties of the theta series are:

$$\vartheta_{\Lambda_1 \oplus \Lambda_2} = \vartheta_{\Lambda_1}\vartheta_{\Lambda_2}; \quad \vartheta_\Lambda(\Omega_1 \oplus \Omega_2) = \vartheta_\Lambda(\Omega_1)\vartheta_\Lambda(\Omega_2); \quad \Phi\{\,\vartheta_\Lambda \text{ on } \mathcal{H}_n\,\} = \vartheta_\Lambda \text{ on } \mathcal{H}_{n-1}.$$

If Λ is a Type II Lattice then M is necessarily a multiple of 8. For $M = 8$ there is one isometry class E_8, and it can be shown that $M_n^4 = \mathbb{C}\vartheta_{E_8}$ for all n, see [6]. For $M = 16$ there are two isometry classes, $E_8 \oplus E_8$ and D_{16}^+, and the discussion of the linear dependencies between their theta series is prototypical for the whole subject. We know that $\vartheta_{E_8}^2 - \vartheta_{D_{16}^+}$ is a cusp form of weight 8 in $n = 1$ and so must be identically zero. Therefore $\vartheta_{E_8}^2 - \vartheta_{D_{16}^+}$ is a cusp form of weight 8 in $n = 2$ and observation of Table 1 shows that it is also identically zero. This is a theorem of Witt. Turning to Table 2 in $n = 3$ we see that a cusp form of weight 8 is uniquely determined by its Fourier coefficient for the unique class of dyadic trace two, $A_3 = \begin{bmatrix} 1 & x & x \\ x & 1 & 0 \\ x & 0 & 1 \end{bmatrix}$. A computation, see [28, p. 353], shows that $E_8 \oplus E_8$ and D_{16}^+ both represent A_3 $480 \cdot 56 \cdot 27 = 725{,}760$ times so that $\vartheta_{E_8}^2 - \vartheta_{D_{16}^+}$ is again trivial in $n = 3$. This fact, Witt's Conjecture, was first proven by Igusa [13] and Kneser [19]. In $n = 4$, $\vartheta_{E_8}^2 - \vartheta_{D_{16}^+}$ is a cusp form of weight 8 and Table 3 tells us that the modular form is determined by two Fourier coefficients, those for $D_4 = \begin{bmatrix} 1 & x & x & x \\ x & 1 & 0 & 0 \\ x & 0 & 1 & 0 \\ x & 0 & 0 & 1 \end{bmatrix}$ and $A_4 = \begin{bmatrix} 1 & x & x & 0 \\ x & 1 & 0 & x \\ x & 0 & 1 & 0 \\ 0 & x & 0 & 1 \end{bmatrix}$. A theorem of Igusa [15] shows that $\vartheta_{E_8}^2 - \vartheta_{D_{16}^+} = \frac{3^2 \cdot 5 \cdot 7}{2^2} J$ is really Schottky's modular form. We can prove this here by evaluating these two Fourier coefficients. We have that $E_8 \oplus E_8$ represents D_4 $480 \cdot 56 \cdot 27 \cdot 10 = 7{,}257{,}600$ times and A_4 $480 \cdot 56 \cdot 27 \cdot 16 = 11{,}612{,}160$ times; whereas D_{16}^+ represents D_4 $480 \cdot 56 \cdot 26 \cdot 3 = 2{,}096{,}640$ times and A_4 $480 \cdot 56 \cdot 26 \cdot 24 = 16{,}773{,}120$ times. Therefore, by subtraction, we have

$$\vartheta_{E_8}^2 - \vartheta_{D_{16}^+} = 5{,}160{,}960 e\left(\langle D_4, \Omega\rangle\right) - 5{,}160{,}960 e\left(\langle A_4, \Omega\rangle\right) + \dots$$

whereas from the expression defining J we compute

$$J = 2^{16} e\left(\langle D_4, \Omega\rangle\right) - 2^{16} e\left(\langle A_4, \Omega\rangle\right) + \dots.$$

Noting that $5{,}160{,}960 = 2^{14} \cdot 3^2 \cdot 5 \cdot 7$ we have Igusa's result: $\vartheta_{E_8}^2 - \vartheta_{D_{16}^+} = \frac{3^2 \cdot 5 \cdot 7}{2^2} J$.

For $M = 24$, there are 24 isometry classes of Type II lattices, the Niemeier lattices. Classifying the linear relations among the theta series of the Niemeier lattices is a very interesting problem. The best results are due to Erokhin [9], see also [3], and we revisit these results in the light of our present estimates for weight 12 cusp forms.

The span of the ϑ_Λ is 2 dimensional for $n = 1$ and a cusp form is completely determined by the coefficients a_s with $s \leq \frac{2}{\sqrt{3}} \cdot 1 \cdot \frac{12}{4\pi} \leq 1.103$, that is by a_1. Here a_1 has the interpretation as the number of lattice elements of norm 2; zero in the case of the Leech lattice and the "kissing number" for the remaining 23 Niemeier lattices. Examining Table 1, we see that for $n = 2$ a cusp form of weight 12 is determined by the two classes with dyadic trace less than or equal to 2, that is by $\begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. We know that $\dim S_2^{12} = 1$ so that the Fourier coefficients of cusp forms for these two classes must always bear the same ratio; in fact this ratio is $1 : 10$. Hence the span of the theta series attached to the Niemeier lattices is 3 dimensional for $n = 2$. Examining Table 2, we see that for $n = 3$ a cusp form of weight 12 is determined by the five classes with dyadic trace less than or equal to 3; namely

$$\begin{bmatrix} 1 & x & x \\ x & 1 & 0 \\ x & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & x & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & x & x \\ x & 1 & 0 \\ x & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & x \\ x & x & 2 \end{bmatrix}$$

Again, the calculation that the Fourier coefficients of any cusp form for these fives classes are always in the same proportion is equivalent to the fact that $\dim S_3^{12} = 1$. Hence the span of the theta series attached to the Niemeier lattices is 4 dimensional for $n = 3$. Examining Table 4, we see that for $n = 4$ a cusp form of weight 12 is determined by the 23 classes with dyadic trace less than or equal to 4, listed in Table 5. We know that $\dim S_4^{12} = 2$, see [27], but this fact is known as a corollary of the work of Erokhin that the span of the theta seires attached to the Niemeier lattices in $n = 4$ is 6 dimensional. The two papers of Erokhin [9] [10] that provide this result are intricate and it would be nice to give a straightforward alternate proof of $\dim S_4^{12} = 2$ by computing the rank of a certain 24×27 matrix to be 6. The 24 rows of this matrix are indexed by the 24 Niemeier lattices and the 27 columns are indexed by the matrices: 0, 1, $\begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & x & x \\ x & 1 & 0 \\ x & 0 & 1 \end{bmatrix}$, and the 23 4×4 forms of dyadic trace less than or equal to 4 listed in Table 5. The ij-entry of this matrix is the representation number of the i-th Niemeier lattice on the j-th form. This computation, although within the realm of tractability, is beyond our computational resources. The 27 Fourier coefficients needed for each Niemeier lattice using this method stand in stark contrast to the more than $48,000$ forms of trace less than or equal to 17 that are required by the old method. Nipp's extensive computer generated tables of quaternary form do not even exhaust all of the forms with trace less than or equal to 17, so the actual figure is probably closer to 100,000 forms.

Table 5. (The 23 Quaternary Forms with dyadic trace ≤ 4)
See Table 1 for the first 10; this table lists the other 13.

D	$w(s)$	s
20	4	1 1 1 3 1 1 0 1 0 0
20	3.5	1 1 1 2 0 0 0 1 1 1
20	4	1 1 1 2 1 0 0 1 0 0
21	4	1 1 1 2 1 0 0 0 0 1
24	4	1 1 1 2 0 0 0 1 1 0
25	3.5	1 1 2 2 1 1 0 1 1 2
28	4	1 1 2 2 1 1 0 0 1 1
32	4	1 1 2 2 0 0 0 1 1 2
32	4	1 1 2 2 1 1 0 1 0 0
33	4	1 1 2 2 0 1 1 1 0 2
32	4	1 1 2 2 0 1 1 1 1 1
33	4	1 1 2 2 0 0 0 1 2 2
64	4	2 2 2 2 0 0 0 2 2 2

To apply our estimates in another way to lattices of lower rank and to obtain some new results, consider theta series with harmonic coefficients. Let Λ be a Type II lattice of rank m and let $Q : M^{k\times n}(\mathbb{C}) \to \mathbb{C}$ be a pluri-harmonic polynomial [11, p. 161] of degree ν and define $\vartheta_{\Lambda,Q} : \mathcal{H}_n \to \mathbb{C}$ by

$$\vartheta_{\Lambda,Q}(\Omega) = \sum_{L\in\Lambda^n} Q(L) e^{i\pi\langle\, {}^tLL,\, \Omega\rangle}.$$

The function $\vartheta_{\Lambda,Q}$ is then a Siegel modular cusp form of weight $\frac{m}{2} + \nu$ and degree n. Furthermore we know that $Q(X) = \det(BX)^\nu$ is pluri-harmonic whenever B satisfies $B\,{}^tB = 0$. Here $B \in M_{n\times m}(\mathbb{C})$ and $X \in M_{m\times n}(\mathbb{C})$. Set B_1 to be the 4×8 matrix $[I\ iI]$ and B_2 to be the 4×16 matrix $[I\ 0\ iI\ 0]$; set $Q_1(X) = \det(B_1X)^6$ and $Q_2(X) = \det(B_2X)^2$. Then both ϑ_{E_8,Q_1} and $\vartheta_{E_8\oplus E_8,Q_2}$ are in S_4^{10} and their Fourier coefficients for the ten classes with dyadic trace less than or equal to 3.5 are given in Table 6. Since all of the coefficients are in the ratio $-5 :: 96$ we conclude that $96\vartheta_{E_8,Q_1} + 5\vartheta_{E_8\oplus E_8,Q_2} = 0$. As a final example we construct an element of S_4^{10} from thetanullwerte.

A fundamental system in $\mathbb{F}_2^{2n}$ is a sequence of $2n + 2$ characteristics where all triplets are azygetic [15, p. 534]. For $n = 4$ the number of fundamental systems composed entirely of even characteristics is 13056. For any fundamental system, FS, the function $\prod_{\zeta\in FS} \theta[\zeta](0,\Omega)$ is a cusp form for $\Gamma_n(2)$. If we define G_{10} by

$$G_{10} = \sum_{FS:FS \text{ is an even fund. sys.}}^{13056} \prod_{\zeta\in FS} \theta[\zeta](0,\Omega)^2$$

then G_{10} is in S_4^{10}. An examination of the ten Fourier coefficients a_s with $w(s) \le 3.5$ shows that they are proportional to those listed in Table 6. The coefficient of G_{10} for D_4 is $100663296 = 2^{25} \cdot 3$ so that we have $3^2 \cdot 5^2\, G_{10} = -2^{12}\, \vartheta_{E_8,Q_1}$, an identity between modular forms arising from quite different sources.

Table 6. (Fourier Coefficients)

D	$w(s)$	s	a_s: ϑ_{E_8,Q_1}	a_s: $\vartheta_{E_8 \oplus E_8,Q_2}$
4	2	1 1 1 1 0 0 0 1 1 1	-5529600	106168320
5	2.5	1 1 1 1 1 0 0 1 0 1	-11059200	212336640
8	3	1 1 1 1 0 0 0 1 1 0	121651200	-2335703040
9	3	1 1 1 1 1 0 0 0 0 1	-398131200	7644119040
12	3	1 1 1 2 1 1 0 1 0 0	199065600	-3822059520
12	3.5	1 1 1 1 0 0 0 1 0 0	199065600	-3822059520
13	3.5	1 1 1 2 1 1 0 0 1 0	-143769600	2760376320
17	3.5	1 1 1 2 1 0 0 1 0 1	1282867200	-24631050240
20	3.5	1 1 1 2 0 0 0 1 1 1	-6635520000	127401984000
25	3.5	1 1 2 2 1 1 0 1 1 2	-13713408000	263297433600

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