RESTRICTION OF SIEGEL MODULAR FORMS TO MODULAR CURVES

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Abstract. We study homomorphisms from the ring of Siegel modular forms of a given degree to the ring of elliptic modular forms for a congruence subgroup. These homomorphisms essentially arise from the restriction of Siegel modular forms to modular curves. These homomorphisms give rise to linear relations among the Fourier coefficients of a Siegel modular form. We use this technique to prove that $\dim S_4^{10} = 1$.

§1. Introduction.

A Siegel modular cusp form of degree n has a Fourier series $f(\Omega) = \sum_t a(t)e(\mathrm{tr}(\Omega t))$ where t runs over $\mathcal{X}_n$, the set of positive definite semi-integral $n \times n$ forms. If we restrict attention to cusp forms of even weight then the Fourier coefficients are class functions of t. The vector space S_n^k of cusp forms of weight k is finite dimensional and so there exist finite subsets $\mathcal{S} \subset \mathrm{classes}(\mathcal{X}_n)$ such that the projection map $\mathrm{FS}_{\mathcal{S}} : S_n^k \to \mathbb{C}^{\mathcal{S}}$ given by $f \mapsto \prod_{[t]\in\mathcal{S}} a(t)$ is injective. The following Theorem [13, p. 218] gives one such $\mathcal{S}$ that is readily computable from n and k. Instead of ordering semi-integral forms t by their determinant $\det(t)$ we order them by their dyadic trace $w(t)$. Denote by $\mathcal{P}_n(\mathbb{F})$ the positive definite $n \times n$ symmetric matrices with coefficients in $\mathbb{F} \subseteq \mathbb{R}$. The dyadic trace $w : \mathcal{P}_n(\mathbb{R}) \to \mathbb{R}^+$ is a class function and only a finite number of classes from $\mathcal{X}_n$ will have a dyadic trace below any fixed bound, see [13].

Theorem 1.1. *Let $n, k \in \mathbb{Z}^+$. Let $\mathcal{S} = \{[t] : t \in \mathcal{X}_n$ and $w(t) \le n\frac{2}{\sqrt{3}}\frac{k}{4\pi}\}$. The map $\mathrm{FS}_{\mathcal{S}} : S_n^k \to \mathbb{C}^{\mathcal{S}}$ is injective.*

This Theorem allows one to deduce equality in S_n^k from equality on the Fourier coefficients for $\mathcal{S}$. There are two obvious avenues for improvement. First, as is evident from Table 1, the bound $\dim S_n^k \le \mathrm{card}(\mathcal{S})$ is tractable but crude and we would like to trim down the set $\mathcal{S}$ to make $\mathrm{card}(\mathcal{S})$ closer to $\dim S_n^k$. Second, the image $\mathrm{FS}_{\mathcal{S}}(f)$ determines f and one would like to compute some Fourier coefficients outside of $\mathcal{S}$ directly from the Fourier coefficients in $\mathcal{S}$. This paper realizes both improvements. We give a method for producing linear relations on the Fourier coefficients of the elements in S_n^k. Table 1 gives $\dim S_4^k$, $\mathrm{card}(\mathcal{S})$ and examples of linear relations for even $k \le 12$. These are the only even weights for which $\dim S_4^k$ is known and the result $\dim S_4^{10} = 1$ is a new one.

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Table 1.

k	$\dim S_4^k$	card($\mathcal{S}$)	linear relations
2	0	0	
4	0	0	
6	0	1	$a(\frac{1}{2}D_4) = 0$
8	1	2	$a(\frac{1}{2}D_4) + a(\frac{1}{2}A_4) = 0$
10	1	10	see equations (3.3)
12	2	23	21 uncomputed relations

For $k \leq 4$ we have $\mathcal{S} = \emptyset$ and so Theorem 1.1 by itself proves $S_4^k = 0$, results due to Christian [2] and Eichler [4][5]. For $k = 6$ we have $\mathcal{S} = \{[\frac{1}{2}D_4]\}$ and the method in this paper provides the linear relation $a(\frac{1}{2}D_4) = 0$ so that we conclude $\dim S_4^6 = 0$. For $k = 8$ we have $\mathcal{S} = \{[\frac{1}{2}D_4], [\frac{1}{2}A_4]\}$ and the method provides the linear relation $a(\frac{1}{2}D_4) + a(\frac{1}{2}A_4) = 0$ showing that $\dim S_4^8 \leq 1$. The Schottky form J is in S_4^8 [9] so we have $\dim S_4^8 = 1$, see [14][11][3] for these results. For $k = 10$ the $\mathcal{S}$ consists of the ten classes in Table 3 and the method provides the nine linearly independent relations given in equation 3.3. We know the cusp form G_{10} is in S_4^{10}, see [13, p. 232], so that we have $\dim S_4^{10} = 1$, a result that has been beyond the reach of other methods [12][3]. By the work of Erokhin $\dim S_4^{12} = 2$ is already known, see [6][7][11]. Linear relations among Fourier coefficients for semi-integral forms not solely in $\mathcal{S}$ allow the computation of Fourier coefficients outside of $\mathcal{S}$.

The method of producing linear relations on Fourier coefficients from S_n^k relies on the homomorphisms $\phi_s^* : S_n^k \to S_1^{nk}(\Gamma_0(\ell))$ which exist for any $s \in \mathcal{P}_n(\mathbb{Z})$ and any $\ell \in \mathbb{Z}^+$ with ℓs^{-1} integral. We write elements of $\Gamma_1 = \mathrm{Sp}_1(\mathbb{Z})$ as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and define the subgroup $\Gamma_0(\ell)$ by $\ell | c$ and the subgroup Δ_1 by $c = 0$. We define $\phi_s(\tau) = s\tau$ so that for $f \in M_n^k$ we have $(\phi_s^* f)(\tau) = f(s\tau)$. There are three important points about these homomorphisms: (1) The image ring $M_1(\Gamma_0(\ell))$ is amenable to computation. (2) The Fourier coefficients of $\phi_s^* f$ at each cusp are linear combinations of the Fourier coefficients of f, see Proposition 2.3. (3) There are lots of $n \times n$ integral forms s. The first point allows us to work out the linear relations among the Fourier coefficients at all cusps of elements in $S_1^{nk}(\Gamma_0(\ell))$. The second point induces linear relations on the Fourier coefficients of elements in S_n^k from the linear relations on $S_1^{nk}(\Gamma_0(\ell))$. The third point allows us to continue producing linear relations if more are desired.

We illustrate the technique in weights 6 and 8 when the number of Fourier coefficients remains small. Let $f \in S_4^k$ have the Fourier expansion $f(\Omega) = \sum_t a(t) e(\langle \Omega, t \rangle)$ where $\langle \Omega, t \rangle = \mathrm{tr}(\Omega t)$. Let D_4 represent the 4×4 form of this root lattice ($D_4 = 2B_0$ from Table 3). We compute the Fourier expansion of $\phi_{D_4}^* f$ in powers of $q = e(\tau)$. For any $s \in \mathcal{P}_n(\mathbb{Q})$ we expand $\phi_s^* f$ into a Fourier series as

$$(\phi_s^* f)(\tau) = \sum_{j \in \mathbb{Q}^+} \left(\sum_{t : \langle s, t \rangle = j} a(t) \right) q^j.$$

For simplicity we will henceforth assume that k is even. If we introduce the notation

$\mathcal{V}(j,s,t) = \text{card}\{v \in \mathcal{X}_n : [v]=[t],\ \langle v,s\rangle = j\}$ then we can write

$$(\phi_s^* f)(\tau) = \sum_{j\in\mathbb{Q}^+} \left(\sum_{[t]} \mathcal{V}(j,s,t)\, a(t) \right) q^j. \tag{1.2}$$

Table 2 is a table of the representation numbers $\mathcal{V}(j, D_4, t)$ for $j \le 7$, omitted entries are zero. See Table 3 for the list of $B_0, B_1, \dots, B_9$.

Table 2. $\mathcal{V}(j, D_4, t)$.

j	B_0	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9
4	1									
5	16	48								
6	144	288	216	48	12					
7	384	1488	864	288	144	432	240	288	48	16

Thus we have the following expansion:

$$\begin{aligned}\left(\phi_{D_4}^* f\right)(\tau) =& a(B_0)q^4 + \left(16a(B_0) + 48a(B_1)\right) q^5 \\ &+ \left(144a(B_0) + 288a(B_1) + 216a(B_2) + 48a(B_3) + 12a(B_4)\right) q^6 \\ &+ (384a(B_0) + 1488a(B_1) + 864a(B_2) + 288a(B_3) + 144a(B_4) + 432a(B_5) \\ &\quad + 240a(B_6) + 288a(B_7) + 48a(B_8) + 16a(B_9))q^7 + \cdots .\end{aligned} \tag{1.3}$$

The function $\phi_{D_4}^* f \in S_1^{4k}(\Gamma_0(2))$ is invariant under the Fricke operator because D_4^{-1} is equivalent to $\frac{1}{2}D_4$, see Proposition 2.2. The ring $M_1(\Gamma_0(2))$ is generated by $E_{2,2}^- \in M_1^2(\Gamma_0(2))$ and $E_{4,2}^- \in M_1^4(\Gamma_0(2))$ and the ring of cusp forms is principally generated by $C_{8,2}^+ \in S_1^8(\Gamma_0(2))$. The $\pm$ superscript indicates an eigenvalue of ± 1 under the Fricke operator. In general we define $E_{k,d}^\pm(\tau) = (E_k(\tau) \pm d^{\frac{k}{2}} E_k(d\tau))/(1 \pm d^{\frac{k}{2}})$ where the $E_k(\tau) = 1 - \frac{2k}{B_k}\sum_{n=1}^\infty \sigma_{k-1}(n)q^n$ are the Eisenstein series and the B_k are given by $t/(e^t-1) = \sum_{k=0}^\infty B_k t^k/k!$. We have $E_{k,d}^\pm \in M_1^k(\Gamma_0(d))$ except in the case of $E_{2,d}^+$. The Fourier expansions of these generators are given by

$$E_{2,2}^-(\tau) = 1 + 24\sum_{n=1}^\infty \left(\sigma_1(n) - 2\sigma_1(n/2)\right) q^n = 1 + 24q + 24q^2 + 96q^3 + 24q^4 + 144q^5 + \dots$$

$$E_{4,2}^-(\tau) = 1 - 80\sum_{n=1}^\infty \left(\sigma_3(n) - 4\sigma_3(n/2)\right) q^n = 1 - 80q - 400q^2 - 2240q^3 - 2960q^4 - \dots$$

$$C_{8,2}^+(z) = \frac{1}{256}\left(E_{2,2}^-(\tau)^4 - E_{4,2}^-(\tau)^2\right) = q - 8q^2 + 12q^3 + 64q^4 - 210q^5 - 96q^6 - \dots$$

The vanishing order of $\phi_{D_4}^* f$ at the cusp $[I]$ is at least 4 and because $\phi_{D_4}^* f$ is an eigenfunction of the Fricke operator the vanishing order at the cusp $[J]$ is the same. Thus we have $(C_{8,2}^+)^4 | \phi_{D_4}^* f$ in $M_1(\Gamma_0(2))$. For $k=6$ this means $\phi_{D_4}^* f = 0$ and so every coefficient in equation 1.3 gives a homogeneous linear relation; in particular we must have $a(B_0)=0$ (or $a(\frac{1}{2}D_4)=0$) and hence by Theorem 1.1 we have $S_4^6 = 0$. For $k=8$ there is a parameter $c \in \mathbb{C}$ such that

$$\phi_{D_4}^* f = c(C_{8,2}^+)^4 = c\left(q^4 - 32q^5 + 432q^6 - 2944q^7 + 7192q^8 + 39744q^9 - \dots\right).$$

Elimination of the parameter c provides the following 3 linear relations for any $f \in S_4^8$.

$$\begin{aligned} a(B_0) + a(B_1) &= 0; \\ -24a(B_0) + 24a(B_1) + 18a(B_2) + 4a(B_3) + a(B_4) &= 0; \\ 208a(B_0) + 93a(B_1) + 54a(B_2) + 18a(B_3) + 9a(B_4) + 27a(B_5) & \\ + 15a(B_6) + 18a(B_7) + 3a(B_8) + a(B_9) &= 0. \end{aligned} \tag{1.4}$$

As mentioned, the first relation alone, $a(\frac{1}{2}D_4) + a(\frac{1}{2}A_4) = 0$ (note $B_1 = \frac{1}{2}A_4$), implies that $\dim S_4^8 \leq 1$.

For $k = 10$ there are parameters $\alpha, \beta \in \mathbb{C}$ such that $\phi_{D_4}^* f = (C_{8,2}^+)^4 \left(\alpha(E_{2,2}^-)^4 + \beta C_{8,2}^+\right)$. The element $(E_{2,2}^-)^2 E_{4,2}^-$ cannot occur in this representation because it has eigenvalue -1 under the Fricke operator. Elimination of the parameters α and β provides two linear relations:

$$\begin{aligned} 224a(B_0) &= 184a(B_1) + 18a(B_2) + 4a(B_3) + a(B_4); \\ 21376a(B_1) &= -16110a(B_2) - 3916a(B_3) - 1231a(B_4) - 1512a(B_5) - 840a(B_6) \\ &\quad - 1008a(B_7) - 168a(B_8) - 56a(B_9). \end{aligned} \tag{1.5}$$

In conjunction with Theorem 1.1 these two relations imply $\dim S_4^{10} \leq 8$ but it will require another homomorphism $\phi_H^* : S_4^{10} \to S_1^{40}(\Gamma_0(6))$ and a more extensive computation to prove that $\dim S_4^{10} \leq 1$.

§2. Propositions.

We let $\Gamma_n = \mathrm{Sp}_n(\mathbb{Z})$. We write elements of $\mathrm{Sp}_n(\mathbb{R})$ as $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. The group $\mathrm{Sp}_n(\mathbb{R})$ acts on functions from the right via $(f|_k \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right))(\Omega) = \det(C\Omega + D)^{-k} f\left((A\Omega + B)(C\Omega + D)^{-1}\right)$.

Proposition 2.1. *Let $n, \ell \in \mathbb{Z}^+$. Let $s, \ell s^{-1} \in \mathcal{P}_n(\mathbb{Z})$. The map $\phi_s^* : M_n^k \to M_1^{nk}(\Gamma_0(\ell))$ is a graded ring homomorphism.*

Proof. For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_1(\mathbb{R})$ we have

$$\begin{aligned} (\phi_s^* f \mid_{nk} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))(\tau) &= (c\tau + d)^{-nk} f(\frac{a\tau + b}{c\tau + d} s) \\ &= (c\tau + d)^{-nk} f((a\tau s + bs)(cs^{-1}\tau s + dI)^{-1}) \\ &= (c\tau + d)^{-nk} f(\left(\begin{smallmatrix} aI & bs \\ cs^{-1} & dI \end{smallmatrix}\right) \cdot \tau s) \\ &= (c\tau + d)^{-nk} \det(cs^{-1}\tau s + dI)^k (f|_k \left(\begin{smallmatrix} aI & bs \\ cs^{-1} & dI \end{smallmatrix}\right))(\tau s) = (f|_k \left(\begin{smallmatrix} aI & bs \\ cs^{-1} & dI \end{smallmatrix}\right))(\tau s). \end{aligned}$$

If we now assume that $\sigma \in \Gamma_0(\ell)$ then cs^{-1} is integral and so $\left(\begin{smallmatrix} aI & bs \\ cs^{-1} & dI \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{Z})$. Therefore we have $(f|\left(\begin{smallmatrix} aI & bs \\ cs^{-1} & dI \end{smallmatrix}\right))(\tau s) = f(\tau s) = \phi_s^* f(\tau)$. It is straightforward to see that $\phi_s^* f$ is holomorphic on $\mathcal{H}_1$ and that it is bounded on domains of type $\{\tau \in \mathcal{H}_1 : \mathrm{Im}\tau > y_0\}$. Thus we have $\phi_s^* : M_n^k \to M_1^{nk}(\Gamma_0(\ell))$. $\square$

For $\ell \in \mathbb{Z}^+$ let $W_\ell = \frac{1}{\sqrt{\ell}} \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix}$ denote the Fricke involution.

Proposition 2.2. *Let n, $\ell \in \mathbb{Z}^+$. Let s, $\ell s^{-1} \in \mathcal{P}_n(\mathbb{Z})$. Let $f \in M_n^k$. Assume that s is $\mathrm{GL}_n(\mathbb{Z})$-equivalent to ℓs^{-1}. Then $\phi_s^* f \in M_1^{nk}(\Gamma_0(\ell))$ is an eigenfunction of the Fricke operator W_ℓ. The eigenvalue is $+1$ unless s is improperly equivalent to ℓs^{-1} and k is odd in which case $\phi_s^* f$ has eigenvalue -1 under W_ℓ.*

Proof. When s is equivalent to ℓs^{-1} we have $UsU' = \ell s^{-1}$ for some $U \in \mathrm{GL}_n(\mathbb{Z})$. We will show that $(\phi_s^* f)\,|W_\ell = \det(U)^k \phi_s^* f$. The factor $\det(U)^k$ is one except in the case noted. We first check that $\phi_s \circ W_\ell = \begin{pmatrix} 0 & U^* \\ -U & 0 \end{pmatrix} \circ \phi_s$. For every $\tau \in \mathcal{H}_1$ we have

$$\begin{aligned}(\phi_s \circ W_\ell)(\tau) &= \phi_s\left(-\frac{1}{\ell\tau}\right) = -\frac{1}{\ell} s\tau^{-1} = -U^* s^{-1} U^{-1} \tau^{-1} = U^*\left(-Us\tau\right)^{-1} \\ &= \begin{pmatrix} 0 & U^* \\ -U & 0 \end{pmatrix}(s\tau) = \left(\begin{pmatrix} 0 & U^* \\ -U & 0 \end{pmatrix} \circ \phi_s\right)(\tau).\end{aligned}$$

Noting that $\begin{pmatrix} 0 & U^* \\ -U & 0 \end{pmatrix} \in \Gamma_n$ we compute

$$\begin{aligned}&\left[(\phi_s^* f) \underset{nk}{|} W_\ell\right](\tau) = (\sqrt{\ell}\tau)^{-nk}\,(\phi_s^* f)(W_\ell(\tau)) = (\sqrt{\ell}\tau)^{-nk}\,(f \circ \phi_s \circ W_\ell)(\tau) \\ =&(\sqrt{\ell}\tau)^{-nk}\left(f \circ \begin{pmatrix} 0 & U^* \\ -U & 0 \end{pmatrix} \circ \phi_s\right)(\tau) = (\sqrt{\ell}\tau)^{-nk} \det(-Us\tau)^k f(\phi_s(\tau)) \\ =&(-\sqrt{\ell})^{-nk} \det(U)^k \det(s)^k\,(\phi_s^* f)(\tau) = \det(U)^k\,(\phi_s^* f)(\tau).\end{aligned}$$

In the last line above we have used the fact that $\det(s)^2 = \ell^n$ and that when nk is odd we must have f identically zero. $\square$

The next Proposition shows how to develop the Fourier expansion of $\phi_s^* f$ at any cusp.

Proposition 2.3. *Let $n \in \mathbb{Z}^+$. Let $s \in \mathcal{P}_n(\mathbb{Q})$. Let $f \in S_n^k$ have the Fourier expansion $f(\Omega) = \sum_t a(t) e(\langle \Omega, t\rangle)$. Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1$. There exist $\mathcal{A}$, $\mathcal{B} \in \mathbb{Q}^{n\times n}$ such that $\left(\begin{smallmatrix} aI & bs \\ cs^{-1} & dI \end{smallmatrix}\right) \in \Gamma_n \left(\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ 0 & \mathcal{A}^* \end{smallmatrix}\right)$ and for any such $\mathcal{A}$, $\mathcal{B}$ we have*

$$(\phi_s^* f \underset{nk}{|} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))(\tau) = (\det \mathcal{A})^k f(\tau \mathcal{A} s \mathcal{A}' + \mathcal{B}\mathcal{A}') = (\det \mathcal{A})^k \sum_{j \in \mathbb{Q}^+} \left(\sum_{t : \langle \mathcal{A} s \mathcal{A}', t\rangle = j} a(t) e\left(\langle t, \mathcal{B}\mathcal{A}'\rangle\right)\right) q^j$$

Proof. We now wish to study $(\phi_s^* f | \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_1(\mathbb{Z})$. Then as in the proof of Proposition 2.1 we have

$$(\phi_s^* f \underset{nk}{|} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))(\tau) = (f \underset{k}{|} \left(\begin{smallmatrix} aI & bs \\ cs^{-1} & dI \end{smallmatrix}\right))(\tau s).$$

Now, we can always decompose any matrix in $\mathrm{Sp}_n(\mathbb{Q})$ as something in $\mathrm{Sp}_n(\mathbb{Z})$ times something in $\mathrm{Sp}_n(\mathbb{Q})$ with $C = 0$ [8, p. 125]. So let $\left(\begin{smallmatrix} aI & bs \\ cs^{-1} & dI \end{smallmatrix}\right) \in \Gamma_n \left(\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ 0 & \mathcal{A}^* \end{smallmatrix}\right)$. Since f is automorphic with respect to Γ_n we have

$$(\phi_s^* f \underset{nk}{|} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))(\tau) = (f \underset{k}{|} \left(\begin{smallmatrix} aI & bs \\ cs^{-1} & dI \end{smallmatrix}\right))(\tau s) = (f \underset{k}{|} \left(\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ 0 & \mathcal{A}^* \end{smallmatrix}\right))(\tau s) = (\det \mathcal{A})^k f(\tau \mathcal{A} s \mathcal{A}' + \mathcal{B}\mathcal{A}').$$

The Fourier expansion for $(\phi_s^* f|\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))(\tau)$ follows from the Fourier expansion for f under the substitution $\Omega = \tau \mathcal{A} s \mathcal{A}' + \mathcal{B}\mathcal{A}'$. $\square$

The above Proposition provides for the computation of the Fourier expansion of $\phi_s^* f|\sigma$ in general. When ℓ is squarefree however the computation of the character $e\left(\langle t, \mathcal{B}\mathcal{A}'\rangle\right)$ may be finessed. We introduce a new notation: Notice that $\mathcal{A}$ in Proposition 2.3 is determined up to $u\mathcal{A}$ with $u \in \mathrm{GL}_n(\mathbb{Z})$. Thus $\mathcal{A}s\mathcal{A}'$ is determined up to equivalence class. We define

$$s\square\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \mathcal{A}s\mathcal{A}'$$

with the understanding that this is well-defined only up to equivalence class. Since f is automorphic with respect to $\left(\begin{smallmatrix} u & 0 \\ 0 & u^* \end{smallmatrix}\right)$, we have $f(usu'\tau) = f(s\tau)$ and it makes sense to talk about $f((s\square\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))\tau)$ and $\phi^*_{s\square\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} f$.

Proposition 2.4. *Let $s \in \mathcal{P}_n(\mathbb{Z})$. Let $\ell \in \mathbb{Z}^+$ such that ℓs^{-1} is integral and primitive. Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1$. Suppose $\gcd(c, \frac{\ell}{c}) = 1$. Let $\hat{c} \in \mathbb{Z}$ such that $\hat{c}c \equiv 1 \mod \frac{\ell}{c}$. For any $\mathcal{A}$ with $\left(\begin{smallmatrix} aI & bs \\ cs^{-1} & dI \end{smallmatrix}\right) \in \Gamma_n\left(\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ 0 & \mathcal{A}^* \end{smallmatrix}\right)$ we have*

$$(\phi_s^* f|\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))(\tau) = (\det \mathcal{A})^k \phi^*_{s\square\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} f(\tau + d\hat{c}).$$

Proof. We have $\left(\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ 0 & \mathcal{A}^* \end{smallmatrix}\right)\left(\begin{smallmatrix} aI & bs \\ cs^{-1} & dI \end{smallmatrix}\right)^{-1} \in \mathrm{Sp}_n(\mathbb{Z})$. Thus we have

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ 0 & \mathcal{A}^* \end{pmatrix}\begin{pmatrix} dI & -bs \\ -cs^{-1} & aI \end{pmatrix} = \begin{pmatrix} d\mathcal{A} - c\mathcal{B}s^{-1} & -b\mathcal{A}s + a\mathcal{B} \\ -c\mathcal{A}^*s^{-1} & a\mathcal{A}^* \end{pmatrix} \in \mathrm{Sp}_n(\mathbb{Z}). \tag{2.5}$$

Note that each of the four blocks must be in $\mathbb{Z}^{n\times n}$. Multiplying $d\mathcal{A} - c\mathcal{B}s^{-1}$ by the integral s implies $d\mathcal{A}s - c\mathcal{B}$ is integral. Both $\mathcal{A}s$ and $\mathcal{B}$ are integral because we have

$$\mathcal{A}s = a(d\mathcal{A}s - c\mathcal{B}) + c(-b\mathcal{A}s + a\mathcal{B}),$$
$$\mathcal{B} = b(d\mathcal{A}s - c\mathcal{B}) + d(-b\mathcal{A}s + a\mathcal{B}).$$

Since $c\mathcal{B}s^{-1} = \frac{c}{\ell}\mathcal{B}\ell s^{-1}$ and $\ell s^{-1} \in \mathbb{Z}^{n\times n}$, we have $c\mathcal{B}s^{-1} \in \frac{c}{\ell}\mathbb{Z}^{n\times n}$. This combined with $d\mathcal{A} - c\mathcal{B}s^{-1} \in \mathbb{Z}^{n\times n}$ implies $d\mathcal{A} \in \frac{c}{\ell}\mathbb{Z}^{n\times n}$. Also we have $\mathcal{A} = \frac{1}{\ell}(\mathcal{A}s)\ell s^{-1} \in \frac{1}{\ell}\mathbb{Z}^{n\times n}$ and consequently $\mathcal{A} = a(d\mathcal{A}) - b(c\mathcal{A}) \in \frac{c}{\ell}\mathbb{Z}^{n\times n}$. Since $\mathcal{A}s$ is integral, its transpose $s\mathcal{A}'$ is also integral. Then multiplying $d\mathcal{A} - c\mathcal{B}s^{-1}$ by the integral $\hat{c}s\mathcal{A}'$ implies that $d\hat{c}\mathcal{A}s\mathcal{A}'$ and $\hat{c}c\mathcal{B}\mathcal{A}'$ differ by an integer matrix. But $\hat{c}c \equiv 1 \mod \frac{\ell}{c}$ and $\mathcal{B}\mathcal{A}' \in \frac{c}{\ell}\mathbb{Z}^{n\times n}$ imply that $\hat{c}c\mathcal{B}\mathcal{A}'$ and $\mathcal{B}\mathcal{A}'$ differ by an integer matrix. Hence $d\hat{c}\mathcal{A}s\mathcal{A}'$ and $\mathcal{B}\mathcal{A}'$ differ by an integer matrix. Finally, from Proposition 2.3 we have $(\det \mathcal{A})^{-k}(\phi_s^* f|\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))(\tau) =$

$$f(\tau\mathcal{A}s\mathcal{A}' + \mathcal{B}\mathcal{A}') = f(\tau\mathcal{A}s\mathcal{A}' + d\hat{c}\mathcal{A}s\mathcal{A}') = f(\mathcal{A}s\mathcal{A}'(\tau + d\hat{c})) = \phi^*_{s\square\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} f(\tau + d\hat{c}). \quad \square$$

§3. The space S_4^{10}.

We will apply the technique of the Introduction to S_4^{10}. Theorem 1.1 says a form in S_4^{10} is determined by its coefficients $a(t)$ with $w(t) \leq 3.5$. Table 3 gives the list of these 10 quadratic forms, see [10][13]. For uniformity of notation we will refer to these as $B_0, \ldots, B_9$. Here the number under ℓ for B_i is the smallest positive integer such that $\ell(2B_i)^{-1}$ is integral.

Table 3. Semi-integral quaternary forms with dyadic trace ≤ 3.5.

Name	Form	Dyadic trace	16·Determinant	ℓ
B_0	$\frac{1}{2}\begin{pmatrix} 2&1&1&1\\ 1&2&0&0\\ 1&0&2&0\\ 1&0&0&2 \end{pmatrix}$	2	4	2
B_1	$\frac{1}{2}\begin{pmatrix} 2&1&0&1\\ 1&2&0&0\\ 0&0&2&1\\ 1&0&1&2 \end{pmatrix}$	2.5	5	5
B_2	$\frac{1}{2}\begin{pmatrix} 2&0&0&1\\ 0&2&0&1\\ 0&0&2&0\\ 1&1&0&2 \end{pmatrix}$	3	8	4
B_3	$\frac{1}{2}\begin{pmatrix} 2&1&0&0\\ 1&2&0&0\\ 0&0&2&1\\ 0&0&1&2 \end{pmatrix}$	3	9	3
B_4	$\frac{1}{2}\begin{pmatrix} 2&1&1&1\\ 1&2&0&0\\ 1&0&2&0\\ 1&0&0&4 \end{pmatrix}$	3	12	6
B_5	$\frac{1}{2}\begin{pmatrix} 2&0&0&1\\ 0&2&0&0\\ 0&0&2&0\\ 1&0&0&2 \end{pmatrix}$	3.5	12	6
B_6	$\frac{1}{2}\begin{pmatrix} 2&1&1&0\\ 1&2&0&1\\ 1&0&2&0\\ 0&1&0&4 \end{pmatrix}$	3.5	13	13
B_7	$\frac{1}{2}\begin{pmatrix} 2&1&0&1\\ 1&2&0&0\\ 0&0&2&1\\ 1&0&1&4 \end{pmatrix}$	3.5	17	17
B_8	$\frac{1}{2}\begin{pmatrix} 2&0&0&1\\ 0&2&0&1\\ 0&0&2&1\\ 1&1&1&4 \end{pmatrix}$	3.5	20	10
B_9	$\frac{1}{2}\begin{pmatrix} 2&1&1&1\\ 1&2&0&1\\ 1&0&4&2\\ 1&1&2&4 \end{pmatrix}$	3.5	25	5

We will apply the technique to $H = 2B_4$ for which $6H^{-1}$ is integral. By Proposition 2.1 we have $\operatorname{Im} \phi_H^* f \subset M_1(\Gamma_0(6))$ and our calculations will occur inside this ring. The ring $M_1(\Gamma_0(6))$ is generated by three forms A, B, C of weight 2. There is one relation $C^2 = 9B^2 - 8A^2$. The ring of cusp forms is principally generated by a form of weight 4, $D = \frac{1}{4}(A^2 - B^2)$. There are 4 cusps in $\Gamma_0(6)\backslash\Gamma_1/\Delta_1$, represented by I, $\sigma_2 = \left(\begin{smallmatrix} 1&0\\ 2&1 \end{smallmatrix}\right)$, $\sigma_3 = \left(\begin{smallmatrix} 1&0\\ 3&1 \end{smallmatrix}\right)$ and J with respective widths 1, 3, 2 and 6. We now give the Fourier expansions of the generators at all four cusps. The definition of $E_{2,2}^-$ has already been given, similarly define $E_{2,3}^-(\tau) = 1 + 12\sum_{n=1}^{\infty}(\sigma_1(n) - 3\sigma_1(n/3))q^n = 1 + 12(q + 3q^2 + q^3 + 7q^4 + 6q^5 + \cdots)$. Define the following elements in $M_1(\Gamma_0(6))$:

$$
\begin{aligned}
A(\tau) &= (3/4)E_{2,2}^-(3\tau) + (1/4)E_{2,2}^-(\tau) = 1 + 6q + 6q^2 + 42q^3 + \cdots, \\
B(\tau) &= (2/3)E_{2,3}^-(2\tau) + (1/3)E_{2,3}^-(\tau) = 1 + 4q + 20q^2 + 4q^3 + \cdots, \\
C(\tau) &= (3/2)E_{2,2}^-(3\tau) - (1/2)E_{2,2}^-(\tau) = 1 - 12q - 12q^2 - 12q^3 + \cdots.
\end{aligned}
$$

The elliptic modular forms A, B, C transform nicely as

$$
\begin{array}{lll}
(A|J)(\tau) = -\frac{1}{6}A(\tau/6), & (A|\sigma_2)(\tau) = +\frac{1}{3}A((\tau-1)/3), & (A|\sigma_3)(\tau) = -\frac{1}{2}A((\tau-1)/2) \\
(B|J)(\tau) = -\frac{1}{6}B(\tau/6), & (B|\sigma_2)(\tau) = -\frac{1}{3}B((\tau-1)/3) & (B|\sigma_3)(\tau) = +\frac{1}{2}B((\tau-1)/2) \\
(C|J)(\tau) = +\frac{1}{6}C(\tau/6), & (C|\sigma_2)(\tau) = -\frac{1}{3}C((\tau-1)/3) & (C|\sigma_3)(\tau) = -\frac{1}{2}C((\tau-1)/2)
\end{array}
$$

We use Propostions 2.3 and 2.4 to work out the Fourier expansion of $\phi_H^* f|\sigma$ for $\sigma = I$, σ_2, σ_3, J. We implement the algorithms from [8, pp.125, 322-328] to produce a factorization $\left(\begin{smallmatrix} aI & bH\\ cH^{-1} & dI \end{smallmatrix}\right) \in \Gamma_n\left(\begin{smallmatrix} \mathcal{A} & \mathcal{B}\\ 0 & \mathcal{A}^* \end{smallmatrix}\right)$ and obtain $\det(\mathcal{A})$ and $H\square\sigma = \mathcal{A}H\mathcal{A}'$. We display $H\square\sigma_2$, $H\square\sigma_3$, $H\square J$ and mention that the associated $|\det(\mathcal{A})|$ equals 3, 4, 12, respectively:

$$H\square\sigma_2=\frac{1}{3}\begin{pmatrix} 4 & 2 & 1 & -1 \\ 2 & 4 & -1 & 1 \\ 1 & -1 & 4 & -1 \\ -1 & 1 & -1 & 4 \end{pmatrix};H\square\sigma_3=\frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 3 \end{pmatrix};H\square J=\frac{1}{6}\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 5 & -1 & 2 \\ 1 & -1 & 5 & 2 \\ 1 & 2 & 2 & 5 \end{pmatrix}$$

Note that all three of the cases σ_2, σ_3, J satisfy the hypotheses of Proposition 2.4. Note that for $c = 2$, we can take $\hat{c} = -1$ so that $c\hat{c} = 1 \mod 3$; for $c = 3$, we can take $\hat{c} = -1$ so that $c\hat{c} = 1 \mod 2$. Thus we have

$$\begin{aligned} (\phi_H^* f|I)(\tau) &= \phi_H^* f(\tau), \\ (\phi_H^* f|\sigma_2)(\tau) &= 3^{-10}\phi_{H\square\sigma_2}^* f(\tau-1), \\ (\phi_H^* f|\sigma_3)(\tau) &= 4^{-10}\phi_{H\square\sigma_3}^* f(\tau-1), \\ (\phi_H^* f|J)(\tau) &= 12^{-10}\phi_{H\square J}^* f(\tau). \end{aligned} \tag{3.1}$$

Hence the Fourier expansions may be computed from the numbers $\mathcal{V}(j, H\square\sigma, t)$ given in Tables 4, 5, 6 and 7. Among the computations we perform, the computation of these representation numbers is by far the most expensive.

Table 4. $\mathcal{V}(j, H, t)$

j	B_0	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9
6	1									
7	12	36								
8	96	168	114	24	6					
9	196	760	384	108	60	168	108	96	12	4

Table 5. $\mathcal{V}(j, H\square\sigma_2, t)$

j	B_0	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9
10/3	1									
11/3	4	12								
12/3	24	24	6							
13/3	12	96	24	12						
14/3	78	192	120	24	6	12				1
15/3	144	312	192	24	28	84	48	36		

Table 6. $\mathcal{V}(j, H\square\sigma_3, t)$

j	B_0	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9
6/2	5	4								
7/2	24	72	12							
8/2	120	264	138	48	15	12	12			

Table 7. $\mathcal{V}(j, H\Box J, t)$

j	B_0	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9
10/6	1									
11/6		8								
12/6	24	12	6		1					
13/6	24	96	12				8			

From these expansions we see that $\phi_H^* f$ vanishes to order at least 6 at every cusp so that there are pararmeters $\alpha_0, \dots, \alpha_8$ and $\beta_0, \dots, \beta_7 \in \mathbb{C}$ such that

$$\phi_H^* f = (D)^6 \left(\alpha_0 A^8 + \alpha_1 A^7 B + \cdots + \alpha_8 B^8 + C(\beta_0 A^7 + \beta_1 A^6 B + \cdots + \beta_7 B^7) \right).$$

Without introducing any new parameters we also have equalities for any $\sigma \in \Gamma_1$:

$$\phi_H^* f|\sigma = (D|\sigma)^6 \left(\alpha_0 (A|\sigma)^8 + \ldots + \alpha_8 (B|\sigma)^8 + (C|\sigma)(\beta_0 (A|\sigma)^7 + \ldots + \beta_7 (B|\sigma)^7) \right). \tag{3.2}$$

For $\sigma = I, \sigma_2, \sigma_3, J$ the left side of equation 3.2 is computed from equations 3.1, equation 1.2 and Tables 4 through 7. The right side is computed from the expansions of the elliptic modular forms A, B and C. At the cusp $[I]$ we equate the coefficents for $j = 6, \dots, 9$; at the cusp $[\sigma_2]$ for $j = 6/3, \dots, 15/3$; at the cusp $[\sigma_3]$ for $j = 6/2, \dots, 8/2$ and at the cusp $[J]$ for $j = 6/6, \dots, 13/6$. Elimination of the 17 parameters α_i, β_i from the $4+10+3+8 = 25$ linear equations results in 8 linearly independent equations:

$$\begin{aligned}
a(B_2) &= -86/21 a(B_0) - 188/21 a(B_1) \\
a(B_3) &= 100/3 a(B_0) + 58/3 a(B_1) \\
a(B_4) &= -300/7 a(B_0) + 24/7 a(B_1) \\
a(B_5) &= -1892/21 a(B_0) + 568/21 a(B_1) \\
a(B_6) &= 288/7 a(B_0) - 53/7 a(B_1) \\
a(B_7) &= 2860/63 a(B_0) - 8738/63 a(B_1) \\
a(B_8) &= 656/7 a(B_0) + 3872/7 a(B_1) \\
a(B_9) &= 21016/21 a(B_0) + 15532/21 a(B_1).
\end{aligned}$$

When we combine these 8 linear relations with the 2 linear relations in equation 1.5 obtained by considering $\phi_{D_4}^*$, we see that the rank is actually 9, so that we have a total of 9 linearly independent relations in $a(B_0), \ldots, a(B_9)$:

$$\begin{aligned}
a(B_1) &= 2a(B_0) \\
a(B_2) &= -22a(B_0) \\
a(B_3) &= 72a(B_0) \\
a(B_4) &= -36a(B_0) \\
a(B_5) &= -36a(B_0) \\
a(B_6) &= 26a(B_0) \\
a(B_7) &= -232a(B_0) \\
a(B_8) &= 1200a(B_0) \\
a(B_9) &= 2480a(B_0).
\end{aligned} \tag{3.3}$$

These relations and Theorem 1.1 imply that $\dim S_4^{10} \le 1$. Since we can come up with one nonzero cusp form G_{10} in S_4^{10} we have a theorem.

Theorem 3.4. *We have* $\dim S_4^{10} = 1$ *and* $S_4^{10} = \mathbb{C}G_{10}$.

§4. Final Comments.

The computations that have been performed for the form H are largely independent of the weight k. Applied to the space S_4^8 we may extend the Fourier expansion of the Schottky form J beyond that given in [1]. Table 8 gives the Fourier coefficients $a(B_i)$ for $J/2^{16}$ and $G_{10}/2^{18}3^45$.

Table 8. (Fourier Coefficients)

f	B_0	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9
$J/2^{16}$	1	-1	2	6	-12	-12	11	2	-72	116
$G_{10}/2^{18}3^45$	1	2	-22	72	-36	-36	26	-232	1200	2480

Although the parameters α_i and β_i were simply eliminated in section §3, their values are also determined by this process. It may be of interest to present the images of $\phi_s^* f$ for $s = D_4, H$ and $f = J, G_{10}$.

$$\begin{aligned}
\phi_{D_4}^* J &= 2^{16}(C_{8,2}^+)^4 \\
\phi_{D_4}^* G_{10} &= 2^{18}3^45(C_{8,2}^+)^4\left((E_{2,2}^-)^4 + 48C_{8,2}^+\right) \\
\phi_H^* J &= 2^{12}D^6(A+C)^4 \\
\phi_H^* G_{10} &= 2^{14}3^35D^6(A+C)^4(25A^4 - 8A^3B - 7A^3C - 8A^2BC - ABC^2 + 4AC^3 - BC^3 - C^4)
\end{aligned}$$

It is interesting to note that the image of G_{10} comes out to a multiple of the image of J under both $\phi_{D_4}^*$ and ϕ_H^*. As a final comment we note that linear relations among Fourier coefficients can be viewed as linear relations among Poincare series.

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