RELATIONS ON THE PERIOD MAPPING GIVING EXTENSIONS OF MIXED HODGE STRUCTURES ON COMPACT RIEMANN SURFACES

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Abstract. We consider a period map Ψ from Teichmüller space to $\mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{R}}$, which is a real vector bundle over the Siegel upper half space. This map lifts the Torelli map. We study the action of the mapping class group on this period map. We show that the period map from Teichmüller space modulo the Johnson kernel is generically injective. We derive relations that the quadratic periods must satisfy. These identities are generalizations of the symmetry of the Riemann period matrix. Using these higher bilinear relations, we show that the period map factors through a translation of the subbundle $(\wedge^3\mathcal{H}_1)_{\mathbb{R}}$ and is completely determined by the purely holomorphic quadratic periods. We apply this result to strengthen some theorems in the literature. One application is that the quadratic periods, along with the abelian periods, determine a generic marked compact Riemann surface up an element of the kernel of Johnson's homomorphism. Another application is that we compute the cocycle that exhibits the mapping class group modulo the Johnson kernel as an extension of the group $\mathrm{Sp}_g(\mathbb{Z})$ by the group $(\wedge^3 H_1)_{\mathbb{Z}}$.

§0. Introduction.

The goal of this paper is to produce "higher bilinear period relations" for a compact Riemann surface from a point of view which combines that of Gunning [4] and Jablow [9] on quadratic periods with that of Hain [6] and Pulte [13] on variations of mixed Hodge structure. The primary object arising from abelian periods on pure Hodge structures is the Torelli map $\Omega : \mathfrak{T}_g \to \mathfrak{h}_g$ which sends a marked Riemann surface (f, M) from Teichmüller space $\mathfrak{T}_g$ to the period matrix Ω_f in the Siegel upper half space $\mathfrak{h}_g$; the notation (f, M) refers to a map f from a fixed topological reference surface S to a Riemann surface M, thus giving a marking on M. The Torelli map Ω is equivariant with respect to the action of the mapping class group $\mathcal{M}_g$ on $\mathfrak{T}_g$ and the action of the symplectic group $\mathrm{Sp}_g(\mathbb{Z})$ on $\mathfrak{h}_g$. The symplectic group arises as the quotient of $\mathcal{M}_g$ by the Torelli group I_g, the normal subgroup of $\mathcal{M}_g$ inducing the identity on $H_1(S, \mathbb{Z})$. The map $[\Omega] : \mathfrak{T}_g/\mathcal{M}_g \to \mathfrak{h}_g/\mathrm{Sp}_g(\mathbb{Z})$ injects by Torelli's theorem; alternatively, noting that $\mathfrak{T}_g/\mathcal{M}_g$ can be identified with the moduli space of Riemann surfaces of genus g, one says the abelian periods Ω_{ij} completely determine the Riemann surface. Our emphasis will be on the symmetry $\Omega_{ij} = \Omega_{ji}$ of the period map Ω; this symmetry is a consequence of Riemann's bilinear relations which we henceforth view as a period relation restricting the possible abelian periods. Gunning [4, p.14] pointed out that the bilinear period relations of Riemann follow from the existence of iterated integrals which are homotopy functionals. It is the generalization of these period

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relations to the period map giving classes of extensions of mixed Hodge structure that we discuss in this paper.

The "higher periods" of a marked Riemann surface (f, M, z) are the values of iterated integrals of 1-forms on M over loops in $\pi_1(M, z)$. We only consider iterated integrals which are path homotopy functionals in order to restrict attention to homotopy classes of loops. The classical abelian periods fit into this picture as the values $\int_\gamma \omega$, for $\gamma \in \pi_1(M, z)$ and $\omega \in H^1(M, \mathbb{C})$, of iterated integrals of length one. For a 1-form ω, the integral $\int \omega$ is a homotopy functional precisely when ω is closed. A standard basis $\omega_1, \ldots, \omega_g$ for the holomorphic 1-forms on M is chosen in the following way. If $a_1, \ldots, a_g, b_1, \ldots, b_g$ are standard generators of $\pi_1(M, z)$ satisfying $\prod_{i=1}^g (a_i, b_i) = e$, then the requirement $\int_{a_j} \omega_i = \delta_{ij}$ for $1 \leq i, j \leq g$ uniquely determine the ω_i. The period matrix Ω_f is then given by $(\Omega_f)_{ij} = \int_{b_j} \omega_i$ for $1 \leq i, j \leq g$. Since $\omega_1, \ldots, \omega_g, \overline{\omega}_1, \ldots, \overline{\omega}_g$ give a basis for $H^1(M, \mathbb{C})$, the abelian period matrix Ω_f fully describes the periods in the length one case. The length two case is immediately more complicated because iterated integrals of the form $I_2 = \sum_i \int \xi_i \eta_i + \int \mu$, for ξ_i, η_i, μ 1-forms on M, may not be homotopy path functionals even when the ξ_i, η_i are closed. The condition that a μ exists such that I_2 is a homotopy functional is in fact that $\sum_i \xi_i \otimes \eta_i$ lies in the kernel of the cup product $K_2(M) = \ker(\cup : H^1 \otimes H^1 \to H^2(M))$. Although $K_2(M)$ does not seem to have a natural basis in terms of $\omega_i \otimes \omega_j$, $\omega_i \otimes \overline{\omega}_j$, etc., we can make some convenient choices and obtain the corresponding homotopy functionals: the purely holomorphic homotopy functionals $\sigma_{ij} = \int \omega_i \omega_j$, the mixed functionals τ_{ij} (see Remark 2.11), and so on. The periods of the $\omega_i, \sigma_{ij}, \tau_{ij}$ and their conjugates over loops in $\pi_1(M, z)$ fully describe the periods in the length two case. Homotopy functionals of length two, moreover, provide information about the length one periods; for example, if the σ_{ij} for $1 \leq i, j \leq g$ are applied to the commutator relation $\prod_{i=1}^g (a_i, b_i) = e$, we obtain Riemann's Equality $(\Omega_f)_{ij} = (\Omega_f)_{ji}$. The question arises whether relations among the length two periods may be derived by considering homotopy functionals of length three. This is indeed feasible, and all such relations are generated by those in equations 5.15 and 5.16. The choice of a basis for $K_2(M)$ is an unpleasant aspect of these equations, and one may ask for an intrinsic version. The intrinsic meaning of these equations is better perceived in the context of mixed Hodge structures, and it is in these terms that the paper is written.

A mixed Hodge structure (see [6]) may be put on homotopy functionals which are iterated integrals of a fixed length. Let $B_s(M)$ denote the iterated integrals on M of length at most s, and let $\overline{B}_s(M)$ be those with no constant term. Let $H^0(\overline{B}_s(M), z)$ denote the homotopy functionals on $\pi_1(M, z)$ from $\overline{B}_s(M)$. The weight filtration on $H^0(\overline{B}_s(M), z)$ is given by the length of a representative iterated integral, and the Hodge filtration is given by the "number of dz's" in the iterated integral. To every pointed Riemann surface (M, z), there is an associated extension of mixed Hodge structures given by

$$(*) \qquad 0 \to H^1(M, \mathbb{C}) \to H^0(\overline{B}_2(M), z) \to K_2(M) \to 0.$$

The congruence class of this extension is given by $\tilde{\Psi}_{(M,z)} \in \operatorname{Ext}(K_2(M), H^1(M, \mathbb{C}))$ (see [2] or [6]), which is computed via $\tilde{\Psi}_{(M,z)} = r_{\mathbb{Z}} \circ s_2$ for a Hodge filtration preserving section $s_2 : K_2(M) \to H^0(\overline{B}_2(M), z)$ and an integral retraction $r_{\mathbb{Z}} : H^0(\overline{B}_2(M), z) \to H^1(M, \mathbb{C})$. The important information given by picking a basis for $K_2(M)$ and computing higher periods is equally well given by a section s_2 from $K_2(M)$ to $H^0(\overline{B}_2(M), z)$. Riemann's

Equality, for example, is obtained by applying the section s_2 to $\sum_{i=1}^g [a_i, b_i] \equiv 0 \mod J^3$ in the group ring $\mathbb{C}\pi_1(M, z)$ with augmentation ideal J. Since the element $\tilde{\Psi}_{(M,z)} = r_{\mathbb{Z}} \circ s_2 \in \mathrm{Ext}(K_2(M), H^1(M))$ embodies all the length two periods and is intrinsic to M, we may use $\tilde{\Psi}_{(M,z)}$ to construct an intrinsic version of the "higher bilinear period relations" found in equations 5.15 and 5.16. In order to construct homotopy functionals of length three it was necessary to consider $K_3(M) = (K_2(M) \otimes H^1(M)) \cap (H^1(M) \otimes K_2(M))$ and the exact sequence $0 \to H^0(\overline{B}_2(M), z) \to H^0(\overline{B}_3(M), z) \to K_3(M) \to 0$. In Definition 5.7, a section $s_3 : K_3(M) \to H^0(\overline{B}_3(M), z)$ is defined which provides homotopy functionals of length three. The section s_3 was constructed using Chen's path functional derivative (see Proposition 5.8 or [3]). Applying this section s_3 to $\sum_{i=1}^g ([a_i, b_i] - [a_i, b_i](a_i - 1 + b_i - 1)) \equiv 0 \mod J^4$ provides the intrinsic version of the "higher bilinear relations" given in Proposition 5.12. Instead of working with a fixed Riemann surface, however, we work over Teichmüller space $\mathfrak{T}_{g,*}$ (see beginning of §1).

We globalize these constructions using the abelian periods and the Torelli map as a model. The Torelli map arises from trying to send each Riemann surface to its polarized Hodge structure; however, it is easier to map marked Riemann surfaces and allow an appropriate group action to identify changes of marking. The Torelli map $\Omega : \mathfrak{T}_{g,*} \to \mathfrak{h}_g$ is equivariant with respect to $\mathcal{M}_{g,*}$ and $\mathrm{Sp}_g(\mathbb{Z})$; the group $\mathcal{M}_{g,*}$ identifies equivalent Riemann surfaces and the group $\mathrm{Sp}_g(\mathbb{Z})$ identifies equivalent polarized Hodge structures. The Torelli group $\mathrm{I}_{g,*}$ acts trivially on Ω, and we have $\mathrm{Sp}_g(\mathbb{Z}) \cong \mathcal{M}_{g,*}/\mathrm{I}_{g,*}$. To each $Z \in \mathfrak{h}_g$, we may construct a principally polarized abelian variety A_Z (see beginning of §1) whose first cohomology group gives a weight one polarized Hodge structure $H^1(Z)$ with polarization form q_Z. We let $K_2(Z) = \ker(q_Z : H^1 \otimes H^1 \to \mathbb{C})$. We replace the Siegel upper half space $\mathfrak{h}_g$ with a real vector bundle over $\mathfrak{h}_g$, $\mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{R}}$, whose fiber over each $Z \in \mathfrak{h}_g$ is $\mathrm{Hom}(K_2(Z), H^1(Z))_{\mathbb{R}}$ (see 1.7). We define a period map $\Psi : \mathfrak{T}_{g,*} \to \mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{R}}$ which takes each marked Riemann surface to its congruence class of extension of mixed Hodge structures given by (*). In Definition 2.8, we define $\tilde{\Psi}_f \in \mathrm{Hom}(K_2, H^1)_{\mathbb{R}}$ by $\tilde{\Psi}_f = r_{\mathbb{Z}} \circ s_2$ using the section s_2 and a retraction $r_{\mathbb{Z}}$ which depends on the marking f. In Definition 2.10, we define $\Psi_f = (w_f)_* \tilde{\Psi}_f$ using the Abel-Jacobi map $w_f : M \to A_{\Omega_f}$ (see §1 prior to 1.8). Then $[\Psi_f] \in \mathrm{Hom}(K_2(\Omega_f), H^1(\Omega_f))_{\mathbb{R}}/\mathrm{Hom}(K_2(\Omega_f), H^1(\Omega_f))_{\mathbb{Z}} \cong \mathrm{Ext}(K_2(\Omega_f), H^1(\Omega_f))$ gives the correct extension class in Ext. Thus we have chosen not to map into a torus bundle over $\mathfrak{h}_g$ with $\mathrm{Ext}(K_2, H^1)$ as fibers but into a real vector bundle over $\mathfrak{h}_g$ with $\mathrm{Hom}(K_2, H^1)$ as fibers. The subgroup $\mathcal{N}_{g,*}$ of $\mathcal{M}_{g,*}$ which acts trivially on Ψ turns out to be the kernel of Johnson's homomorphism τ (see 3.1). The map Ψ is equivariant with respect to $\mathcal{M}_{g,*}$ and the group $\mathcal{M}_{g,*}/\mathcal{N}_{g,*}$ which plays the role in the mixed Hodge structure case that $\mathrm{Sp}_g(\mathbb{Z})$ plays in the pure Hodge structure case. We explicitly give the structure of $\mathcal{M}_{g,*}/\mathcal{N}_{g,*}$ in Proposition 3.10. A global version of the "higher period relations" is given by the factorization of Ψ through a subset of $\mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{R}}$ in Theorem 5.24, part of which we state here. The definitions of the various symbols are given in the main sections, so we mention them only briefly here: $\hat{\delta}$ is the cocycle extension (see §3) of th0and λ identifies $\mathrm{Hom}(K_2, H^1) \xrightarrow{\cong} ((H_1 \otimes H_1)/q) \otimes H^1$.

Theorem 5.24. *Let $y \in \mathcal{M}_{g,*}$ be any homology involution and $\jmath\lambda^{-1}\hat{\delta}(y)$ the corresponding global section of $\mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{Z}}$. The period map $\Psi : \mathfrak{T}_{g,*} \to \mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{R}}$ factors through the translation by $-\frac{1}{2}\jmath\lambda^{-1}\hat{\delta}(y)$ of the subbundle $\lambda^{-1}\imath^{-1}(\bigwedge^3 \mathcal{H}_1)_{\mathbb{R}}$ so that we have:*

$$\Psi : \mathfrak{T}_{g,*}/\mathcal{N}_{g,*} \to (-\tfrac{1}{2}\jmath\lambda^{-1}\hat{\delta}(y) + \lambda^{-1}\imath^{-1}\textstyle\bigwedge^3 \mathcal{H}_1)_{\mathbb{R}} \to \mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{R}},$$

and Ψ is equivariant with respect to the action of $\mathcal{M}_{g,*}/\mathcal{N}_{g,*}$.

We also have the factorization from Torelli space through the translation of a torus bundle:

$$\Psi : \mathfrak{T}_{g,*}/\mathrm{I}_{g,*} \to \frac{-\frac{1}{2}\jmath\lambda^{-1}\hat{\delta}(y) + \lambda^{-1}\imath^{-1}(\bigwedge^3\mathcal{H}_1)_{\mathbb{R}}}{\lambda^{-1}\imath^{-1}(\bigwedge^3\mathcal{H}_1)_{\mathbb{Z}}} \to \frac{\mathcal{H}om(\mathcal{K}_2,\mathcal{H}^1)_{\mathbb{R}}}{\lambda^{-1}\imath^{-1}(\bigwedge^3\mathcal{H}_1)_{\mathbb{Z}}},$$

and Ψ is equivariant with respect to the affine action of $\mathrm{Sp}_g(\mathbb{Z}) \cong \mathcal{M}_{g,*}/\mathrm{I}_{g,*}$.

The result of the bilinear relations in the case of the Torelli map Ω is that a corresponding tensor in $H_1 \otimes H_1$ is symmetric. The result of the higher bilinear relations in the case of Ψ is that a corresponding tensor in $H_1 \otimes H_1 \otimes H_1$ is antisymmetric. In Theorem 5.24 we see that $\Psi : \mathfrak{T}_{g,*} \to \mathcal{E}xt(\mathcal{K}_2, \mathcal{H}^1)$ factors through the translation of $\bigwedge^3\mathcal{H}_1$ by a half-lattice element, and we see that $\Psi : \mathfrak{T}_{g,*}/\mathrm{I}_{g,*} \to \mathcal{E}xt(\mathcal{K}_2, \mathcal{H}^1)$ factors through the translation of a torus subbundle of $\mathcal{E}xt(\mathcal{K}_2, \mathcal{H}^1)$ by a two-torsion element. That Ψ factors in this manner over Torelli space $\mathfrak{T}_{g,*}/\mathrm{I}_{g,*}$ is a result due to Harris, Hain, and Pulte. Pulte [13] shows that for each fixed Riemann surface M the difference of $\tilde{\Psi}$ at any two basepoints factors through the intermediate Jacobian $\mathrm{J}_2(\mathrm{Jac}(M))$ and also that $2\tilde{\Psi} \in \mathrm{J}_2(\mathrm{Jac}(M))$:

$$\begin{array}{ccccc} & & M & \to & \mathcal{E}xt(\mathcal{K}_2,\mathcal{H}^1) \\ & & x & \mapsto & [\tilde{\Psi}_{M,x}] \\ M \times M & \to & \mathrm{J}_2(\mathrm{Jac}(M)) & \hookrightarrow & \mathcal{E}xt(\mathcal{K}_2,\mathcal{H}^1) \\ (x,y) & \mapsto & M_x - M_y & \mapsto & [\tilde{\Psi}_{M,x}] - [\tilde{\Psi}_{M,y}] \end{array}$$

The meaning of the higher bilinear period relations given in Proposition 5.12 and Lemma 5.1 is now clear: they give a new proof and a global version of the factorization of $\tilde{\Psi}$ through the intermediate Jacobian due to Harris, Hain and Pulte. They also show that this factorization constitutes a global period relation in strict analogy with Riemann's bilinear relations. In §3, we also compute a global section of the $\frac{1}{2}\mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{Z}}$ bundle which gives the two-torsion element needed to translate $(\bigwedge^3\mathcal{H}_1)_{\mathbb{R}}/(\bigwedge^3\mathcal{H}_1)_{\mathbb{Z}}$ and show how it arises from a homology involution. This allows us to exhibit the quotient of $\mathcal{M}_{g,*}$ which acts on Ψ nontrivially, $\mathcal{M}_{g,*}/\mathcal{N}_{g,*}$, explicitly as an extension of $\mathrm{Sp}_g(\mathbb{Z})$ by $(\bigwedge^3 H_1)_{\mathbb{Z}}$ by computing the cocycle in $H^2(\mathrm{Sp}_g(\mathbb{Z}), (\bigwedge^3 H_1)_{\mathbb{Z}})$ that does this in Proposition 3.10.

We turn to the relation our results have to other results in the literature. In Proposition 4.7 we use the result of Harris, Hain, Pulte and Koizumi to show that the map $\Psi : \mathfrak{T}_{g,*}/\mathcal{N}_{g,*} \to \mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{R}}$ is generically injective. In Lemma 6.2 and Corollary 6.5 we use the higher period relations to show that the map Ψ is completely determined by the purely holomorphic quadratic periods. This result has implications for the earlier work on higher periods which usually studied only the purely holomorphic periods; it shows that on a Riemann surface the purely holomorphic periods already determine the congruence class of the extension of mixed Hodge structure and hence determine all the periods of mixed type. This result allows us to weaken the assumptions in two of Jablow's results [9]. Proposition 6.6 says that if an element $h \in \mathrm{I}_{g,*}$ fixes all the purely holomorphic quadratic periods then $h \in \mathcal{N}_{g,*} = \mathrm{Ker}\,\tau$. Theorem 6.7 says that the abelian and purely holomorphic quadratic periods generically determine a pointed marked Riemann surface up to elements of $\mathcal{N}_{g,*}$. We end the paper by showing how to effectively compute any period of mixed type in terms of the purely holomorphic periods.

Finally, we try to give an indication of the importance of these results for the program that we are following. The torus-bundle of translated intermediate Jacobians J_2 over $\mathfrak{h}_g$ and the affine action of $\mathcal{M}_{g,*}/\mathcal{N}_{g,*}$ give a moduli space for a class of (polarized) nilmanifolds which have (polarized) abelian varieties as quotients and which are "like" the second albanese manifold of a compact Riemann surface. Questions such as a global pointed Torelli theorem, "Does $\Psi : \mathfrak{T}_{g,*}/\mathcal{M}_{g,*} \to \mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{R}}/(\mathcal{M}_{g,*})$ inject?", become "Is (M, z) determined by the equivalence class of its second albanese as a polarized nilmanifold in $\mathcal{J}_2/(\mathcal{M}_{g,*})$?". It is to study the second albanese directly as a member of a specific class of nilmanifolds that we have endeavored to explicitly determine all of the polynomial relations which bind the periods of Ψ.

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The main part of the paper is organized into six sections:

In §1, we make some preliminary definitions and prove some preliminary propositions. We define the important bundle $\mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{R}}$. We also define various identification maps.

In §2, we define our period map Ψ, which embodies the quadratic periods.

In §3, we define our cocycle $\hat{\delta}$ that extends Johnson's homomorphism. We compute the cocycle that gives the structure of the mapping class group modulo Johnson's kernel.

In §4, we compute the action of mapping class group $\mathcal{M}_{g,*}$ on Ψ. We define an affine action of the mapping class group on $\mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{R}}$, and we show that $\mathcal{M}_{g,*}$ acts equivariantly on Ψ. We also show that Ψ is generically injective modulo Johnson's kernel.

In §5, we derive the "higher bilinear relations" and prove the main theorem that Ψ factors through the translation of a certain subbundle isomorphic to $\bigwedge^3 \mathcal{H}_1$.

In §6, we show that Ψ is actually determined by the holomorphic quadratic periods. We use this to strengthen some results in the literature. Finally, we show how to effectively compute any period of mixed type in terms of the holomorphic periods.

§1. Preliminaries.

We let S be a compact real 2-manifold of genus $g \geq 1$, and $s \in S$ be a distinguished point. We denote by β_i, α_i for $i = 1, \dots, g$, or by γ_i for $i = 1, \dots, 2g$ if one symbol is desired, the standard generators of $\pi_1(S, s)$. We note that the $[\gamma_i]$ then give a standard basis for $H_1(S, \mathbb{Z})$. Here, we use $[\cdot]$ to denote the element in homology corresponding to an element in homotopy. We place a fixed complex structure on S and use it as a fixed surface for the construction of the Teichmüller space $\mathfrak{T}_{g,*}$ of marked, pointed Riemann surfaces. Let $\mathfrak{A}$ be the set of triples (f, M, z) where M is a compact Riemann surface of genus g, $z \in M$ and $f : (S, s) \to (M, z)$ is an orientation preserving homeomorphism of S onto M taking s to z. Two elements of $\mathfrak{A}$ are termed equivalent if there is a conformal map $\phi : (M_1, z_1) \to (M_2, z_2)$ such that $f_2^{-1} \circ \phi \circ f_1 : (S, s) \to (S, s)$ is isotopic to the identity. $\mathfrak{T}_{g,*}$ is the set of equivalence classes in $\mathfrak{A}$ and we view $\mathfrak{T}_{g,*}$ as the moduli space of "marked" Riemann surfaces. For $(f, M, z) \in \mathfrak{T}_{g,*}$, a standard set of generators for $\pi_1(M, z)$ is given by $a_i = f_*(\alpha_i)$, $b_i = f_*(\beta_i)$ for $i = 1, \dots, g$ and this is what is meant when f is referred to as a "marking". The mapping class group $\mathcal{M}_{g,*}$ is the group of isotopy classes of orientation preserving homeomorphisms of (S, s) onto (S, s). We let the group $\mathcal{M}_{g,*}$ act on the right on $\mathfrak{T}_{g,*}$ via:

$$\text{for } h \in \mathcal{M}_{g,*}, \quad h: \begin{array}{ccc} \mathfrak{T}_{g,*} & \to & \mathfrak{T}_{g,*} \\ (f, M, z) & \mapsto & (f \circ h, M, z) \end{array}. \tag{1.1}$$

A class $h \in \mathcal{M}_{g,*}$ is given by an orientation preserving homeomorphism $h : (S, s) \to (S, s)$, and h induces an automorphism h_* of $\pi_1(S, s)$ which induces the identity on $H_2(S, \mathbb{Z})$ as well as a symplectic automorphism of $H_1(S, \mathbb{Z})$ which we also denote by h_*. We let $\rho_h \in \mathrm{Sp}_g(\mathbb{Z})$ be the $2g \times 2g$ matrix which represents h_* on $H_1(S, \mathbb{Z})$ with respect to the standard basis $[\gamma_i]$. This gives a homomorphism $\rho : \mathcal{M}_{g,*} \to \mathrm{Sp}_g(\mathbb{Z})$, which is onto by a theorem of Manger [12]; its kernel $\mathrm{Ker}\,(\rho) = \mathrm{I}_{g,*}$ is known as the Torelli group.

We let $\mathfrak{h}_g$ denote the Siegel upper half space of $g \times g$ symmetric complex matrices with positive definite imaginary part. For $Z \in \mathfrak{h}_g$ we let $\mathcal{L}_Z = Z\mathbb{Z}^g + \mathbb{Z}^g \subseteq \mathbb{C}^g$ be a lattice and recall that $A_Z = \mathbb{C}^g/\mathcal{L}_Z$ has the structure of a principally polarized abelian variety. The identification of lattice elements in $\mathcal{L}_Z$ with loops based at 0 in A_Z gives the canonical identification $H_1(A_Z, \mathbb{Z}) \cong \mathcal{L}_Z$. Given $Z \in \mathfrak{h}_g$, we may use this identification to construct a standard basis for $H_1(A_Z, \mathbb{Z})$. We let $A_i \in H_1(A_Z, \mathbb{Z})$ be given by the loop $t \in [0, 1]$, $t \mapsto (\delta_{ij}t)_{j=1}^g \in A_Z$, and B_i by the loop $t \in [0, 1]$, $t \mapsto (Z_{ij}t)_{j=1}^g \in A_Z$, for $i = 1, \dots, g$. We note that the polarization form is dual to the element $q_Z = \sum_i A_i \wedge B_i \in H_1 \otimes H_1$. The standard left action of $\mathrm{Sp}_g(\mathbb{Z})$ on $\mathfrak{h}_g$ is given by $\sigma \cdot Z = (aZ + b)(cZ + d)^{-1}$, for $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_g(\mathbb{Z})$ and $Z \in \mathfrak{h}_g$. We will use a right action of $\mathrm{Sp}_g(\mathbb{Z})$ on $\mathfrak{h}_g$ given by $Z \cdot \sigma = {}^t\sigma \cdot Z$.

Definition 1.2. We define our action on the right of $\mathrm{Sp}_g(\mathbb{Z})$ on the trivial bundle $\mathfrak{h}_g \times H_1(S, \mathbb{Z})$ as follows: $(Z, ([\beta]\,[\alpha])\binom{n}{m}) \cdot \sigma = ({}^t\sigma \cdot Z, ([\beta]\,[\alpha])\sigma^{-1}\binom{n}{m})$ for $\sigma \in \mathrm{Sp}_g(\mathbb{Z})$ and $n, m \in \mathbb{Z}^g$. Here, $[\alpha]$, $[\beta]$ stand for the row vector with components α_i and β_i, respectively.

The right action of $\mathrm{Sp}_g(\mathbb{Z})$ on $\mathfrak{h}_g \times H_1(S, \mathbb{Z})$ defined above is easily motivated. We are imitating the right action of $\mathcal{M}_{g,*}$ on $\mathfrak{T}_{g,*} \times H_1(S, \mathbb{Z})$ given by $((f, M), [\gamma]) \cdot h = ((f \circ h, M), h_*^{-1}[\gamma])$ for $h \in \mathcal{M}_{g,*}$.

Definition 1.3. Let $\mathcal{H}_1(\mathbb{Z}) = \coprod_{Z \in \mathfrak{h}_g} H_1(A_Z, \mathbb{Z})$.

$$\begin{array}{cccc} \text{Define} & \jmath : & \mathfrak{h}_g \times H_1(S, \mathbb{Z}) & \to & \mathcal{H}_1(\mathbb{Z}) \\ \textit{by} & & (Z, ([\beta]\,[\alpha])\binom{n}{m}) & \mapsto & (B \;\; A)\binom{n}{m} \in H_1(A_Z, \mathbb{Z}). \end{array}$$

Here, A, B stand for the row vector with components A_i and B_i, respectively.

Definition 1.4. Let $\mathrm{Sp}_g(\mathbb{Z})$ act on $\mathcal{H}_1(\mathbb{Z})$ on the right via

$$[(B \;\; A)\begin{pmatrix} n \\ m \end{pmatrix}] \cdot \sigma = (B \;\; A)\sigma^{-1}\begin{pmatrix} n \\ m \end{pmatrix} \in H_1(A_{{}^t\sigma \cdot Z}, \mathbb{Z})$$

for any given $\sigma \in \mathrm{Sp}_g(\mathbb{Z})$ and $(B \;\; A)\binom{n}{m} \in H_1(A_Z, \mathbb{Z})$.

Lemma 1.5. *$\jmath$ is a bijection and gives $\mathcal{H}_1(\mathbb{Z})$ the structure of a trivial bundle. For $\sigma \in \mathrm{Sp}_g(\mathbb{Z})$, the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{h}_g \times H_1(S,\mathbb{Z}) & \xrightarrow{\ \jmath\ } & \mathcal{H}_1(\mathbb{Z}) \\ \downarrow{\cdot\sigma} & & \downarrow{\cdot\sigma} \\ \mathfrak{h}_g \times H_1(S,\mathbb{Z}) & \xrightarrow{\ \jmath\ } & \mathcal{H}_1(\mathbb{Z}). \end{array}$$

Proof. Left to the reader. ∎

Note also that for $i = 1, \dots, g$, $\jmath(\bullet, [\beta_i])$ and $\jmath(\bullet, [\alpha_i])$ are global sections of $\mathcal{H}_1(\mathbb{Z})$ over $\mathfrak{h}_g$. If we tensor $\mathcal{H}_1(\mathbb{Z})$ over $\mathbb{Z}$ in each fiber with $\mathbb{C}$ or $\mathbb{R}$, we obtain a trivial complex or real vector bundle $\mathcal{H}_1(\mathbb{C})$ or $\mathcal{H}_1(\mathbb{R})$, whose fibers may be treated as $H_1(A_Z,\mathbb{C})$ or $H_1(A_Z,\mathbb{R})$, respectively. Furthermore the homology group $H_1(A_Z,\mathbb{C})$, often abbreviated $H_1(Z)$, has a polarized Hodge structure of weight -1 dual to the weight 1 polarized Hodge structure on $H^1(A_Z,\mathbb{C})$. The Hodge filtration on $H_1(A_Z,\mathbb{C})$ is given by $F^0H_1 = \mathrm{Ann}(F^1H^1) = \mathrm{Ann}(H^{1,0}) = \{(B\ A)\binom{\mu}{\nu} \in H_1 \otimes_{\mathbb{Z}} \mathbb{C} : \mu,\nu \in \mathbb{C}^g \text{ and } (Z\ \ I)\binom{\mu}{\nu} = 0\}$. The action of $\sigma \in \mathrm{Sp}_g(\mathbb{Z})$ on each fiber is actually a morphism of polarized Hodge structures. To check this, let $(B\ A)\binom{\mu}{\nu} \in H_1(A_Z,\mathbb{C})$ and assume $(Z\ \ I)\binom{\mu}{\nu} = 0$. Then $((B\ A)\binom{\mu}{\nu})\cdot\sigma = (B\ A)\sigma^{-1}\binom{\mu}{\nu} \in H_1(A_{{}^t\sigma\cdot Z},\mathbb{C})$, and we verify that $({}^t\sigma\cdot Z\ \ I)\sigma^{-1}\binom{\mu}{\nu} = (Zb+d)^{-1}(Z\ \ I)\binom{\mu}{\nu} = (Zb+d)^{-1}0 = 0$. Furthermore, $(q_Z)\cdot\sigma = q_{Z\cdot\sigma}$ because the symplectic group fixes the symplectic form.

Proposition 1.6. *$\mathcal{H}_1(\mathbb{C})$ is a trivial vector bundle of polarized Hodge structures over $\mathfrak{h}_g$. If $\Gamma \subseteq \mathrm{Sp}_g(\mathbb{Z})$ is a subgroup that acts without fixed points on $\mathfrak{h}_g$, then $\mathcal{H}_1(\mathbb{C})/\Gamma$ is also a vector bundle of polarized Hodge structures over $\mathfrak{h}_g/\Gamma$.*

Proof. Everything has been verified already except the part about the action of the subgroup Γ. $\mathrm{Sp}_g(\mathbb{Z})/\pm I$ acts properly discontinuously on $\mathfrak{h}_g$ [8, p.25]. If Γ is fixed point free then $\mathcal{H}_1(\mathbb{C}) \to \mathcal{H}_1(\mathbb{C})/\Gamma$ will be a covering map so that $\mathfrak{h}_g/\Gamma$ will be a complex manifold and $\mathcal{H}_1(\mathbb{C})/\Gamma$ will be a vector bundle [8, p.117]. ∎

We have described the construction of the bundle $\mathcal{H}_1(\mathbb{C})$ in some detail because we will omit most details when constructing further bundles. The abelian category of polarized Hodge structures admits both duals and tensor products. These two operations on vector spaces induce corresponding operations on vector bundles. We have the vector bundle $\mathcal{H}_1(\mathbb{C}) \otimes \mathcal{H}_1(\mathbb{C})$, which will be a vector bundle of Hodge structures, and $\mathrm{Sp}_g(\mathbb{Z})$ acts naturally on the right as morphisms on fibers. We continue to write the right action of $\sigma \in \mathrm{Sp}_g(\mathbb{Z})$ as $\cdot\sigma$ even when $\cdot(\sigma\otimes\sigma)$ or $\cdot(\sigma \otimes {}^t\sigma^{-1})$ would be more precise; this convention avoids the need to change our notation every time an intermediate isomorphism such as $\mathcal{H}_1(\mathbb{C})\otimes\mathcal{H}_1(\mathbb{C}) \cong \mathcal{H}_1(\mathbb{C})\otimes\mathcal{H}^1(\mathbb{C})$ is applied. Note that $Z \mapsto (Z, q_Z)$ will be a section of the bundle $\mathcal{H}_1(\mathbb{C}) \otimes \mathcal{H}_1(\mathbb{C})$; call this section q. Since each q_Z may be thought of as a bilinear map $q : H^1 \otimes H^1 \to \mathbb{C}$, then q is a vector bundle morphism from $\mathcal{H}^1(\mathbb{C})\otimes\mathcal{H}^1(\mathbb{C}) \to \mathfrak{h}_g \times \mathbb{C}$. The symplectic group leaves q invariant; so the symplectic group restricts to an action on the kernel of q. Namely, we can construct vector bundles of polarized Hodge structures $\mathcal{K}_2$ and $\mathcal{K}_3$ where the fibers are given by the exact sequences of Hodge structures:

(1.7)

$$0 \to K_2(Z) \to H^1(A_Z,\mathbb{C}) \otimes H^1(A_Z,\mathbb{C}) \xrightarrow{q_Z} \mathbb{C} \to 0,$$

$$0 \to K_3(Z) \to H^1(A_Z,\mathbb{C}) \otimes H^1(A_Z,\mathbb{C}) \otimes H^1(A_Z,\mathbb{C}) \xrightarrow{(q_Z\otimes\mathrm{id})\oplus(-\mathrm{id}\otimes q_Z)} H^1 \oplus H^1 \to 0.$$

So $\mathrm{Sp}_g(\mathbb{Z})$ will act on the vector bundles $\mathcal{K}_2$ and $\mathcal{K}_3$. Our main interest will be in the real vector bundle $\mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{R}}$ over $\mathfrak{h}_g$, whose fibers are $\mathrm{Hom}(K_2(Z), H^1(Z))_{\mathbb{R}}$. The space $\mathrm{Hom}(K_2(Z), H^1(Z))$ has a Hodge structure of weight -1. Because $\mathrm{Hom}(K_2, H^1)$ has negative weight, the real space $\mathrm{Hom}(K_2, H^1)_{\mathbb{R}}$ injects into $\mathrm{Hom}(K_2, H^1)_{\mathbb{C}}/F^0\mathrm{Hom}(K_2, H^1)$. For a quick proof, let A be a Hodge structure of negative weight. Then $F^0A \cap \overline{F}^0A = 0$. Since $A_{\mathbb{R}} \cap F^0A \subseteq A_{\mathbb{R}}$ is invariant under conjugation, we have $A_{\mathbb{R}} \cap F^0A = A_{\mathbb{R}} \cap \overline{F}^0A = A_{\mathbb{R}} \cap (F^0A \cap \overline{F}^0A) = 0$, and so $A_{\mathbb{R}}$ imbeds in $A_{\mathbb{C}}/F^0A$ indeed. We define the torus

$$\mathrm{Ext}(K_2, H^1) = \frac{\mathrm{Hom}(K_2, H^1)_{\mathbb{C}}}{F^0\mathrm{Hom}(K_2, H^1) + \mathrm{Hom}(K_2, H^1)_{\mathbb{Z}}} \cong \frac{\mathrm{Hom}(K_2, H^1)_{\mathbb{R}}}{\mathrm{Hom}(K_2, H^1)_{\mathbb{Z}}}$$

which gives the set of congruence classes of separated extensions of mixed Hodge structures of K_2 by H^1 (see [2] or [6]). We may also define the torus bundle $\mathcal{E}xt(\mathcal{K}_2, \mathcal{H}^1)$ over $\mathfrak{h}_g$ having these $\mathrm{Ext}(K_2, H^1)$ as fibers.

Consider $(f, M, z) \in \mathfrak{T}_{g,*}$ with its marking $a_i = f_*(\alpha_i)$, $b_i = f_*(\beta_i)$ and its corresponding abelian differentials ω_i normalized by $\int_{a_j} \omega_i = \delta_{ij}$ and $\int_{b_j} \omega_i = (\Omega_f)_{ij}$. We also have the Abel-Jacobi map $w_f : M \to A_{\Omega_f}$ given by $x \mapsto (\int_z^x \omega_i)_{i=1}^g$ and the induced isomorphism on homology $(w_f)_* : H_1(M, \mathbb{Z}) \to H_1(A_{\Omega_f}, \mathbb{Z})$. Since $(w_f)_*(a_i) = \int_{a_i} \omega_\bullet = \delta_{\bullet i} = \jmath(\Omega_f, \alpha_i)$ and $(w_f)_*(b_i) = \int_{b_i} \omega_\bullet = \Omega_{\bullet i} = \jmath(\Omega_f, \beta_i)$, the following diagram commutes:

$$\begin{array}{ccc} H_1(M, \mathbb{Z}) & \xrightarrow{(w_f)_*} & H_1(A_{\Omega_f}, \mathbb{Z}) \\ {\scriptstyle f_*}\uparrow & & \| \\ H_1(S, \mathbb{Z}) & \xrightarrow{\jmath\Omega_f} & H_1(A_{\Omega_f}, \mathbb{Z}). \end{array} \tag{1.8}$$

If the homology basis $\binom{[b]}{[a]}$ for $H_1(M, \mathbb{Z})$ is changed to $\binom{[\tilde{b}]}{[\tilde{a}]} = \sigma\binom{[b]}{[a]}$ for $\sigma \in \mathrm{Sp}_g(\mathbb{Z})$, then the period matrix Ω_f changes to $\sigma \cdot \Omega_f$. We let $(f, M, z) \in \mathfrak{T}_{g,*}$ have period matrix Ω_f and let $h \in \mathcal{M}_{g,*}$ act on (f, M, z) as $(f, M, z) \cdot h = (f \circ h, M, z)$. The change of marking $f \circ h \circ f^{-1}$ has the homology matrix ρ_h with respect to the homology basis $f_*[\gamma] \in H_1(M, \mathbb{Z})$ that was used to compute Ω_f. Accordingly the action of $f \circ h \circ f^{-1}$ on the homology basis $\binom{[b]}{[a]}$ is ${}^t\rho_h\binom{[b]}{[a]}$, and we conclude that $\Omega_{f \circ h} = {}^t\rho_h \cdot \Omega_f$. This additional piece of information is enough to show that the following diagram commutes. For $(f, M, z) \in \mathfrak{T}_{g,*}$ and $h \in \mathcal{M}_{g,*}$,

$$
\begin{array}{ccccc}
H_1(S,\mathbb{Z}) & & \xrightarrow{\qquad\qquad \jmath_{\Omega_f} \qquad\qquad} & & H_1(A_{\Omega_f},\mathbb{Z}) \\
\Vert & & & & \Vert \\
H_1(S,\mathbb{Z}) & \xrightarrow{f_*} & H_1(M,\mathbb{Z}) & \xrightarrow{(w_f)_*} & H_1(A_{\Omega_f},\mathbb{Z}) \\
\uparrow{\scriptstyle h_*} & & \Vert & & \downarrow{\scriptstyle \cdot\rho_h} \\
H_1(S,\mathbb{Z}) & \xrightarrow{(f\circ h)_*} & H_1(M,\mathbb{Z}) & \xrightarrow{(w_{f\circ h})_*} & H_1(A_{\Omega_{f\circ h}},\mathbb{Z}) \\
\Vert & & & & \Vert \\
H_1(S,\mathbb{Z}) & & \xrightarrow{\qquad\qquad \jmath_{\Omega_{f\circ h}} \qquad\qquad} & & H_1(A_{\Omega_{f\circ h}},\mathbb{Z}).
\end{array}
\tag{1.9}
$$

Also, for any $Z \in \mathfrak{h}_g$, we have the diagram

$$
\begin{array}{ccc}
H_1(S,\mathbb{Z}) & \xrightarrow{\jmath_Z} & H_1(A_Z,\mathbb{Z}) \\
\uparrow{\scriptstyle h_*} & & \downarrow{\scriptstyle \cdot\rho_h} \\
H_1(S,\mathbb{Z}) & \xrightarrow{\jmath_{Z\cdot\rho_h}} & H_1(A_{Z\cdot\rho_h},\mathbb{Z})
\end{array}
\tag{1.10}
$$

We now introduce certain isomorphisms, λ, $\imath$, and θ, which will be defined whenever we have a principally polarized Hodge structure of weight one. In general, we use angular brackets $\langle\,\cdot\,,\,\cdot\,\rangle$ to denote the pairing between a mixed Hodge structure and its dual.

Definition 1.11. Let (H^1, q) be a principally polarized Hodge structure of weight one with $K_2 = \operatorname{Ker} q \subseteq H^1 \otimes H^1$ and $H_1 =$ dual of H^1. The map

$$\lambda : \operatorname{Hom}(K_2, H^1) \xrightarrow{\cong} ((H_1 \otimes H_1)/q) \otimes H^1$$

is defined by the property that $\langle\, \lambda\phi\,,\, k \otimes c \,\rangle = \langle\, \phi(k)\,,\, c \,\rangle$ for all $\phi \in \operatorname{Hom}(K_2, H^1)$ and all $k \otimes c \in K_2 \otimes H_1$. Also, define

$$
\begin{array}{cccc}
\imath : & ((H_1 \otimes H_1)/q) \otimes H^1 & \xrightarrow{\cong} & ((H_1 \otimes H_1)/q) \otimes H_1 \\
\text{by} & [a \otimes b] \otimes c & \mapsto & [a \otimes b] \otimes ((c \otimes \mathrm{id}_{H_1})(q)).
\end{array}
$$

Let $\theta : H_1 \otimes H_1 \otimes H_1 \to H_1 \otimes H_1 \otimes H_1$ be the map defined by $\theta : a \otimes b \otimes c \mapsto b \otimes a \otimes c$. Since $\theta(q \otimes c) = -q \otimes c$, we have the induced map

$$
\begin{array}{cccc}
\theta : & ((H_1 \otimes H_1)/q) \otimes H_1 & \xrightarrow{\cong} & ((H_1 \otimes H_1)/q) \otimes H_1 \\
\text{defined by} & [a \otimes b] \otimes c & \mapsto & [b \otimes a] \otimes c.
\end{array}
$$

We now prove a useful lemma.

Lemma 1.12. *The following diagram of vector spaces commutes, and the last three terms of the middle row form an exact sequence. Here, we write $\bigwedge^2 H_1$ to denote the vector subspace of $H_1 \otimes H_1$ spanned by elements of the form $a \wedge b = a \otimes b - b \otimes a$; and more generally, the exterior product $\bigwedge^3 H_1$ is the subspace of $\bigotimes^3 H_1$ of elements which are antisymmetric with respect to transpositions under the action of the symmetric group.*

$$
\begin{array}{ccccccc}
\bigwedge^3 H_1 & \xrightarrow{\text{injects}} & \bigwedge^2 H_1 \otimes H_1 & \xrightarrow{\text{injects}} & H_1 \otimes H_1 \otimes H_1 & & \\
\Vert & & \text{surjects}\downarrow & & \text{surjects}\downarrow & & \\
\bigwedge^3 H_1 & \xrightarrow{\text{injects}} & (\bigwedge^2 H_1/q) \otimes H_1 & \xrightarrow{\text{injects}} & ((H_1 \otimes H_1)/q) \otimes H_1 & \xrightarrow{\text{id}+\theta} & \text{Sym}(H_1 \otimes H_1) \otimes H_1 \\
\Vert & & \Vert & & \text{surjects}\downarrow & & \\
\bigwedge^3 H_1 & \xrightarrow{\text{injects}} & (\bigwedge^2 H_1/q) \otimes H_1 & \xrightarrow{\text{injects}} & (H_1 \otimes H_1 \otimes H_1)/(q \otimes H_1 + H_1 \otimes q) & &
\end{array}
$$

Proof. The injections in the top row are clear. To show that $\bigwedge^3 H_1 \hookrightarrow (\bigwedge^2 H_1/q) \otimes H_1$ is an injection, we need to show that $(\bigwedge^3 H_1) \cap (q \otimes H_1) = 0$. Pick a standard basis $A_j, B_j \in H_1$ so that $q = \sum A_j \wedge B_j$, and suppose $q \otimes h \in \bigwedge^3 H_1$ where $h = \sum m_j A_j + n_j B_j$, with $m_j, n_j \in \mathbb{C}$. We can cycle the three tensor components in $q \otimes h$ to obtain equal representations of $2q \otimes h$:

$$2(\sum_{i=1}^{g} A_i \wedge B_i) \otimes (\sum_{j=1}^{g} m_j A_j + n_j B_j) = \sum_{j=1}^{g} h \wedge A_j \otimes B_j + B_j \wedge h \otimes A_j.$$

Equating elements in $\bigwedge^2 H_1$ paired with A_j or B_j in the third tensor component, we would have $2m_j q = B_j \wedge h$ and $2n_j q = h \wedge A_j$. Since q is not decomposable, we conclude $m_j, n_j = 0$, and hence $h = 0$. The subspace $(\bigwedge^2 H_1/q) \otimes H_1 \subseteq ((H_1 \otimes H_1)/q) \otimes H_1$ is clearly the kernel of $\text{id}+\theta$ since $\text{id}+\theta$ is twice the projection onto $\text{Sym}(H_1 \otimes H_1) \otimes H_1$. This shows the exactness of the last three terms. The third row contains the map $(\bigwedge^2 H_1/q) \otimes H_1 \to (H_1 \otimes H_1 \otimes H_1)/(q \otimes H_1 + H_1 \otimes q)$, and to show this is an injection, we must show

$$(\bigwedge{}^2 H_1 \otimes H_1) \cap (q \otimes H_1 + H_1 \otimes q) = q \otimes H_1. \tag{1.13}$$

Suppose $q \otimes h + h' \otimes q \in \bigwedge^2 H_1 \otimes H_1$. Then $h' \otimes q \in \bigwedge^2 H_1 \otimes H_1$. This implies that $h' \otimes q$ is alternating when the first two tensor components are switched. Since $h' \otimes q$ is alternating when the last two tensor components are switched, we conclude that $h' \otimes q \in \bigwedge^3 H_1$. By our previous argument, $(H_1 \otimes q) \cap (\bigwedge^3 H_1) = 0$, and so $h' \otimes q = 0$. This proves equation 1.13, and completes the proof. ∎

For more details about Teichmüller space, see Bers [1]; about the Torelli group, see Johnson [10]; about $\mathfrak{h}_g$, see Igusa [8]; about extensions of mixed Hodge structures, see Carlson [2].

§2. The map Ψ.

In this section we define the map $\Psi : \mathfrak{T}_{g,*} \to \mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{R}}$ which associates to $(f, M, z) \in \mathfrak{T}_{g,*}$ the period matrix Ω_f and the element $\Psi_f \in \text{Hom}(K_2(A_{\Omega_f}), H^1(A_{\Omega_f}))_{\mathbb{R}}$.

The element Ψ_f is the image of a $\tilde{\Psi}_f \in \mathrm{Hom}(K_2(M), H^1(M))_{\mathbb{R}}$ giving the extension of mixed Hodge structures for $0 \to H^1(M) \to (J/J^3)^* \to K_2(M) \to 0$.

For a compact Riemann surface M with basepoint z, we have the following exact sequences of mixed Hodge structures [6]:

$$0 \to H^1(M) \to H^0(\overline{B}_2(M), z) \xrightarrow{p_2} K_2(M) \to 0, \tag{2.1}$$

and the dual sequence,

$$0 \to (H_1(M) \otimes H_1(M))/q_M \xrightarrow{p_2^*} (J(M,z)/J(M,z)^3) \to (J(M,z)/J(M,z)^2) \to 0. \tag{2.2}$$

Here, p_2^* is defined by $p_2^*([c_1] \otimes [c_2]) = (c_1 - 1)(c_2 - 1)$ for $c_i \in \pi_1(M, z)$; the complex vector space $H^0(\overline{B}_2(M), z)$ is the space of homotopy functionals which can be expressed as a reduced length two iterated integral; and the ring $J(M, z)$ is the augmentation ideal of the group ring $\mathbb{C}\pi_1(M, z)$. We will denote the pairing (via integration) of an iterated integral $I \in H^0(\overline{B}_2(M), z)$ with a loop $c \in \pi_1$ by $\langle\, I\, ,\, c\,\rangle$.

We denote the vector space of C^∞ k-forms on M by $A^k(M)$, and the subspace of $(p+q)$-forms of type p, q by $A^{p,q}(M)$. Also, denote the subspace of closed forms by $Z^{p,q}$.

Definition 2.3. $\hat{K}_2(M) = \{k \in A^1(M) \otimes A^1(M) : \wedge k \text{ is exact}\}$.

Lemma 2.4. *There is a unique linear map $u : \hat{K}_2(M) \to A^{1,0}(M)$ such that for all $k \in \hat{K}_2(M)$, $\wedge k + du(k) = 0$ and $u(k) - \overline{u(\overline{k})}$ is exact.*

Proof. We will first prove the claim that to each $k \in \hat{K}_2(M)$ there exists a unique pair $u(k)$ and $\tilde{u}(k)$ in $A^{1,0}(M)$ such that $du(k) + \wedge k = 0$, $d\tilde{u}(k) + \wedge \overline{k} = 0$, and $u(k) - \overline{\tilde{u}(k)}$ is exact. Since $\wedge k$ is exact, there exists a $1,0$-form $w \in A^1(M)$ such that $\wedge k + dw = 0$. This can be proven as follows. Consider the following diagram with exact rows,

$$\begin{array}{ccccc} A^{1,0}(M) & \xrightarrow{\bar{\partial}} & A^{1,1}(M) & \to & 0 \\ \cap & & \| & & \\ A^1(M) & \xrightarrow{d} & A^2(M) & \to & 0. \end{array}$$

This induces a surjective linear map,

$$\mathbb{C} \cong H^{1,1}_{\bar{\partial}}(M) \cong \frac{A^{1,1}(M)}{\bar{\partial} A^{1,0}(M)} \to \frac{A^2(M)}{dA^1(M)} \cong \mathbb{C}.$$

So this map is an isomorphism, which means that $\wedge k \in dA^1(M)$ implies $\wedge k \in \bar{\partial} A^{1,0}(M)$. Hence there exists a $1,0$-form $w \in A^1(M)$ such that $\wedge k + dw = 0$, as claimed. There is also a $1,0$-form $v \in A^1(M)$ such that $\wedge \overline{k} + dv = 0$. Then we have $dw - d\overline{v} = 0$, and so $w - \overline{v}$ is closed. By the Hodge decomposition, we have

$$w - \overline{v} = w' - \overline{v'} + \text{exact},$$

where $w', v' \in Z^{1,0}(M)$ are holomorphic 1-forms. Now let $u(k) = w - w'$, and let $\tilde{u}(k) = v - v'$. Certainly, $u(k)$ and $\tilde{u}(k)$ are in $A^{1,0}(M)$, with $du(k) + \wedge k = 0$ and $d\tilde{u}(k) + \wedge \overline{k} = 0$.

Also, we have $u(k)-\overline{\tilde{u}(k)} = w-w'-\overline{(v-v')} =$ (an exact form). To prove the uniqueness in this claim, suppose also that u' and $\tilde{u}'$ are in $A^{1,0}(M)$ such that $du'+\wedge k=0$, $d\tilde{u}'+\wedge\overline{k}=0$, and $u'-\overline{\tilde{u}'}$ is exact. Then we have $du-du'=0$, so that $u-u'\in Z^{1,0}(M)$. Similarly, we have $\tilde{u}-\tilde{u}'\in Z^{1,0}(M)$. Also, since $u-\overline{\tilde{u}}$ and $u'-\overline{\tilde{u}'}$ are exact, then their difference, $u-u'-\overline{(\tilde{u}-\tilde{u}')}$ is exact. But $(u-u')$ is holomorphic and $\overline{(\tilde{u}-\tilde{u}')}$ is antiholomorphic; hence by the Hodge decomposition of closed 1-forms, they must both be 0. Thus $u=u'$ and $\tilde{u}=\tilde{u}'$, proving uniqueness and the claim.

Note that since $d\tilde{u}(k)+\wedge\overline{k}=0$, $du(k)+\wedge\overline{\overline{k}}=0$, and $\tilde{u}(k)-\overline{u(k)}$ is exact, we must have that $\tilde{u}(k)=u(\overline{k})$ by uniqueness in the claim. Thus we have proven that there exists a set-theoretic map $u:\hat{K}_2(M)\to A^{1,0}(M)$ such that $du(k)+\wedge k=0$ and and $u(k)-\overline{u(\overline{k})}$ is exact. Such a map must be unique since any such map u would generate a pair u and $\tilde{u}$ that satisfies the above claim.

The uniqueness in the claim forces this map u to be linear since corresponding to any element $k_1+k_2\in\hat{K}_2(M)$, the pair $(u(k_1)+u(k_2))$ and $(\tilde{u}(k_1)+\tilde{u}(k_2))$ and the pair $(u(k_1+k_2))$ and $(\tilde{u}(k_1+k_2))$ both satisfy the claim and hence must be the same. ∎

This map u was constructed and used by Pulte in section 3 of [13]. We have repeated the existence proof here because u is important in our computations.

Remark 2.5. By using harmonic representatives, we may view $K_2(M)\subseteq\hat{K}_2(M)$: if $\sum[\mu_i]\otimes[\nu_i]\in K_2(M)\subseteq H^1\otimes H^1$ and μ_i and ν_i are the unique harmonic 1-forms representing $[\mu_i],[\nu_i]\in H^1$, then $\sum\mu_i\otimes\nu_i\in\hat{K}_2(M)$. This allows us to use the restricted map $u: K_2(M)\to A^{1,0}(M)$.

Proposition 2.6. *Let (M,z) be a compact Riemann surface with basepoint. A Hodge filtration preserving section of p_2 is defined by*

$$\begin{array}{cccc} s_2: & K_2(M) & \to & H^0(\overline{B}_2(M),z) \\ & k & \mapsto & \int(k+u(k)). \end{array}$$

Proof. (Pulte [13, p.730]) The iterated integral $\int(k+u(k))$ is a homotopy functional since $\wedge k+du(k)=0$. It preserves the Hodge filtration since $u(k)\in A^{1,0}$ and $u|_{F^2K_2}=0$. ∎

Remark 2.7. The section s_2 also preserves the weight filtration. But s_2 is not a morphism of mixed Hodge structures because s_2 does not preserve the lattice. For example, if $h\in(H^1(M))_{\mathbb{Z}}$, then $s_2(h\otimes h)=\int hh=\frac{1}{2}\int h\int h$, which may take the value $\frac{1}{2}$ on $H_1(M,\mathbb{Z})$.

Definition 2.8. Let $(f,M,z)\in\mathfrak{T}_{g,*}$ be a marked Riemann surface. Define an integral retraction $r_{\mathbb{Z}}: H^0(\overline{B}_2(M),z)\to H^1(M)$ for all $I\in H^0(\overline{B}_2(M),z)$ by

$$r_{\mathbb{Z}}(I)=\sum_{i=1}^{g}\langle\, I\,,\,a_i\,\rangle[a_i]^*+\langle\, I\,,\,b_i\,\rangle[b_i]^*,$$

where $[a_i]^*,[b_i]^*$ is the dual basis in H^1, of the basis $[a_i],[b_i]$ in H_1. Also, define

$$\tilde{\Psi}_f=r_{\mathbb{Z}}\circ s_2\in\operatorname{Hom}(K_2(M),H^1(M))_{\mathbb{C}}.$$

Lemma 2.9. *Let* $(f, M, z) \in \mathfrak{T}_{g,*}$. *Then* $\tilde{\Psi}_f \in \mathrm{Hom}(K_2(M), H^1(M))_{\mathbb{R}}$ *and* $[\tilde{\Psi}_f] \in \mathrm{Ext}(K_2(M), H^1(M))$ *gives the congruence class of the extension of mixed Hodge structures in* $0 \to H^1(M) \to H^0(\overline{B}_2(M), z) \xrightarrow{p_2} K_2(M) \to 0$.

Proof. Since $r_{\mathbb{Z}}(\overline{I}) = \overline{r_{\mathbb{Z}}(I)}$ and $s_2(\overline{k}) = \overline{s_2(k)}$, we have $\tilde{\Psi}_f(\overline{k}) = \overline{\tilde{\Psi}_f(k)}$ for all $k \in K_2(M)$ and this means that $\tilde{\Psi}_f \in \mathrm{Hom}(K_2, H^1)_{\mathbb{R}}$. That $\tilde{\Psi}_f$ gives the correct class in $\mathrm{Ext}(K_2, H^1)$ is proven in Pulte [13, Theorem 2.9] or Carlson [2, pp.116–117]. ∎

Definition 2.10. For $(f, M, z) \in \mathfrak{T}_{g,*}$, define $\Psi_f \in \mathrm{Hom}(K_2(\Omega_f), H^1(\Omega_f))_{\mathbb{R}}$ by $\Psi_f = (w_f)_* \tilde{\Psi}_f$ where $(w_f)_* : \mathrm{Hom}(K_2(M), H^1(M))_{\mathbb{R}} \to \mathrm{Hom}(K_2(\Omega_f), H^1(\Omega_f))_{\mathbb{R}}$ is the map induced by the Abel-Jacobi map $w_f : M \to A_{\Omega_f}$.

Remark 2.11. If we wish to consider $\tilde{\Psi}$ as a period map, we select a basis for $K_2(M)$. Let $\omega_1, \ldots, \omega_g$ be a normalized basis of abelian differentials on M so that $\langle \int \omega_i \,,\, a_j \rangle = \delta_{ij}$ and $\langle \int \omega_i \,,\, b_j \rangle = \Omega_{ij}$. Then $\omega_i \otimes \omega_j$, $\overline{\omega}_i \otimes \overline{\omega}_j$, $\overline{\omega}_i \otimes \omega_j - \Lambda_{ij} \overline{\omega}_1 \otimes \omega_1$, $\omega_i \otimes \overline{\omega}_j - \Lambda_{ij} \omega_1 \otimes \overline{\omega}_1$, and $\omega_1 \otimes \overline{\omega}_1 + \overline{\omega}_1 \otimes \omega_1$ form a basis for $K_2(M)$ where $\Lambda_{ij} = \mathrm{Im}\Omega_{ij}/\mathrm{Im}\Omega_{11}$. Hence we have the homotopy functionals, σ_{ij} and τ_{ij} defined by

$$\sigma_{ij}(c) = \int_c \omega_i \omega_j, \qquad (1 \le i, j \le g)$$

$$\tau_{ij}(c) = \int_c \overline{\omega}_i \omega_j - \Lambda_{ij} \overline{\omega}_1 \omega_1 + u(\overline{\omega}_i \otimes \omega_j - \Lambda_{ij} \overline{\omega}_1 \otimes \omega_1), \qquad ((i, j) \ne (1, 1))$$

etc.

for loops $c \in \pi_1(M, z)$. We may explictly give $\tilde{\Psi}_f$ as follows. For simplicity, write $c_1, \ldots, c_{2g}$ for $b_1, \ldots, b_g, a_1, \ldots, a_g$. Then

$$\begin{aligned} \lambda \tilde{\Psi}_f = \sum_{i,j,k} & (\sigma_{ij}(c_k)[\omega_i \otimes \omega_j]^* \otimes [c_k]^* + \tau_{ij}(c_k)[\overline{\omega}_i \otimes \omega_j - \Lambda_{ij} \overline{\omega}_1 \otimes \omega_1]^* \otimes [c_k]^* \\ & + \overline{\tau}_{ij}(c_k)[\omega_i \otimes \overline{\omega}_j - \Lambda_{ij} \omega_1 \otimes \overline{\omega}_1]^* \otimes [c_k]^* + \overline{\sigma}_{ij}(c_k)[\overline{\omega}_i \otimes \overline{\omega}_j]^* \otimes [c_k]^*) \\ & + \sum_k \langle \textstyle\int \omega_1 \int \overline{\omega}_1 \,,\, c_k \rangle [\omega_1 \otimes \overline{\omega}_1 + \overline{\omega}_1 \otimes \omega_1]^* \otimes [c_k]^*. \end{aligned}$$

The functions $\sigma_{ij}(c_k)$, $\tau_{ij}(c_k)$, etc. are well-defined functions on $\mathfrak{T}_{g,*}$. The functions $\sigma_{ij}(a_k)$, $\sigma_{ij}(b_k)$ arising from the purely holomorphic part of K_2 are called the quadratic periods, and the $\tau_{ij}(a_k)$, $\tau_{ij}(b_k)$ are called the mixed periods (see [9], [4]). The basis used here is merely convenient and we know of no "natural" basis for $K_2(M)$. When one wishes to present a statement about periods in a basis-free form, it is best to use the map $\tilde{\Psi}$.

§3. The cocycle $\hat{\delta}$.

In this section, we define a cocycle $\hat{\delta} \in Z^1((\mathcal{M}_{g,*}, (H_1 \otimes H_1)/q) \otimes H^1)_{\mathbb{Z}})$, a global section $\eta \in \bigotimes^3 \mathcal{H}_1(\mathbb{Z})$, and a cocycle $\phi \in H^2(\mathrm{Sp}_g(\mathbb{Z}), (\bigwedge^3 H_1)_{\mathbb{Z}})$. The cocycle $\hat{\delta}$ extends Johnson's homomorphism δ and will be used in the next section to compute the action of $\mathcal{M}_{g,*}$ on Ψ in Proposition 4.2. The global section η gives the value $\imath \jmath \hat{\delta}(y)$ in $((\mathcal{H}_1 \otimes \mathcal{H}_1)/q) \otimes \mathcal{H}_1)$ for some homology involution $y \in \mathcal{M}_{g,*}$, and among all such sections we show that η is intrinsically distinguished by its retracting to zero in $\bigwedge^3\mathcal{H}_1$ (Lemma 3.9). This explicit section η is then used to compute the cocycle ϕ which gives the group extension $0 \to (\bigwedge^3 H_1)_{\mathbb{Z}} \to \mathcal{M}_{g,*}/\mathcal{N}_{g,*} \to \mathrm{Sp}_g(\mathbb{Z}) \to 0$. Hence the structure of the mapping class group modulo Johnson's kernel, $\mathcal{M}_{g,*}/\mathcal{N}_{g,*}$, is given very explicitly.

We now recall D. Johnson's homomorphism [10]. There are three equivalent versions, which we denote by δ', δ and τ; these maps differ in their image spaces. The homomorphism δ',

$$\delta' : \mathrm{I}_{g,*} \to \mathrm{Hom}(H_1(S), (H_1 \otimes H_1)/q_S)_{\mathbb{Z}},$$

is defined for $h \in \mathrm{I}_{g,*}$ to be the element $\delta'(h) \in \mathrm{Hom}(H_1(S), (H_1 \otimes H_1)/q_S)$ that sends each $[\gamma] \in H_1(S)$ (where $\gamma \in \pi_1(S,s)$) to $\gamma^{-1}h_*(\gamma) \in [\pi_1(S,s), \pi_1(S,s)]$ and then identifying $[\pi_1,\pi_1]/[\pi_1,[\pi_1,\pi_1]] \cong (H_1 \wedge H_1)/q$. The homomorphism δ,

$$\delta : \mathrm{I}_{g,*} \to ((H_1(S,\mathbb{Z}) \otimes H_1(S,\mathbb{Z}))/q_S) \otimes H^1, \tag{3.1}$$

is obtained from δ' through the natural isomorphism $\mathrm{Hom}(H_1, (H_1 \otimes H_1)/q) \cong ((H_1 \otimes H_1)/q) \otimes H^1$. Recalling the identification map of Definition 1.11, $\imath : ((H_1 \otimes H_1)/q) \otimes H^1 \xrightarrow{\cong} ((H_1 \otimes H_1)/q) \otimes H_1$, we have the homomorphism $\tau = \imath\delta : \mathrm{I}_{g,*} \to ((H_1 \otimes H_1)/q) \otimes H_1$. Johnson showed [10, p.170] that this map τ has the same image as $\bigwedge^3 H_1$, and this is the most important fact for us.

We will define a map $\hat{\delta}$ from $\mathcal{M}_{g,*}$ to $(H_1(S) \otimes H_1(S))/q_S \otimes H^1(S)$ which restricts to Johnson's map δ on $\mathrm{I}_{g,*}$. A generalization of $[\pi_1,\pi_1]/[\pi_1,[\pi_1,\pi_1]] \cong (H_1 \wedge H_1)/q$ is that we may identify $J^2/J^3 \xrightarrow{\cong} (H_1 \otimes H_1)/q$. This is accomplished via $(p_2^*)_S^{-1}$ as given in equation 2.2. Thus Johnson's map δ' is given by $\delta'(h)[\gamma] = (p_2^*)_S^{-1}(\gamma^{-1}h_*(\gamma) - 1)$. So $\delta(h)$ could be written as $\delta(h) = \sum_\gamma (p_2^*)_S^{-1}(\gamma^{-1}h_*(\gamma) - 1) \otimes [\gamma]^*$, where the sum runs over a basis $[\gamma]$ of H_1. We use this viewpoint to define our map $\hat{\delta}$; however, $\hat{\delta}$ is no longer a homomorphism but a cocycle of some sort.

Definition 3.2. For $h \in \mathcal{M}_{g,*}$, recall that we let $\rho_h \in \mathrm{Sp}_g(\mathbb{Z})$ be the matrix representing h_* in the homology basis $[\gamma_i]$ of $H_1(S,\mathbb{Z})$. For $i = 1, \ldots, 2g$, define $\hat{\delta}_i : \mathcal{M}_{g,*} \to J(S,s)_{\mathbb{Z}}$ by

$$\hat{\delta}_i(h) = \left(\sum_{j=1}^{2g} [{}^t\rho_h^{-1}]_{ij}(h_*(\gamma_j) - 1) \right) - (\gamma_i - 1).$$

Lemma 3.3. *For each $i = 1, \ldots, 2g$, we have that $\hat{\delta}_i$ maps to $J^2(S,s)_{\mathbb{Z}}$.*

Proof. Let $h \in \mathcal{M}_{g,*}$. Since $\hat{\delta}_i(h) \in J(S,s)$, we only need to show that $\hat{\delta}_i(h) = 0$ in $J/J^2 \cong H^1(S)$. In $H^1(S)$ however, $\hat{\delta}_i(h) = (\sum_{j=1}^{2g} [{}^t\rho_h^{-1}]_{ij}(h_*(\gamma_j) - 1)) - (\gamma_i - 1)$ is given by $({}^t\rho_h^{-1}\,{}^t\rho_h[\gamma])_i - [\gamma_i] = [\gamma_i] - [\gamma_i] = 0$. ∎

Definition 3.4. Define $\hat{\delta} = \sum_{i=1}^{2g} (p_2^*)_S^{-1} \hat{\delta}_i \otimes [\gamma_i]^*$, so that

$$\begin{array}{rccc} \hat{\delta} : & \mathcal{M}_{g,*} & \to & ((H_1(S,\mathbb{Z}) \otimes H_1(S,\mathbb{Z}))/q_S) \otimes H^1(S,\mathbb{Z}) \\ & h & \mapsto & \sum_{i=1}^{2g} ((p_2^*)_S^{-1} \hat{\delta}_i(h)) \otimes [\gamma_i]^*. \end{array}$$

Proposition 3.5. *For $h \in \mathrm{I}_{g,*}$, we have $\hat{\delta}(h) = \delta(h)$. The map $\hat{\delta}$ is a group cocycle in $Z^1((\mathcal{M}_{g,*}, ((H_1 \otimes H_1)/q) \otimes H^1)_{\mathbb{Z}})$, that is, for all $h_1, h_2 \in \mathcal{M}_{g,*}$, we have $\hat{\delta}(h_1h_2) = (h_1)_*\hat{\delta}(h_2) + \hat{\delta}(h_1)$.*

Proof. To show that $\hat{\delta}(h) = \delta(h)$ for $h \in \mathrm{I}_{g,*}$, notice that by definition $\rho_h = I$ so that we have $\hat{\delta}_i(h) = (h_*(\gamma_i) - 1) - (\gamma_i - 1)$. Observe also that $h \in \mathrm{I}_{g,*}$ implies $\gamma_i^{-1}h_*(\gamma_i) - 1 \in J^2$. Then $h_*(\gamma_i) - 1 - (\gamma_i - 1) - (\gamma_i^{-1}h_*(\gamma_i) - 1) = (\gamma_i - 1)(\gamma_i^{-1}h_*(\gamma_i) - 1) \in J^3$, and so we have $(h_*(\gamma_i) - 1) - (\gamma_i - 1) \equiv \gamma_i^{-1}h_*(\gamma_i) - 1$ modulo J^3. Therefore, we have $(p_2^*)_S^{-1}\hat{\delta}_i(h) = (p_2^*)_S^{-1}(\gamma_i^{-1}h_*(\gamma_i) - 1)$, and so $\hat{\delta}(h) = \sum_{i=1}^{2g}(p_2^*)_S^{-1}(\gamma_i^{-1}h_*(\gamma_i) - 1) \otimes [\gamma_i]^*$, which is exactly $\delta(h)$.

To prove the cocycle relation, let $h \in \mathcal{M}_{g,*}$. We have $\hat{\delta}_i(h) = \sum_{j=1}^{2g}({}^t\rho_h^{-1})_{ij}(h_*(\gamma_j) - 1) - (\gamma_i - 1)$. So we have $\hat{\delta}_i(h_1h_2) = \sum_{j=1}^{2g}({}^t\rho_{h_1h_2}^{-1})_{ij}((h_1h_2)_*(\gamma_j) - 1) - (\gamma_i - 1) = \sum_{j=1}^{2g}({}^t\rho_{h_1}^{-1}{}^t\rho_{h_2}^{-1})_{ij}((h_1)_*(h_2)_*(\gamma_j) - 1) - (\gamma_i - 1)$. Then

$$
\begin{aligned}
(p_2^*)_S^{-1}\hat{\delta}_i(h_1h_2) =& (p_2^*)_S^{-1}\left(\sum_{j,k=1}^{2g}[{}^t\rho_{h_1}^{-1}]_{ij}[{}^t\rho_{h_2}^{-1}]_{jk}(h_{1*}(h_{2*}(\gamma_k)) - 1)\right.\\
&\left. - \sum_j [{}^t\rho_{h_1}^{-1}]_{ij}(h_{1*}(\gamma_j) - 1) + \sum_j [{}^t\rho_{h_1}^{-1}]_{ij}(h_{1*}(\gamma_j) - 1) - (\gamma_i - 1)\right)\\
=& (h_1)_*(p_2^*)_S^{-1}\left(\sum_j ({}^t\rho_{h_1}^{-1})_{ij}\left(\sum_k ({}^t\rho_{h_2}^{-1})_{jk}(h_{2*}(\gamma_k) - 1) - (\gamma_j - 1)\right)\right) + (p_2^*)_S^{-1}\hat{\delta}_i(h_1)\\
=& (h_1)_*(p_2^*)_S^{-1}\left(\sum_j ({}^t\rho_{h_1}^{-1})_{ij}\hat{\delta}_j(h_2)\right) + (p_2^*)_S^{-1}\hat{\delta}_i(h_1).
\end{aligned}
$$

Therefore,

$$\begin{aligned}
\hat{\delta}(h_1h_2) - \hat{\delta}(h_1) &= \sum_{i=1}^{2g} \left((p_2^*)_S^{-1} (\hat{\delta}_i(h_1h_2) - \hat{\delta}_i(h_1)) \right) [\gamma_i]^* \\
&= \sum_{i=1}^{2g} (h_1)_* (p_2^*)_S^{-1} \left(\sum_{j=1}^{2g} ({}^t\rho_{h_1}^{-1})_{ij} \hat{\delta}_j(h_2) \right) [\gamma_i]^* \\
&= \sum_{j=1}^{2g} (h_1)_* (p_2^*)_S^{-1} \hat{\delta}_j(h_2) \left(\sum_{i=1}^{2g} (\rho_{h_1}^{-1})_{ji} [\gamma_i]^* \right) \\
&= \sum_{j=1}^{2g} (h_1)_* (p_2^*)_S^{-1} \hat{\delta}_j(h_2) \left[\sum_{i=1}^{2g} ({}^t\rho_{h_1})_{ji} \gamma_i \right]^* \\
&= \sum_{j=1}^{2g} (h_1)_* (p_2^*)_S^{-1} \hat{\delta}_j(h_2) ([(h_1)_* \gamma_j]^*) \\
&= (h_1)_* \left(\sum_{j=1}^{2g} (p_2^*)_S^{-1} \hat{\delta}_j(h_2) [\gamma_j]^* \right) \\
&= (h_1)_* \hat{\delta}(h_2). \blacksquare
\end{aligned}$$

Remark 3.6. The cocycle $\hat{\delta}$ depends upon the marking of S. If the basis $[\gamma_i]$ of $H_1(S, \mathbb{Z})$ is replaced by $[\tilde{h}_* \gamma_i]$ for some $\tilde{h} \in \mathcal{M}_{g,*}$, then $\hat{\delta}(h)$ will be replaced by $\tilde{h}_* \hat{\delta}(h) + \tilde{h}_* \delta((\tilde{h}^{-1}, h))$ for all $h \in \mathcal{M}_{g,*}$, where $(a, b) = aba^{-1}b^{-1}$ denotes the group commutator.

We now compute $\hat{\delta}(y)$ for a specific homology involution y.

Proposition 3.7. *There is a $y \in \mathcal{M}_{g,*}$ such that y_* maps $\alpha_i \mapsto \alpha_i^{-1}(\alpha_i, \beta_i)$ and $\beta_i \mapsto (\beta_i, \alpha_i)\beta_i^{-1}$, and*

$$\hat{\delta}(y) = \sum_{i=1}^{g} ((-[\beta_i] \wedge [\alpha_i] - [\beta_i] \otimes [\beta_i]) \otimes [\beta_i]^* + (-[\alpha_i] \wedge [\beta_i] - [\alpha_i] \otimes [\alpha_i]) \otimes [\alpha_i]^*);$$

$$\jmath\hat{\delta}(y) = \sum_{i=1}^{g} ((-B_i \wedge A_i - B_i \otimes B_i) \otimes B_i^* + (-A_i \wedge B_i - A_i \otimes A_i) \otimes A_i^*);$$

$$\imath\jmath\hat{\delta}(y) = \sum_{i=1}^{g} (B_i \wedge A_i \otimes A_i + B_i \otimes B_i \otimes A_i - A_i \wedge B_i \otimes B_i - A_i \otimes A_i \otimes B_i).$$

Proof. Thinking of $\pi_1(S, s)$ as the quotient of the free group on α_i, β_i modulo the commutator relation $\prod_i (\alpha_i, \beta_i) = 1$, we let $v \in \mathrm{Aut}(\pi_1(S, s))$ be defined by

$$\alpha_i \mapsto \alpha_i^{-1}(\alpha_i, \beta_i)$$
$$\beta_i \mapsto (\beta_i, \alpha_i)\beta_i^{-1}.$$

One needs to check that the commutator relation is preserved. Since it is easy to see that each $(\alpha_i,\beta_i) \mapsto (\alpha_i,\beta_i)$, the commutator relation is actually preserved exactly, and v is indeed an automorphism of π_1 which induces $+1$ on H_2. Since S is a $K(\pi_1, 1)$, there exists a homeomorphism $y : (S,s) \to (S,s)$ which induces $y_* = v$ on π_1 and hence is orientation preserving. Therefore, we have $y \in \mathcal{M}_{g,*}$. Finally we note that the action of y_* on $H_1 \cong \pi_1/[\pi_1,\pi_1]$ is $\alpha_i \mapsto -\alpha_i$ and $\beta_i \mapsto -\beta_i$ so that y is a homology involution, that is, $\rho_y = -I = (-\delta_{ij})$. By Definition 3.2, we have

$$\begin{aligned}\hat{\delta}_i(y) &= \sum_{j=1}^{2g} [{}^t\rho_y^{-1}]_{ij}(y_*(\gamma_j)-1) - (\gamma_i - 1)\\ &= \sum_{j=1}^{2g} (-\delta_{ij})(v(\gamma_j)-1) - (\gamma_i-1)\\ &= -(v(\gamma_i)-1) - (\gamma_i - 1).\end{aligned}$$

Hence for $i = 1, \ldots, g$, we have

$$\begin{aligned}\hat{\delta}_i(y) &= -(v(\beta_i)-1) - (\beta_i - 1)\\ &= -((\beta_i,\alpha_i)\beta_i^{-1} - 1) - (\beta_i - 1)\\ &= -((\beta_i,\alpha_i) - 1) - (\beta_i - 1)^2 \mod J^3\end{aligned}$$

so that we have $(p_2^*)^{-1}\hat{\delta}_i(y) = -[\beta_i] \wedge [\alpha_i] - [\beta_i] \otimes [\beta_i]$. In the same way, we have for $i = 1, \ldots, g$,

$$\begin{aligned}\hat{\delta}_{i+g}(y) &= -(v(\alpha_i)-1) - (\alpha_i - 1)\\ &= -(\alpha_i^{-1}(\alpha_i,\beta_i) - 1) - (\alpha_i - 1)\\ &= -((\alpha_i,\beta_i) - 1) - (\alpha_i - 1)^2 \mod J^3\end{aligned}$$

so that we have $(p_2^*)^{-1}\hat{\delta}_{i+g}(y) = -[\alpha_i] \wedge [\beta_i] - [\alpha_i] \otimes [\alpha_i]$. This proves the first equation in the proposition. The other two equations follow by applying the maps $\jmath$ and $\imath$. ∎

Lemma 3.8. *For any homology involution y, and for any $h \in \mathcal{M}_{g,*}$, we have*

$$\hat{\delta}(h) = \tfrac{1}{2}(\hat{\delta}(y) - h_*\hat{\delta}(y)) + \tfrac{1}{2}\delta((h,y)).$$

Proof. From $\hat{\delta}(hy) = h_*\hat{\delta}(y) + \hat{\delta}(h)$, $\hat{\delta}(yh) = y_*\hat{\delta}(h) + \hat{\delta}(y)$, and $y_* = -\mathrm{id}_{H_1}$, we have $\hat{\delta}(hy) - \hat{\delta}(yh) = h_*\hat{\delta}(y) - \hat{\delta}(y) + 2\hat{\delta}(h)$, or $\hat{\delta}(h) = \frac{1}{2}(\hat{\delta}(y) - h_*\hat{\delta}(y)) + \frac{1}{2}(\hat{\delta}(hy) - \hat{\delta}(yh))$. For any commutator $(a,b) = aba^{-1}b^{-1}$, we have $\hat{\delta}((a,b)) = \hat{\delta}(ab) - (a,b)_*\hat{\delta}(ba)$. The centrality of $y_* = -\mathrm{id}_{H_1}$ implies that $(h,y)_* = \mathrm{id}_{H_1}$, so that $\hat{\delta}((h,y)) = \hat{\delta}(hy) - \hat{\delta}(yh)$. This completes the proof by noting that, since $(h,y) \in \mathrm{I}_{g,*}$, we have $\hat{\delta}((h,y)) = \delta((h,y))$. ∎

Lemma 3.9. *There is a unique global section η of $\bigotimes^3\mathcal{H}_1(\mathbb{Z})$ whose retraction to $\bigwedge^3\mathcal{H}_1(\mathbb{Q})$ is zero and whose image in $((\mathcal{H}_1\otimes\mathcal{H}_1)/q)\otimes\mathcal{H}_1$ is $\imath\jmath\hat{\delta}(y)$ for some homology involution y. This section is given by*

$$\eta = -\sum_{j=1}^{g}(A_j\otimes A_j\otimes B_j + A_j\wedge B_j\otimes B_j - B_j\otimes B_j\otimes A_j - B_j\wedge A_j\otimes A_j).$$

Proof. The injection of $\bigwedge^3 H_1$ into $\bigotimes^3 H_1$ is given by $c_1\wedge c_2\wedge c_3 = \sum_{\pi\in S_3}\operatorname{sgn}(\pi)c_{\pi(1)}\otimes c_{\pi(2)}\otimes c_{\pi(3)}$, and the mentioned retraction, say r, of $\bigotimes^3 H_1$ onto $\bigwedge^3 H_1$ is given by $r(c_1\otimes c_2\otimes c_3) = \frac{1}{6}\sum_{\pi\in S_3}\operatorname{sgn}(\pi)c_{\pi(1)}\otimes c_{\pi(2)}\otimes c_{\pi(3)} = \frac{1}{6}c_1\wedge c_2\wedge c_3$. Clearly $r(\eta)=0$ because each summand of η has repeated factors. There is a homology involution y such that $[\eta] = \imath\jmath\hat{\delta}(y)$ in $((H_1\otimes H_1)/q)\otimes H_1$ by proposition 3.7. Now assume that $\eta'\in\bigotimes^3 H_1$ exists such that $r(\eta')=0$ and $[\eta'] = \imath\jmath\hat{\delta}(y')$ in $((H_1\otimes H_1)/q)\otimes H_1$ for some homology involution y'. By proposition 3.8, we have $\hat{\delta}(y') = \hat{\delta}(y) + \frac{1}{2}\delta((y',y))$ so that $[\eta'-\eta] = \imath\jmath\hat{\delta}(y') - \imath\jmath\hat{\delta}(y) = \frac{1}{2}\imath\jmath\delta((y',y))\in\bigwedge^3 H_1$ in $((H_1\otimes H_1)/q)\otimes H_1$. Lemma 1.12 then implies that $\eta'-\eta\in\bigwedge^3 H_1\subseteq\bigotimes^3 H_1$. Since $r(\eta'-\eta)=0$, we have $\eta'=\eta$. ∎

As an application of the techniques used in the above proofs, we compute the cocycle in $H^2(\mathrm{Sp}_g(\mathbb{Z}),\bigwedge^3 H_1(S,\mathbb{Z}))$ which gives the extension

$$0\to(\textstyle\bigwedge^3 H_1)_{\mathbb{Z}}\xrightarrow{\tau^{-1}}\mathcal{M}_{g,*}/\mathcal{N}_{g,*}\xrightarrow{\rho}\mathrm{Sp}_g(\mathbb{Z})\to 0.$$

Recall that the existence of this extension follows from the two exact sequences

$$0\to \mathrm{I}_{g,*}\to\mathcal{M}_{g,*}\xrightarrow{\rho}\mathrm{Sp}_g(\mathbb{Z})\to 0,$$
$$0\to\mathcal{N}_{g,*}\to \mathrm{I}_{g,*}\xrightarrow{\tau}(\textstyle\bigwedge^3 H_1)_{\mathbb{Z}}\to 0.$$

We mention some facts from group cohomology which can be directly verified. Let M be a $\mathbb{Z}$-module with no two-torsion, and let G be a group with a central element y. Assume that M is also a G-module and that y acts on M as -1. We employ the usual boundary operators: $(\delta_0\eta)(\sigma)=\sigma\eta-\eta$, $(\delta_1 m)(\sigma_1,\sigma_2)=\sigma_1 m(\sigma_2)-m(\sigma_1\sigma_2)+m(\sigma_1)$, etc.; we also define a homotopy $y_i: C^i(G,M)\to C^{i-1}(G,M)$ of degree -1 via $y_1 m = -m(y)$, $(y_2\phi)(\sigma)=\phi(\sigma,y)-\phi(y,\sigma)$, $(y_3\rho)(\sigma_1,\sigma_2) = -\rho(y,\sigma_1,\sigma_2)+\rho(\sigma_1,y,\sigma_2)-\rho(\sigma_1,\sigma_2,y)$, etc. One computes that on $C^n(G,M)$, we have $y_{n+1}\delta_n+\delta_{n-1}y_n = 2\,\mathrm{id}_{C^n}$, and this shows that multiplication by 2 is homotopic to zero. Hence $H^*(G,M)$ consists solely of two-torsion elements. From the long exact cohomology sequence for the short exact sequence $0\to 2M\to M\to M/2M\to 0$, we conclude further that $H^n(G,M/2M)\cong H^n(G,M)\oplus H^{n+1}(G,M)$. We will apply these results to $G=\mathrm{Sp}_g(\mathbb{Z})$ with $y=-I$ being central and with $M=\bigwedge^3 H_1(S,\mathbb{Z})$. Here we use brackets, [ ], for classes mod 2. Let $V=((H_1(S,\mathbb{Z})\otimes H_1(S,\mathbb{Z}))/q_S)\otimes H_1(S,\mathbb{Z})$.

Proposition 3.10. *Let $g\geq 2$. Let $\phi\in H^2(\mathrm{Sp}_g(\mathbb{Z}),(\bigwedge^3 H_1)_{\mathbb{Z}})$ be the cohomology class giving the extension*

$$0\to(\textstyle\bigwedge^3 H_1)_{\mathbb{Z}}\xrightarrow{\tau^{-1}}\mathcal{M}_{g,*}/\mathcal{N}_{g,*}\xrightarrow{\rho}\mathrm{Sp}_g(\mathbb{Z})\to 0$$

where $\mathrm{Sp}_g(\mathbb{Z})$ *acts naturally on* $H_1(S,\mathbb{Z})$*. Let* $\bar\eta = \imath\hat\delta(y) \in V$ *be given by equation 3.7:*

$$\bar\eta = \sum_{i=1}^{g}([\beta_i]\wedge[\alpha_i]\otimes[\alpha_i] + [\beta_i]\otimes[\beta_i]\otimes[\alpha_i] - [\alpha_i]\wedge[\beta_i]\otimes[\beta_i] - [\alpha_i]\otimes[\alpha_i]\otimes[\beta_i]).$$

We have

$$\begin{aligned}
&H^0(\mathrm{Sp}_g(\mathbb{Z}), ({\textstyle\bigwedge}^3 H_1)_{\mathbb{Z}}) = 0,\\
&H^1(\mathrm{Sp}_g(\mathbb{Z}), ({\textstyle\bigwedge}^3 H_1)_{\mathbb{Z}}) = 0, \ \textit{and}\\
&H^2(\mathrm{Sp}_g(\mathbb{Z}), ({\textstyle\bigwedge}^3 H_1)_{\mathbb{Z}}) \underset{\cong}{\xrightarrow{y_2}} H^1(\mathrm{Sp}_g(\mathbb{Z}), ({\textstyle\bigwedge}^3 H_1)_{\mathbb{Z}}\otimes\mathbb{Z}_2).
\end{aligned}$$

Under the isomorphism y_2*,*

$$H^2(\mathrm{Sp}_g(\mathbb{Z}), ({\textstyle\bigwedge}^3 H_1)_{\mathbb{Z}}) \xrightarrow{y_2} H^1(\mathrm{Sp}_g(\mathbb{Z}), ({\textstyle\bigwedge}^3 H_1)/2({\textstyle\bigwedge}^3 H_1)) \cong H^1(\mathrm{Sp}_g(\mathbb{Z}), ({\textstyle\bigwedge}^3 H_1 + 2V)/2V),$$

the extension class of ϕ *is given by* $y_2\phi = [\delta_0\bar\eta]$*. This class is nontrivial; so the extension does not split.*

Proof. The cocycle $\imath\hat\delta \in Z^1(\mathcal{M}_{g,*}, V)$ gives a homomorphism Θ,

$$\begin{array}{rccc}
\Theta: & \mathcal{M}_{g,*} & \to & V \rtimes \mathrm{Sp}_g(\mathbb{Z})\\
& h & \mapsto & (\imath\hat\delta(h), \rho_h)
\end{array}$$

from $\mathcal{M}_{g,*}$ to the semidirect product $V\rtimes \mathrm{Sp}_g(\mathbb{Z})$. We have $\mathrm{Ker}\,\Theta = \mathcal{N}_{g,*}$ because $\mathrm{Ker}\,(\rho) = \mathrm{I}_{g,*}$ and because for $h \in \mathrm{I}_{g,*}$, we have $\imath\hat\delta(h) = \tau(h) = 0$ if and only if $h \in \mathcal{N}_{g,*}$. The restriction to $\mathrm{I}_{g,*}$ of Θ is $\tau\times\{I\}$, and the projection to $\mathrm{Sp}_g(\mathbb{Z})$ of Θ gives ρ. Therefore we may compute a cocycle $\phi \in Z^2(\mathrm{Sp}_g(\mathbb{Z}), \bigwedge^3 H_1)$ defining the extension by using any section of ρ, $s : \mathrm{Sp}_g(\mathbb{Z}) \to \mathcal{M}_{g,*}$, and letting $\phi = \delta_1(\imath\hat\delta\circ s)$.

We choose $s(-I) = y$, where $y \in \mathcal{M}_{g,*}$ is the homology involution of Proposition 3.7. We use the identity $y_2\delta_1 + \delta_0 y_1 = 2\,\mathrm{id}$ on $C^1(\mathrm{Sp}_g(\mathbb{Z}), \bigwedge^3 H_1)$ to compute that

$$\begin{aligned}
y_2\phi &= y_2\delta_1(\imath\hat\delta\circ s) = 2\imath\hat\delta\circ s - \delta_0 y_1(\imath\hat\delta\circ s)\\
&= 2\imath\hat\delta\circ s + \delta_0(\imath\hat\delta\circ s(-I)) = 2\imath\hat\delta\circ s + \delta_0(\imath\hat\delta(y)) = 2\imath\hat\delta\circ s + \delta_0\bar\eta.
\end{aligned}$$

We see that the image of $y_2\phi$ in $H^1(\mathrm{Sp}_g(\mathbb{Z}), \bigwedge^3 H_1/2\bigwedge^3 H_1)$ is given by $[2\imath\hat\delta\circ s + \delta_0\bar\eta]$. If we make the simple identification $\bigwedge^3 H_1/2\bigwedge^3 H_1 \cong (\bigwedge^3 H_1 + 2V)/2V$, then $[2\imath\hat\delta\circ s + \delta_0\bar\eta]$ is given by $[\delta_0\bar\eta]$. This completes the computation of the class of ϕ.

In order to show that $[\delta_0\bar\eta]$ is a nontrivial element in $H^1(\mathrm{Sp}_g(\mathbb{Z}), (\bigwedge^3 H_1 + 2V)/2V) \cong H^1(\mathrm{Sp}_g(\mathbb{Z}), \bigwedge^3 H_1/2\bigwedge^3 H_1)$, we first compute the values of the representative cycle $(\delta_0\bar\eta)(\sigma) = \sigma\bar\eta - \bar\eta$ on generators of $\mathrm{Sp}_g(\mathbb{Z})$. Recall that a 1-cycle is determined by its values on group generators (though an arbitrary prescription of such values need not define a 1-cycle). For the remainder of this proof only we write a_i instead of $[\alpha_i]$, and b_i instead of $[\beta_i]$. If σ is given by (type 1)

$$a_i \mapsto b_i; \quad b_i \mapsto -a_i,$$

then $\sigma\bar{\eta} - \bar{\eta} = 2\sum_{i=1}^{g}(a_i \otimes a_i \otimes b_i - b_i \wedge a_i \otimes a_i)$. Therefore the value in $\bigwedge^3 H_1 \otimes \mathbb{Z}_2$ is 0. Next, if σ is given by (type 2)

$$b_i \mapsto b_i; \quad a_i \mapsto a_i \ (i \neq \ell); \quad a_\ell \mapsto a_\ell + b_\ell,$$

then $\sigma\bar{\eta} - \bar{\eta} = -2a_\ell \otimes b_\ell \otimes b_\ell$. Therefore the value in $\bigwedge^3 H_1 \otimes \mathbb{Z}_2$ is again 0. Finally, if σ is given by (type 3)

$$b_i \mapsto b_i; \quad a_i \mapsto a_i \ (i \neq m, n); \quad a_m \mapsto a_m + b_n; \quad a_n \mapsto a_n + b_m,$$

then $\sigma\bar{\eta} - \bar{\eta} = b_m \wedge a_m \wedge b_n + b_n \wedge a_n \wedge b_m + 2(b_m \wedge b_n \otimes a_m - a_m \otimes b_n \otimes b_m + b_n \wedge b_m \otimes a_n - a_n \otimes b_m \otimes b_n)$. Therefore the value in $\bigwedge^3 H_1 \otimes \mathbb{Z}_2$ is $[b_m \wedge a_m \wedge b_n + b_n \wedge a_n \wedge b_m]$, and so for $g \geq 2$, $\delta_0\bar{\eta}$ is at least not the boundary of the zero 0-cycle. The 1-cycle $\delta_0\bar{\eta}$ is then trivial precisely when there is an element $x \in \bigwedge^3 H_1 \otimes \mathbb{Z}_2$ such that the boundary $\delta_0 x$ takes the above values on the corresponding generators of $\mathrm{Sp}_g(\mathbb{Z})$. We will show that for $x \in \bigwedge^3 H_1 \otimes \mathbb{Z}_2$, no $\delta_0 x$ can take the above values on all three types of generators. We in fact show more: if $\delta_0 x$ achieves the above values on the first two types of generators, then $x = 0$. This will demonstrate that $\delta_0\bar{\eta}$ does not represent a boundary in $Z^1(\mathrm{Sp}_g(\mathbb{Z}), \bigwedge^3 H_1 \otimes \mathbb{Z}_2)$, that the class $[\delta_0\bar{\eta}]$ is nontrivial, and that the extension does not split.

Let $x \in \bigwedge^3 H_1 \otimes \mathbb{Z}_2$ and write

$$x = \sum_{i,j,k}(a_{ijk}\, a_i \wedge a_j \wedge a_k + b_{ijk}\, a_i \wedge a_j \wedge b_k + c_{ijk}\, a_i \wedge b_j \wedge b_k + d_{ijk}\, b_i \wedge b_j \wedge b_k)$$

where the coefficients are in $\mathbb{Z}_2$. Assume that $\sigma x - x = 0$ when σ is the first type of generator (type 1). The resulting condition on x is easily seen to be $a_{ijk} = d_{ijk}$ and $b_{ijk} = c_{kij}$. Now write

$$x = \sum_{i,j,k}(a_{ijk}(a_i \wedge a_j \wedge a_k + b_i \wedge b_j \wedge b_k) + b_{ijk}(a_i \wedge a_j \wedge b_k + b_i \wedge b_j \wedge a_k)),$$

and assume that $\sigma x - x = 0$ when σ is the second type of generator (type 2). The linear map $L_\ell : \bigwedge^3 H_1 \otimes \mathbb{Z}_2 \to \bigwedge^3 H_1 \otimes \mathbb{Z}_2$ given by $L_\ell(x) = [\sigma x - x]$ sends a triple wedge to zero if it has no a_ℓ factor or if it has both a_ℓ and b_ℓ facotrs. Conversely, a moment's thought shows that L_ℓ is nonsingular on the span of the triple wedges with an a_ℓ factor but no b_ℓ factor. From the assumption $L_\ell x = \sigma x - x = 0$, we conclude that x does not contain a term with an a_ℓ factor that does not also have a b_ℓ factor. Therefore x can contain no $a_i \wedge a_j \wedge a_k$ terms, and so $a_{ijk} = 0$. Also, x can contain no $a_i \wedge a_j \wedge b_k$ terms, and so we have $b_{ijk} = 0$. Hence we have $x = 0$. This shows that a nontrivial x cannot be simultaneously fixed under the first two types of generators of $\mathrm{Sp}_g(\mathbb{Z})$ and completes the proof that the class of $\delta_0(\bar{\eta})$ is nontrivial. Notice that this same calculation implies that fixed elements of $\bigwedge^3 H_1 \otimes \mathbb{Z}_2$ are trivial, and so we have $H^0(\mathrm{Sp}_g(\mathbb{Z}), \bigwedge^3 H_1 \otimes \mathbb{Z}_2) = 0$. This vanishing implies the vanishing of $H^1(\mathrm{Sp}_g(\mathbb{Z}), \bigwedge^3 H_1)$ as well, showing that the clearly injective map y_2 in the statement of the theorem is actually an isomorphism. ∎

§4. Equivariance of Ψ.

In this section we use the cocycle $\hat{\delta}$ to compute the action of $\mathcal{M}_{g,*}$ on Ψ in Proposition 4.2. An action of $\mathcal{M}_{g,*}/\mathcal{N}_{g,*}$ on $\mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{R}}$ is defined so that $\Psi : \mathfrak{T}_{g,*}/\mathcal{N}_{g,*} \to$

$\mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{R}}$ is equivariant with respect to $\mathcal{M}_{g,*}/\mathcal{N}_{g,*}$. The structure of $\mathcal{M}_{g,*}/\mathcal{N}_{g,*}$ was given in §3. This equivariance is analogous to the equivariance of the Torelli map $\Omega : \mathfrak{T}_{g,*}/\mathrm{I}_{g,*} \to \mathfrak{h}_g$ with respect to $\mathrm{Sp}_g(\mathbb{Z}) \cong \mathcal{M}_{g,*}/\mathrm{I}_{g,*}$. Finally, we use the work of Harris, Hain, Koizumi and Pulte to show that Ψ is generically injective on $\mathfrak{T}_{g,*}/\mathcal{N}_{g,*}$ in Proposition 4.7.

The following lemma is essential in computing the action of $\mathcal{M}_{g,*}$ on Ψ. It is in this computation that one can see how the cocycle $\hat{\delta}$ arises naturally.

Lemma 4.1. *Let $(f, M, z) \in \mathfrak{T}_{g,*}$ and $h \in \mathcal{M}_{g,*}$. In $(H_1(S) \otimes H_1(S))/q_S \otimes H^1(S)$, we have*

$$\lambda(f_*)^{-1}(\tilde{\Psi}_{f\circ h} - \tilde{\Psi}_f) = \hat{\delta}(h).$$

Proof. Recall from Definition 2.8 that $\tilde{\Psi}_f \in \mathrm{Hom}(K_2(M), H^1(M))_{\mathbb{R}}$ is defined for an element $(f, M, z) \in \mathfrak{T}_{g,*}$ by $\tilde{\Psi}_f(k_1) = \sum_{i=1}^{2g} \langle\, s_2(k_1)\,,\ f_*(\gamma_i)\,\rangle [f_*(\gamma_i)]^*$ for $k_1 \in K_2(M)$ and the fixed standard marking γ_i on (S, s). So $((f_*)^{-1}\tilde{\Psi}_f)(k) = \sum_{i=1}^{2g} \langle\, s_2((f^{-1})^*k)\,,\ f_*(\gamma_i)\,\rangle [\gamma_i]^*$ for $k \in f^*K_2(M)$. For any $h \in \mathcal{M}_{g,*}$, we have

$$\begin{aligned}
((f_*)^{-1}\tilde{\Psi}_{f\circ h})(k) &= \sum_{i=1}^{2g} \langle\, s_2((f^{-1})^*k)\,,\ (f\circ h)_*(\gamma_i)\,\rangle [h_*(\gamma_i)]^* \\
&= \sum_{i=1}^{2g} \langle\, s_2((f^{-1})^*k)\,,\ (f\circ h)_*(\gamma_i)\,\rangle \Big[\sum_{j=1}^{2g} [{}^t\rho_h]_{ij}[\gamma_j]\Big]^* \\
&= \sum_{i=1}^{2g} \langle\, s_2((f^{-1})^*k)\,,\ (f\circ h)_*(\gamma_i)\,\rangle \sum_{j=1}^{2g} [\rho_h^{-1}]_{ij}[\gamma_j]^* \\
&= \sum_{j=1}^{2g} \langle\, s_2((f^{-1})^*k)\,,\ \textstyle\sum_{i=1}^{2g} [{}^t\rho_h^{-1}]_{ji}(f\circ h)_*(\gamma_i)\,\rangle [\gamma_j]^* \\
&= ((f_*^{-1})\tilde{\Psi}_f)(k) + \sum_{j=1}^{2g} \langle\, s_2((f^{-1})^*k)\,,\ f_*\{\textstyle\sum_{i=1}^{2g} [{}^t\rho_h^{-1}]_{ji}h_*(\gamma_i) - \gamma_j\}\,\rangle [\gamma_j]^*.
\end{aligned}$$

We now apply the definition 3.2 of $\hat{\delta}_j(h)$, noting that $s_2((f^{-1})^*k) \in H^0(\overline{B}_2(M), z)$ is zero on constant loops. We have

$$\begin{aligned}
((f_*)^{-1}(\tilde{\Psi}_{f\circ h} - \tilde{\Psi}_f))(k) &= \sum_{j=1}^{2g} \langle\, s_2((f^{-1})^*k)\,,\ f_*\hat{\delta}_j(h)\,\rangle [\gamma_j]^* \\
&= \sum_{j=1}^{2g} \langle\, s_2((f^{-1})^*k)\,,\ (p_2^*)_M (p_2^*)_M^{-1} f_*\hat{\delta}_j(h)\,\rangle [\gamma_j]^* \\
&= \sum_{j=1}^{2g} \langle\, (p_2)_M s_2((f^{-1})^*k)\,,\ f_*(p_2^*)_S^{-1}\hat{\delta}_j(h)\,\rangle [\gamma_j]^* \\
&= \sum_{j=1}^{2g} \langle\, (f^{-1})^*k\,,\ f_*(p_2^*)_S^{-1}\hat{\delta}_j(h)\,\rangle [\gamma_j]^*
\end{aligned}$$

$$= \sum_{j=1}^{2g} \langle\, k\, ,\, (p_2^*)_S^{-1}\hat{\delta}_j(h)\,\rangle [\gamma_j]^*$$
$$= \sum_{j=1}^{2g} (\lambda^{-1}((p_2^*)_S^{-1}\hat{\delta}_j(h) \otimes [\gamma_j]^*))(k)$$
$$= (\lambda^{-1}\hat{\delta}(h))(k).$$

Therefore, we have $(f_*)^{-1}(\tilde{\Psi}_{f\circ h} - \tilde{\Psi}_f) = \lambda^{-1}\hat{\delta}(h)$. In step three of the above calculation we used that the isomorphism $(p_2^*)^{-1} : J^2/J^3 \overset{\sim}{\longrightarrow} (H_1 \otimes H_1)/q$ given by sending $(\gamma_1 - 1)(\gamma_2 - 1) + J^3 \mapsto [\gamma_1] \otimes [\gamma_2]$ commutes with f in the following sense:

$$\begin{array}{ccc} J^2(M,z)/J^3(M,z) & \xleftarrow{(p_2^*)_M} & H_1(M) \otimes H_1(M)/q_M \\ \uparrow f_* \text{ (on } \pi_1) & & \uparrow f_* \text{ (on } H_1) \\ J^2(S,s)/J^3(S,s) & \xleftarrow{(p_2^*)_S} & H_1(S) \otimes H_1(S)/q_S. \end{array}$$ ∎

Recall from section §1 that $\mathrm{Sp}_g(\mathbb{Z})$ has a natural right action on the vector bundles $\mathcal{H}^1$ and $\mathcal{K}_2$. These actions are induced from the action given in Definition 1.4. Accordingly, any $h \in \mathcal{M}_{g,*}$ acts on the vector bundle $\mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{R}}$ by a map on the fibers $\rho_h : \mathrm{Hom}(K_2(Z), H^1(Z))_{\mathbb{R}} \to \mathrm{Hom}(K_2(Z \cdot \rho_h), H^1(Z \cdot \rho_h))_{\mathbb{R}}$.

Proposition 4.2. *Let $\Psi : \mathfrak{T}_{g,*} \to \mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{R}}$, as in Definition 2.10, be given by $(f, M, z) \mapsto \Psi_f$. For all $h \in \mathcal{M}_{g,*}$, we have*

$$\Psi_{f\circ h} = (\Psi_f + \jmath_{\Omega_f}\lambda^{-1}\hat{\delta}(h)) \cdot \rho_h.$$

Proof. From Lemma 4.1, we have $(f_*)^{-1}(\tilde{\Psi}_{f\circ h} - \tilde{\Psi}_f) = \lambda^{-1}\hat{\delta}(h)$. We apply $\jmath_{\Omega_f}$ to both sides and use $\jmath_{\Omega_f}(f_*)^{-1} = (w_f)_*$ from diagram 1.9 to obtain $(w_f)_*(\tilde{\Psi}_{f\circ h} - \tilde{\Psi}_f) = \jmath_{\Omega_f}\lambda^{-1}\hat{\delta}(h)$. Since $\Psi_f = (w_f)_*\tilde{\Psi}_f$, we have $(w_f)_*\tilde{\Psi}_{f\circ h} = \Psi_f + \jmath_{\Omega_f}\lambda^{-1}\hat{\delta}(h)$. Since also $\rho_h \circ (w_f)_* = (w_{f\circ h})_*$ from diagram 1.9, we apply $\cdot\rho_h$ to both sides to obtain $\Psi_{f\circ h} = (w_{f\circ h})_*\tilde{\Psi}_{f\circ h} = (\Psi_f + \jmath_{\Omega_f}\lambda^{-1}\hat{\delta}(h)) \cdot \rho_h$ as desired. ∎

Corollary 4.3. *For any $h \in \mathrm{I}_{g,*}$ and $(f, M, z) \in \mathfrak{T}_{g,*}$, we have*

$$\Psi_{f\circ h} = \Psi_f + \jmath_{\Omega_f}\lambda^{-1}\delta(h) \in \mathrm{Hom}(K_2(\Omega_f), H^1(\Omega_f))_{\mathbb{R}}.$$

Proof. This follows immediately from Proposition 4.2 by noticing that $\rho_h = I$ since $h \in \mathrm{I}_{g,*}$, and also that $\hat{\delta}(h) = \delta(h)$ when $h \in \mathrm{I}_{g,*}$. ∎

We may define an action on $\mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{R}}$ to imitate the action of $\mathcal{M}_{g,*}$ on Ψ. There is no apriori guarantee that such an action exists, but nonexistence would be more interesting as it would furnish a special property of the image of Ω in $\mathfrak{h}_g$.

Definition 4.4. Define an action of $\mathcal{M}_{g,*}/\mathcal{N}_{g,*}$ on the $\mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{R}}$ bundle over $\mathfrak{h}_g$ as follows: for $\phi \in \mathrm{Hom}(K_2(Z), H^1(Z))_{\mathbb{R}}$ and $h \in \mathcal{M}_{g,*}$,

$$\phi \cdot h = (\phi + \jmath_Z \lambda^{-1}\hat{\delta}(h)) \cdot \rho_h \in \mathrm{Hom}(K_2(Z \cdot \rho_h), H^1(Z \cdot \rho_h))_{\mathbb{R}}.$$

We call this the affine action of $\mathcal{M}_{g,*}/\mathcal{N}_{g,*}$.

We need to check that $\phi \cdot h = (\phi + \jmath_Z \lambda^{-1}\hat{\delta}(h)) \cdot \rho_h$ is a right group action and that $\mathcal{N}_{g,*}$ acts trivially. This follows from the cocycle relations that $\hat{\delta}$ satisfies. We have $\phi \cdot (h_1 h_2) = (\phi + \jmath_Z \lambda^{-1}\{(h_1)_*\hat{\delta}(h_2) + \hat{\delta}(h_1)\}) \cdot (\rho_{h_1}\rho_{h_2}) = \phi \cdot (\rho_{h_1}\rho_{h_2}) + \{\jmath_Z \lambda^{-1}((h_1)_*\hat{\delta}(h_2))\} \cdot (\rho_{h_1}\rho_{h_2}) + \{\jmath_Z \lambda^{-1}(\hat{\delta}(h_1))\} \cdot (\rho_{h_1}\rho_{h_2})$. On the other hand, we have $(\phi \cdot h_1) \cdot h_2 = (\phi \cdot h_1 + \jmath_{Z\cdot\rho_{h_1}} \lambda^{-1}\hat{\delta}(h_2)) \cdot \rho_{h_2} = ((\phi + \jmath_Z \lambda^{-1}\hat{\delta}(h_1)) \cdot \rho_{h_1} + \jmath_{Z\cdot\rho_{h_1}} \lambda^{-1}\hat{\delta}(h_2)) \cdot \rho_{h_2} = \phi \cdot \rho_{h_1}\rho_{h_2} + (\jmath_Z \lambda^{-1}\hat{\delta}(h_1)) \cdot \rho_{h_1}\rho_{h_2} + (\jmath_{Z\cdot\rho_{h_1}} \lambda^{-1}\hat{\delta}(h_2)) \cdot \rho_{h_2}$. These are equal because $(\jmath_Z h_* l) \cdot \rho_h = \jmath_{Z\cdot\rho_h} l$ for $l \in \mathrm{Hom}(K_2(Z), H^1(Z))$ by diagram 1.9. Also, $\mathcal{N}_{g,*}$ acts trivially, so that $\mathcal{M}_{g,*}/\mathcal{N}_{g,*}$ acts on $\mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)$. For if $h \in \mathcal{N}_{g,*}$, then $h \in \mathrm{I}_{g,*}$, hence $\rho_h = I$ and $\hat{\delta}(h) = \delta(h) = 0$ so that $\phi \cdot h = (\phi + \jmath_Z \lambda^{-1} 0) \cdot I = \phi$.

Proposition 4.5. *Let $\Psi : \mathfrak{T}_{g,*} \to \mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{R}}$ by $(f, M, z) \mapsto \Psi_f$ be as defined in Definition 2.10. Then $\mathcal{M}_{g,*}/\mathcal{N}_{g,*}$ acts equivariantly on Ψ; for all $h \in \mathcal{M}_{g,*}$,*

$$\Psi_{f\circ h} = (\Psi_f) \cdot h.$$

Proof. This is a result of the action of $\mathcal{M}_{g,*}$ on Ψ_f given in Proposition 4.2, $\Psi_{f\circ h} = (\Psi_f + \jmath_{\Omega_f} \lambda^{-1}\hat{\delta}(h)) \cdot \rho_h$, and the definition of the affine action of h as given in Definition 4.4, $(\Psi_f) \cdot h = (\Psi_f + \jmath_{\Omega_f} \lambda^{-1}\hat{\delta}(h)) \cdot \rho_h$. ∎

We can now use the results of Harris, Hain, Pulte, and Koizumi to state that Ψ is generically injective on $\mathfrak{T}_{g,*}/\mathcal{N}_{g,*}$.

Definition 4.6. Let $\mathfrak{E}_g \subseteq \mathfrak{T}_{g,*}$ be the set of (f, M, z) such that $\mathrm{Jac}(M)$ has no complex multiplication for $g \geq 1$ and M has nonzero harmonic volume for $g \geq 3$.

Proposition 4.7. $\Psi : \mathfrak{E}_g/\mathcal{N}_{g,*} \to \mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{R}}$ *is injective.*

Remark 4.8. B. Harris in [7] has shown that the generic curve has nonzero harmonic volume and S. Koizumi [11] has shown that the generic curve has a Jacobian with no complex multiplication. Since $\mathfrak{E}_g \subseteq \mathfrak{T}_{g,*}$ is a generic subset, we may say that Ψ is generically injective.

Proof of Proposition 4.7. Suppose that we are given $(f_1, M_1, z_1), (f_2, M_2, z_2) \in \mathfrak{T}_{g,*}$ such that $\Omega_{f_1} = \Omega_{f_2}$ and $\Psi_{f_1} = \Psi_{f_2}$. Since $\Omega_{f_1} = \Omega_{f_2}$, we may conclude by the injectivity of the Torelli map that M_1 and M_2 are conformally equivalent, and hence, without loss of generality, that they are the same. Therefore, we have one Riemann surface M with points z_1, z_2 and markings f_1 and f_2. Recall that $\Psi_{f_i} = (w_{f_i})_*\tilde{\Psi}_{f_i}$ where $\tilde{\Psi}_{f_i} \in \mathrm{Hom}(K_2(M), H^1(M))_{\mathbb{R}}$ so that we have $\tilde{\Psi}_{f_1} = (w_{f_1})_*^{-1}\Psi_{f_1} = (w_{f_1})_*^{-1}\Psi_{f_2} = (w_{f_1})_*^{-1}(w_{f_2})_*\tilde{\Psi}_{f_2} = (\jmath_{\Omega_{f_1}}(f_1)_*^{-1})^{-1}(\jmath_{\Omega_{f_1}}(f_2)_*^{-1})\tilde{\Psi}_{f_2} = (f_1)_*(f_2)_*^{-1}\tilde{\Psi}_{f_2}$. Now define $F = f_1 \circ f_2^{-1} : (M, z_2) \to (M, z_1)$ and notice that F is an orientation preserving homeomorphism from M to M. We have a map $F_* : H_1(M, \mathbb{Z}) \to H_1(M, \mathbb{Z})$ which is a symplectic automorphism of $H_1(M, \mathbb{Z})$. Let $\sigma \in \mathrm{Sp}_g(\mathbb{Z})$ be the matrix representing F_* with respect to the basis

$[(f_2)_*(\gamma)]$ in $H_1(M,\mathbb{Z})$. Recall from the discussion in section §1 that $\Omega_{F\circ f_2} = {}^t\sigma\cdot\Omega_{f_2}$, and since $\Omega_{f_1} = \Omega_{F\circ f_2} = {}^t\sigma\cdot\Omega_{f_2} = {}^t\sigma\cdot\Omega_{f_1}$, we see that Ω_{f_1} is a fixed point of ${}^t\sigma$. If $\sigma \notin \{I,-I\}$, then ${}^t\sigma\cdot\Omega_{f_1} = \Omega_{f_1}$ is a special condition on Ω_{f_1} which says that $A_{\Omega_{f_1}}$ has a nontrivial complex multiplication. The assumption that $(M,f_i,z_i)\in\mathfrak{E}_g$ rules out that $\mathrm{Jac}(M)\cong A_{\Omega_{f_1}}$ has nontrivial complex multiplication and implies that ${}^t\sigma = \pm I$; so $F_* = \pm\mathrm{id}$ on $H_1(M,\mathbb{Z})$. Returning to the earlier equation, we see that $\tilde\Psi_{f_1} = (f_1\circ f_2^{-1})_*\tilde\Psi_{f_2} = F_*\tilde\Psi_{f_2} = \pm\tilde\Psi_{f_2}$. In summary, from an image $\Psi_f\in\mathrm{Hom}(K_2(\Omega_f),H^1(\Omega_f))_\mathbb{R}$ in $\mathcal{H}om(\mathcal{K}_2,\mathcal{H}^1)_\mathbb{R}$ and the assumption that $A_{\Omega_f}\cong\mathrm{Jac}(M)$ has no nontrivial complex multiplication, we can recover $\{\tilde\Psi_f,-\tilde\Psi_f\}$ in $\mathrm{Hom}(K_2(M),H^1(M))_\mathbb{R}$. The image $[\tilde\Psi_f]\in\mathrm{Ext}(K_2(M),H^1(M))$ is independent of f and determines the basepoint z of (f,M,z) for a fixed M (Pulte [13, Corollary 5.4]). Furthermore, the proof of the pointed Torelli theorem as proven by Pulte [13, Theorem 5.5] implies that $\{[\tilde\Psi_f],-[\tilde\Psi_f]\}$ determines the basepoint z except perhaps if M is a nonhyperelliptic Riemann surface with zero harmonic volume. Riemann surfaces of genus $1\le g\le 2$ are hyperelliptic; if we assume that M has nonzero harmonic volume for $g\ge 3$, we can determine both the M and z of (f,M,z) from Ω_f and Ψ_f. So we have $z_1 = z_2$. Then we have $f_2 = f_1\circ h$ for $h\in\mathcal{M}_{g,*}$.

We next show that for $(f,M,z)\in\mathfrak{E}_g$ if $\Omega_{f\circ h} = \Omega_f$ and $\Psi_{f\circ h} = \Psi_f$ for $h\in\mathcal{M}_{g,*}$, then $h\in\mathrm{Ker}\,\delta = \mathcal{N}_{g,*}$. The assumption of no complex multiplication on A_{Ω_f} makes $\Omega_{f\circ h} = {}^t\rho_h\cdot\Omega_f = \Omega_f$ imply ${}^t\rho_h = \pm I$. If ${}^t\rho_h = I$, then $h\in\mathrm{I}_{g,*}$. In this case, Corollary 4.3 says $\Psi_{f\circ h} = \Psi_f + \jmath_{\Omega_f}\lambda^{-1}\delta(h) = \Psi_f$, from which we conclude that $\delta(h) = 0$, which is $h\in\mathcal{N}_{g,*}$. On the other hand, if ${}^t\rho_h = -I$, then h is a homology involution and $\Psi_{f\circ h} = (\Psi_f + \jmath_{\Omega_f}\lambda^{-1}\hat\delta(h))\cdot\rho_h = -\Psi_f - \jmath_{\Omega_f}\lambda^{-1}\hat\delta(h) = \Psi_f$ implies that $2\Psi_f = -\jmath_{\Omega_f}\lambda^{-1}\hat\delta(h)\in\mathrm{Hom}(K_2(\Omega_f),H^1(\Omega_f))_\mathbb{Z}$. By the earlier discussion, this implies that $2\tilde\Psi_f = -f_*\lambda^{-1}\hat\delta(h)\in\mathrm{Hom}(K_2(M),H^1(M))_\mathbb{Z}$, and hence that $2\tilde\Psi_f = 0$ in $\mathrm{Ext}(K_2(M),H^1(M))$. An equivalent definition of the harmonic volume of M due to Hain [6, Definition 8.4] shows that $2\tilde\Psi_f = 0$ in Ext implies that the harmonic volume of M is zero. Assuming that M has nonzero harmonic volume consequently rules out the possibility that both $\rho_h = -I$ and $\Psi_{f\circ h} = \Psi_f$. We have shown that for $(M,f_i,z_i)\in\mathfrak{E}_g$ such that $\Psi_{f_1} = \Psi_{f_2}$ and $\Omega_{f_1} = \Omega_{f_2}$, we have $z_1 = z_2$ and $f_2 = f_1\circ h$ with $h\in\mathcal{N}_{g,*}$. The conditions which define $\mathfrak{E}_g\subseteq\mathfrak{T}_{g,*}$ are invariant under $\mathcal{M}_{g,*}$ so that $\mathfrak{E}_g/\mathcal{N}_{g,*}$ is well-defined. ∎

§5. Higher bilinear period relations.

In this section we derive the "higher bilinear period relations" of Proposition 5.12. These relations follow from the existence of the nontrivial commutator relation in $\pi_1(M,z)$ and of nontrivial homotopy functionals of length 3 in $H^0(\overline{B}_3(M),z)$. We first construct a Hodge filtration preserving section s_3 into $H^0(\overline{B}_3(M),z)$ in Proposition 5.8. The "higher bilinear relations" of Proposition 5.12 behave as a 3-cycle symmetry; along with the well-known 2-cycle skew-symmetry of Lemma 5.1, we show that the period map Ψ essentially lives on the third exterior power of H_1. The main theorem, Theorem 5.24, states that the map $\Psi:\mathfrak{T}_{g,*}\to\mathcal{H}om(\mathcal{K}_2,\mathcal{H}^1)_\mathbb{R}$ factors through a translation of the subbundle $\bigwedge^3\mathcal{H}_1$. Over Torelli space $\mathfrak{T}_{g,*}/\mathrm{I}_{g,*}$, Ψ factors through a translation of the torus bundle $(\bigwedge^3\mathcal{H}_1)_\mathbb{R}/(\bigwedge^3\mathcal{H}_1)_\mathbb{Z}$ and we discuss how this follows from the results of Pulte's paper [13].

Let M be a compact Riemann surface. We let $\mathrm{Sym}(H^1(M)\otimes H^1(M))$ denote the symmetric tensors inside of $H^1(M)\otimes H^1(M)$ and let $K_2^{\mathrm{sym}}(M) = K_2(M)\cap\mathrm{Sym}(H^1(M)\otimes H^1(M))$. Actually, we have $K_2^{\mathrm{sym}}(M) = \mathrm{Sym}(H^1(M)\otimes H^1(M))$ because for any $z_1\otimes z_2 +$

$z_2 \otimes z_1 \in \mathrm{Sym}(H^1 \otimes H^1)$, we see that $z_1 \wedge z_2 + z_2 \wedge z_1 = 0$ and so $z_1 \otimes z_2 + z_2 \otimes z_1 \in K_2(M)$. The following simple lemma gives an essential symmetry of length two iterated integrals.

Lemma 5.1. *Let* $(f, M, z) \in \mathfrak{T}_{g,*}$. *We have*

$$\lambda(\tilde{\Psi}_f \Big|_{K_2^{\mathrm{sym}}}) = \tfrac{1}{2} \sum_{j=1}^{g} ([a_j] \otimes [a_j] \otimes [a_j]^* + [b_j] \otimes [b_j] \otimes [b_j]^*).$$

Proof. Recall that $\tilde{\Psi}_f = r_{\mathbb{Z}} \circ s_2$. For $z_1 \otimes z_2 + z_2 \otimes z_1 \in K_2^{\mathrm{sym}}(M)$, we must have $u(z_1 \otimes z_2 + z_2 \otimes z_1) = 0$ because $z_1 \wedge z_2 + z_2 \wedge z_1 + d(0) = 0$, 0 is exact, and u is unique. The lemma then follows from the familiar identity $\int_\gamma (z_1 z_2 + z_2 z_1) = \int_\gamma z_1 \int_\gamma z_2$:

$$\begin{aligned}
\tilde{\Psi}_f(z_1 \otimes z_2 + z_2 \otimes z_1) &= r_{\mathbb{Z}} \circ s_2(z_1 \otimes z_2 + z_2 \otimes z_1) \\
&= r_{\mathbb{Z}}(\int (z_1 z_2 + z_2 z_1)) = r_{\mathbb{Z}}(\int z_1 \int z_2) \\
&= \sum_{j=1}^{g} (\int_{a_j} z_1 \int_{a_j} z_2 \; [a_j]^* + \int_{b_j} z_1 \int_{b_j} z_2 \; [b_j]^*) \\
&= (\lambda^{-1}(\sum_{j=1}^{g} ([a_j] \otimes [a_j] \otimes [a_j]^* + [b_j] \otimes [b_j] \otimes [b_j]^*)))(z_1 \otimes z_2) \\
&= (\lambda^{-1}(\tfrac{1}{2} \sum_{j=1}^{g} ([a_j] \otimes [a_j] \otimes [a_j]^* + [b_j] \otimes [b_j] \otimes [b_j]^*)))(z_1 \otimes z_2 + z_2 \otimes z_1). \blacksquare
\end{aligned}$$

Corollary 5.2. *Let* $(f, M, z) \in \mathfrak{T}_{g,*}$. *We have*

$$\lambda \; \Psi_f |_{K_2^{\mathrm{sym}}(\Omega_f)} = \tfrac{1}{2} \sum_{j=1}^{g} (A_j \otimes A_j \otimes A_j^* + B_j \otimes B_j \otimes B_j^*).$$

Proof. This follows from the preceding lemma, the fact that $(w_f)_*[a_j] = A_j$, $(w_f)_*[b_j] = B_j$, and the following commutative diagram

$$\begin{array}{ccc}
K_2^{\mathrm{sym}}(M) & \xrightarrow{\tilde{\Psi}_f} & H^1(M) \\
\Big\downarrow (w_f)_* & & \Big\downarrow (w_f)_* \\
K_2^{\mathrm{sym}}(\Omega_f) & \xrightarrow{\Psi_f} & H^1(\Omega_f). \blacksquare
\end{array}$$

Remark 5.3. In terms of periods, the lemma 5.1 represents well-known symmetries like

$$\sigma_{ij}(c) + \sigma_{ji}(c) = \omega_i(c)\omega_j(c)$$
$$\tau_{ij}(c) + \overline{\tau}_{ji}(c) = \overline{\omega}_i(c)\omega_j(c) - \Lambda_{ij}\overline{\omega}_1(c)\omega_1(c)$$

for $c \in \pi_1(M, z)$ (see [4]).

We can reformulate the lemma using the map θ that switches tensor components, as defined in Definition 1.11.

Corollary 5.4. *Let $(f, M, z) \in \mathfrak{T}_{g,*}$. In $((H_1(\Omega_f) \otimes H_1(\Omega_f))/q_{\Omega_f}) \otimes H_1(\Omega_f)$ we have*

$$(\mathrm{id} + \theta)\imath\lambda\Psi_f = \sum_{j=1}^{g} (A_j \otimes A_j \otimes B_j - B_j \otimes B_j \otimes A_j).$$

Proof. We have $(\mathrm{id} + \theta)\imath\lambda\Psi_f = 2\,\imath\lambda\Psi_f|_{K_2^{\mathrm{sym}}}$ because $\mathrm{id} + \theta$ is twice the projection onto $\mathrm{Sym}(H_1 \otimes H_1) \otimes H_1$ as in Lemma 1.12. From Corollary 5.2 we have the desired result:

$$\begin{aligned} 2\,\imath\lambda\Psi_f|_{K_2^{\mathrm{sym}}} &= 2\,\imath\tfrac{1}{2} \sum_{j=1}^{g} (A_j \otimes A_j \otimes A_j^* + B_j \otimes B_j \otimes B_j^*) \\ &= \sum_{j=1}^{g} (A_j \otimes A_j \otimes B_j - B_j \otimes B_j \otimes A_j). \ \blacksquare \end{aligned}$$

We proceed to construct a section $s_3 : K_3(M) \to H^0(\overline{B}_3(M), z)$ which preserves the Hodge filtration. Recall that we have the exact sequences of mixed Hodge structures [6]:

$$0 \to H^0(\overline{B}_2(M), z) \to H^0(\overline{B}_3(M), z) \xrightarrow{p_3} K_3(M) \to 0,$$
$$0 \to K_3(M) \to H^1(M) \otimes H^1(M) \otimes H^1(M) \xrightarrow{\delta_3} (H^2 \otimes H^1) \oplus (H^1 \otimes H^2) \to 0,$$

where the second exact sequence can be taken as the definition of $K_3(M)$. The map p_3 for $I \in H^0(\overline{B}_3(M), z)$ is defined as

$$\langle\, p_3 I\, ,\, [\gamma_1] \otimes [\gamma_2] \otimes [\gamma_3] \,\rangle = \langle\, I\, ,\, (\gamma_1 - 1)(\gamma_2 - 1)(\gamma_3 - 1) \,\rangle$$

and is thus dual to the map

$$p_3^* : H_1 \otimes H_1 \otimes H_1 \to J^3/J^4.$$

We mention that the kernel of p_3^* is $\mathrm{Ker}\, p_3^* = q \otimes H_1 \oplus H_1 \otimes q$. The map δ_3 is defined by

$$\delta_3(\xi \otimes \eta \otimes \zeta) = \xi \wedge \eta \otimes \zeta - \xi \otimes \eta \wedge \zeta.$$

Lemma 5.5. $K_3(M) = (K_2(M) \otimes H^1(M)) \cap (H^1(M) \otimes K_2(M))$

Proof. We have $(K_2 \otimes H^1) \cap (H^1 \otimes K_2) \subseteq K_3$ because $\sum \xi_i \otimes \eta_i \otimes \zeta_i \in K_2 \otimes H^1$ implies $\sum \xi_i \wedge \eta_i \otimes \zeta_i = 0$ and $\sum \xi_i \otimes \eta_i \otimes \zeta_i \in H^1 \otimes K_2$ implies $\sum \xi_i \otimes \eta_i \wedge \zeta_i = 0$. Therefore $\sum \xi_i \otimes \eta_i \otimes \zeta_i \in \mathrm{Ker}\,(\delta_3) = K_3(M)$. On the other hand, if we choose ξ_i, η_i, ζ_i from a basis for H^1, then $\sum \xi_i \wedge \eta_i \otimes \zeta_i = 0$ implies that for each j, $\sum_{i:\zeta_j=\zeta_i} \xi_i \wedge \eta_i = 0$, which means that $\sum_{i:\zeta_j=\zeta_i} \xi_i \otimes \eta_i \in K_2$. Hence, $K_3 \subseteq K_2 \otimes H^1$. By the symmetry of the argument, we also have $K_3 \subseteq H^1 \otimes K_2$. This completes the proof. $\blacksquare$

Lemma 5.6. *For all $k \in K_2(M), h \in H^1(M)$, we have $u(k) \otimes h,\ h \otimes u(k) \in \hat{K}_2(M)$.*

Proof. We must show that $u(k) \wedge h$ is exact. Since $u(k) \in A^{1,0}(M)$, we have $u(k) \wedge h = 0$ for the case when $h \in H^{1,0}$. For the case when $h \in H^{0,1}$, we use the fact that $u(k) = \overline{u(\overline{k})} + df$ for some exact 1-form df on M. We have $u(k) \wedge h = (\overline{u(\overline{k})} + df) \wedge h = df \wedge h = d(fh)$, so that $u(k) \wedge h$ is indeed exact. The general case follows from $H^1 = H^{1,0} \oplus H^{0,1}$. ∎

This lemma and the linearity of u permit the following definition.

Definition 5.7. Let $\kappa = \sum k_i \otimes h_i = \sum h_i' \otimes k_i' \in K_3(M)$ where $k_i, k_i' \in K_2(M)$ and $h_i, h_i' \in H^1(M)$. The map s_3 is defined via:

$$\begin{aligned} &s_3 : K_3(M) \to H^0(\overline{B}_3(M), z) \\ &\kappa \mapsto \int \kappa + \sum_i \int (u(k_i)h_i + h_i' u(k_i')) + \sum_i \int u(u(k_i) \otimes h_i + h_i' \otimes u(k_i')). \end{aligned}$$

Proposition 5.8. *The map s_3 does indeed map into $H^0(\overline{B}_3(M), z)$ and is a Hodge filtration preserving section of $p_3 : H^0(\overline{B}_3(M), z) \to K_3(M)$.*

Proof. To show that an interated integral is a homotopy functional we make use of Chen's theory of differential forms on the path space of M, (see [3, p.839] or [6, p.262]). According to this theory $s_3(\kappa)$ is a homotopy functional on loops if it is closed with respect to Chen's functional derivative d_C given by:

$$\begin{aligned} d_C \int \omega_1\omega_2\omega_3 &= \int d\omega_1\omega_2\omega_3 + \int \omega_1 d\omega_2\omega_3 + \int \omega_1\omega_2 d\omega_3 + \int (\omega_1 \wedge \omega_2)\omega_3 + \int \omega_1(\omega_2 \wedge \omega_3) \\ d_C \int \omega_1\omega_2 &= \int d\omega_1\omega_2 + \int \omega_1 d\omega_2 + \int \omega_1 \wedge \omega_2 \\ d_C \int \omega_1 &= \int d\omega_1. \end{aligned}$$

In the computation that $d_C s_3(\kappa) = 0$ we must again use the representations $\kappa = \sum k_i \otimes h_i = \sum h_i' \otimes k_i' \in K_3(M)$ because $K_3(M)$ is not spanned by its decomposable elements. We have

$$\begin{aligned} d_C \int \kappa &= \sum_i \int (\wedge k_i)h_i + \sum_i \int h_i'(\wedge k_i') \\ d_C \int (u(k_i)h_i + h_i' u(k_i')) &= \int u(k_i) \wedge h_i + \int h_i' \wedge u(k_i') - \int (\wedge k_i)h_i - \int h_i'(\wedge k_i') \\ d_C \int u(u(k_i) \otimes h_i + h_i' \otimes u(k_i')) &= -\int (u(k_i) \wedge h_i + h_i' \wedge u(k_i')). \end{aligned}$$

Summing these iterated integrals of 1-forms and 2-forms and using the definition 5.7 of $s_3(\kappa)$ we obtain $d_C s_3(\kappa) = 0$. This shows that $s_3(\kappa) \in H^0(\overline{B}_3(M), z)$.

To show that s_3 is a section of p_3 we compute $\langle\, p_3 s_3(k)\,,\ [c_1]\otimes[c_2]\otimes[c_3]\,\rangle = \langle\, s_3(\kappa)\,,\ (c_1 - 1)(c_2 - 1)(c_3 - 1)\,\rangle = \langle \int \kappa\,,\ (c_1 - 1)(c_2 - 1)(c_3 - 1)\,\rangle = \kappa([c_1] \otimes [c_2] \otimes [c_3])$. This follows from the familiar formulae $\langle \int \omega_1\omega_2\omega_3\,,\ \prod_{j=1}^3 (c_j - 1)\,\rangle = \prod_{j=1}^3 \int_{c_j} \omega_j$; $\langle \int \omega_1\omega_2\,,\ J^3\,\rangle = 0$;

and $\langle \int \omega_1 \, , \, J^3 \rangle = 0$ [6, Proposition 2.13]. That s_3 preserves the $F^\bullet$ filtration is easy and is left to the reader. ▮

Now that the section s_3 has been constructed, we relate it as far as possible to the sections $s_2 : K_2 \to H^0(\overline{B}_2(M), z)$ and $s_1 : H^1 \xrightarrow{\sim} H^0(\overline{B}_1(M), z)$. Here, s_1 is defined by $\langle s_1(\omega) \, , \, c \rangle = \int_c \omega$.

Definition 5.9. Let

$$\begin{array}{rccc} \ell : & (J/J^3) \otimes (J/J^3) & \to & J^2/J^4 \subseteq J/J^4 \\ & c_1 \otimes c_2 & \mapsto & c_1 c_2 \end{array}$$

be defined by multiplication in $\mathbb{C}\pi_1(M, z)$.

Remark 5.10. Using Chen's Theorem that $(J/J^{s+1})^* \cong H^0(\overline{B}_s(M), z)$ [6], it follows that dual map

$$\ell^* : H^0(\overline{B}_3(M), z) \to H^0(\overline{B}_2(M), z) \otimes H^0(\overline{B}_2(M), z)$$

is given by $\langle \ell^* I \, , \, c_1 \otimes c_2 \rangle = \langle I \, , \, c_1 c_2 \rangle$ for $I \in H^0(\overline{B}_3(M), z)$.

Proposition 5.11. *Let $(f, M, z) \in \mathfrak{T}_{g,*}$ be a compact Riemann surface with basepoint z and marking f. The following commutative diagram holds.*

$$\begin{array}{ccc} K_3(M) & \xrightarrow{\qquad s_3 \qquad} & H^0(\overline{B}_3(M), z) \\ \| & & \downarrow \ell^* \\ (K_2 \otimes H^1) \cap (H^1 \otimes K_2) & \xrightarrow{s_2 \otimes s_1 + s_1 \otimes s_2} & H^0(\overline{B}_2(M), z) \otimes H^0(\overline{B}_2(M), z) \\ \| & & \downarrow r_{\mathbb{Z}} \otimes r_{\mathbb{Z}} \\ (K_2 \otimes H^1) \cap (H^1 \otimes K_2) & \xrightarrow{\tilde{\Psi}_f \otimes \mathrm{id} + \mathrm{id} \otimes \tilde{\Psi}_f} & H^1(M) \otimes H^1(M). \end{array}$$

Proof. Recall from definition 5.7 that $s_3(\kappa) = \int \kappa + \sum_i \int (u(k_i)h_i + h_i' u(k_i')) + \sum_i \int u(u(k_i) \otimes h_i + h_i' \otimes u(k_i'))$ for $\kappa \in K_3(M)$. The element $\ell^* s_3(\kappa)$ is given by $\langle \ell^* s_3(\kappa) \, , \, (c_1 - 1) \otimes (c_2 - 1) \rangle = \langle s_3(\kappa) \, , \, (c_1 - 1)(c_2 - 1) \rangle$ for all $c_1, c_2 \in \pi_1(M, z)$. We have

$$\int_{(c_1-1)(c_2-1)} \kappa = \sum_i (\int_{c_1} k_i \int_{c_2} h_i + \int_{c_1} h_i' \int_{c_2} k_i')$$

$$\int_{(c_1-1)(c_2-1)} u(k_i)h_i + h_i' u(k_i') = \int_{c_1} u(k_i) \int_{c_2} h_i + \int_{c_1} h_i' \int_{c_2} u(k_i')$$

$$\int_{(c_1-1)(c_2-1)} u(u(k_i) \otimes h_i + h_i' \otimes u(k_i')) = 0.$$

Hence we have that

$$\begin{aligned} &\langle s_3(\kappa) \, , \, (c_1 - 1)(c_2 - 1) \rangle \\ &\qquad = \sum_i \left(\int_{c_1} k_i \int_{c_2} h_i + \int_{c_1} h_i' \int_{c_2} k_i' + \int_{c_1} u(k_i) \int_{c_2} h_i + \int_{c_1} h_i' \int_{c_2} u(k_i') \right) \\ &\qquad = \sum_i \left((\int_{c_1} k_i + u(k_i)) \int_{c_2} h_i + \int_{c_1} h_i' (\int_{c_2} k_i' + u(k_i')) \right). \end{aligned}$$

This is exactly $\langle\, (s_2{\otimes}s_1{+}s_1{\otimes}s_2)(\kappa)\,,\,(c_1{-}1){\otimes}(c_2{-}1)\,\rangle$ and that proves the commutativity of the diagram's upper rectangle. We have not yet made use of the marking f whose influence is only on the retraction $r_{\mathbb{Z}} : H^0(\overline{B}_2(M), z) \to H^1(M)$ that we used to define $\tilde\Psi_f = r_{\mathbb{Z}}{\circ}s_2$. The formula $(r_{\mathbb{Z}} \otimes r_{\mathbb{Z}})(s_2 \otimes s_1 + s_1 \otimes s_2) = \tilde\Psi_f \otimes \mathrm{id}_{H^1} + \mathrm{id}_{H^1} \otimes \tilde\Psi_f$ completes the diagram. ∎

We will use the above proposition to find symmetries that Ψ must satisfy. Since the identity $e = \prod_{j=1}^{g}(a_j, b_j) \in \pi_1(M, z)$ is annihilated by $H^0(\overline{B}_3(M), z)$, and since the aspect of the pairing with J^3 which is not determined by K_3 factors through ℓ^*, we may in essence apply e to $\Psi \otimes \mathrm{id} + \mathrm{id} \otimes \Psi$. More precisely, we have the following proposition which contains the "higher bilinear period relations" mentioned in the introduction. Here, $(a, b) = aba^{-1}b^{-1}$ denotes a group commutator, whereas $[a, b] = ab - ba$ denotes the ring commutator.

Proposition 5.12 (Higher Bilinear Relations). *Let $(f, M, z) \in \mathfrak{T}_{g,*}$. For all $\kappa \in K_3(M)$, we have*

$$\langle\, (\tilde\Psi_f \otimes \mathrm{id} + \mathrm{id} \otimes \tilde\Psi_f)(\kappa)\,,\, q_M\,\rangle = \langle\, \kappa\,,\, \sum_{j=1}^{g}([a_j] \otimes [b_j] - [b_j] \otimes [a_j]) \otimes ([a_j] + [b_j])\,\rangle.$$

Proof. We recall the following algebra from J/J^4. For $a, b \in \pi_1$, we have $(a, b) - 1 = [a-1, b-1] - [a-1, b-1](a-1+b-1) \mod J^4$. So we also have

$$\prod_{j=1}^{g}(a_j, b_j) - 1 = \sum_{j=1}^{g}([a_j - 1, b_j - 1] - [a_j - 1, b_j - 1](a_j - 1 + b_j - 1)) \mod J^4.$$

Since any homotopy functional on M annihilates $\prod_{j=1}^{g}(a_j, b_j)$, and any iterated integral of length 3 annihilates J^4, we have the following equation for any $\kappa \in K_3$:

$$\begin{aligned}
0 &= \langle\, s_3(\kappa)\,,\, \prod_{j=1}^{g}(a_j, b_j)\,\rangle = \langle\, s_3(\kappa)\,,\, \prod_{j=1}^{g}(a_j, b_j) - 1\,\rangle \\
&= \langle\, s_3(\kappa)\,,\, \sum_{j=1}^{g}([a_j - 1, b_j - 1] - [a_j - 1, b_j - 1](a_j - 1 + b_j - 1)\,\rangle \\
&= \sum_{j=1}^{g}(\langle\, s_3(\kappa)\,,\,(a_j - 1)(b_j - 1) - (b_j - 1)(a_j - 1)\,\rangle \\
&\qquad -\langle\, p_3 s_3(\kappa)\,,\,(p_3^{-1})^*((a_j - 1)(b_j - 1)(a_j - 1) + (a_j - 1)(b_j - 1)(b_j - 1) \\
&\qquad\qquad -(b_j - 1)(a_j - 1)(a_j - 1) - (b_j - 1)(a_j - 1)(b_j - 1))\,\rangle) \\
&= \sum_{j=1}^{g}(\langle\, \ell^* s_3(\kappa)\,,\,(a_j - 1) \otimes (b_j - 1) - (b_j - 1) \otimes (a_j - 1)\,\rangle \\
&\qquad -\langle\, \kappa\,,\,[a_j] \otimes [b_j] \otimes [a_j] + [a_j] \otimes [b_j] \otimes [b_j] - [b_j] \otimes [a_j] \otimes [a_j] - [b_j] \otimes [a_j] \otimes [b_j]\,\rangle)\,.
\end{aligned}$$

Now we make use of the retraction $r_{\mathbb{Z}} : H^0(\overline{B}_2(M), z) \to H^1(M)$, its dual $r_{\mathbb{Z}}^* : H_1(M) \to J/J^3$, and the double tensor of its dual $r_{\mathbb{Z}}^* \otimes r_{\mathbb{Z}}^* : H_1 \otimes H_1 \to J/J^3 \otimes J/J^3$. We have $(r_{\mathbb{Z}}^* \otimes r_{\mathbb{Z}}^*)(q_M) = (r_{\mathbb{Z}}^* \otimes r_{\mathbb{Z}}^*)(\sum_{j=1}^g [a_j] \wedge [b_j]) = \sum_{j=1}^g ((a_j - 1) \otimes (b_j - 1) - (b_j - 1) \otimes (a_j - 1))$. Then we have

$$\sum_{j=1}^{g} \langle\, \kappa\, ,\, [a_j] \otimes [b_j] \otimes [a_j] + [a_j] \otimes [b_j] \otimes [b_j] - [b_j] \otimes [a_j] \otimes [a_j] - [b_j] \otimes [a_j] \otimes [b_j] \,\rangle$$
$$= \langle\, \ell^* s_3(\kappa)\, ,\, (r_{\mathbb{Z}}^* \otimes r_{\mathbb{Z}}^*)(q_M) \,\rangle = \langle\, (r_{\mathbb{Z}} \otimes r_{\mathbb{Z}})\ell^* s_3(\kappa)\, ,\, q_M \,\rangle = \langle\, \tilde{\Psi}_f \otimes \mathrm{id} + \mathrm{id} \otimes \tilde{\Psi}_f\, ,\, q_M \,\rangle,$$

where the last step makes use of Proposition 5.11. ∎

Corollary 5.13. *Let $(f, M, z) \in \mathfrak{T}_{g,*}$. For all $\kappa \in K_3(\Omega_f)$, we have*

$$\langle\, (\Psi_f \otimes \mathrm{id} + \mathrm{id} \otimes \Psi_f)(\kappa)\, ,\, q_{\Omega_f} \,\rangle = \langle\, \kappa\, ,\, \sum_{j=1}^{g} (A_j \otimes B_j - B_j \otimes A_j) \otimes (A_j + B_j) \,\rangle.$$

Proof. This will follow from Theorem 5.12 in the same manner that Corollary 5.2 followed from Lemma 5.1. We use $(w_f)_* q_M = q_{\Omega_f}$ and the commutativity of

$$\begin{array}{ccc} K_3(M) & \longrightarrow & H^1(M) \otimes H^1(M) \otimes H^1(M) \\ \big\downarrow{\scriptstyle (w_f)_*} & & \big\downarrow{\scriptstyle (w_f)_*} \\ K_3(\Omega_f) & \longrightarrow & H^1(\Omega_f) \otimes H^1(\Omega_f) \otimes H^1(\Omega_f). \end{array}$$ ∎

Remark 5.14. As with Lemma 5.1, we may also view this Proposition 5.12 as a set of genuine period relations on the pure and mixed quadratic periods. For example, if we let $\kappa = \omega_i \otimes \omega_j \otimes \omega_k$, then

$$(\tilde{\Psi}_f \otimes \mathrm{id} + \mathrm{id} \otimes \tilde{\Psi}_f)(\kappa) = \sum_{\ell=1}^{g} (\sigma_{ij}(a_\ell)[a_\ell]^* \otimes \omega_k + \sigma_{ij}(b_\ell)[b_\ell]^* \otimes \omega_k$$
$$+ \sigma_{jk}(a_\ell)\omega_i \otimes [a_\ell]^* + \sigma_{jk}(b_\ell)\omega_i \otimes [b_\ell]^*).$$

If we apply this to $q_M = \sum_{m=1}^g [a_m] \wedge [b_m]$, Proposition 5.12 gives us:

$$\sum_{\ell=1}^{g} (\sigma_{ij}(a_\ell)\omega_k([b_\ell]) - \sigma_{ij}(b_\ell)\omega_k([a_\ell]) - \sigma_{jk}(a_\ell)\omega_i([b_\ell]) + \sigma_{jk}(b_\ell)\omega_i([a_\ell]))$$
$$= \langle\, \omega_i \otimes \omega_j \otimes \omega_k\, ,\, \sum_{\ell=1}^{g} ([a_\ell] \otimes [b_\ell] - [b_\ell] \otimes [a_\ell]) \otimes ([a_\ell] + [b_\ell]) \,\rangle.$$

Taking the normalization of the abelian periods $\omega_i([a_j]) = \delta_{ij}$ and $\omega_i([b_j]) = \Omega_{ij}$ into account, this becomes: for all i, j, k,

$$\sigma_{ij}(b_k) - \sum_{\ell=1}^{g} \sigma_{ij}(a_\ell)\Omega_{\ell k} = \sigma_{jk}(b_i) - \sum_{\ell=1}^{g} \sigma_{jk}(a_\ell)\Omega_{\ell i} + \Omega_{ij}\Omega_{jk} - \Omega_{ij}\Omega_{ik} + \delta_{jk}\Omega_{ij} - \delta_{ik}\Omega_{ij}. \tag{5.15}$$

The same process applied to $\kappa = \overline{\omega}_i \otimes \omega_j \otimes \omega_k - \Lambda_{ij}\overline{\omega}_1 \otimes \omega_1 \otimes \omega_k \in K_3(M)$ will produce the following period relation: for any i, j, k,

$$\begin{aligned}\tau_{ij}(b_k) - \sum_{\ell=1}^{g} \tau_{ij}(a_\ell)\Omega_{\ell k} = \sigma_{jk}(b_i) &- \sum_{\ell=1}^{g} \sigma_{jk}(a_\ell)\overline{\Omega}_{\ell i} \\ &+ \delta_{jk}\overline{\Omega}_{ij} - \delta_{ik}\Omega_{ij} + \overline{\Omega}_{ij}\Omega_{jk} - \Omega_{ij}\Omega_{ik} \\ &- \Lambda_{ij}[\sigma_{1k}(b_1) - \sum_{\ell=1}^{g} \sigma_{1k}(a_\ell)\overline{\Omega}_{\ell 1} \\ &\qquad + \delta_{1k}\overline{\Omega}_{11} - \delta_{1k}\Omega_{11} + \overline{\Omega}_{11}\Omega_{1k} - \Omega_{11}\Omega_{1k}].\end{aligned} \tag{5.16}$$

We reformulate the "higher bilinear relations" of Proposition 5.12 by introducing a map which, like θ, permutes tensor components. Also, recall the map $\imath$ of 1.11.

Definition 5.17. Let (H^1, q) be a principally polarized Hodge structure of weight 1, and let H_1 be the dual of H^1. Define

$$z : ((H_1 \otimes H_1)/q) \otimes H_1 \to H_1 \otimes ((H_1 \otimes H_1)/q)$$

by sending $[a \otimes b] \otimes c \mapsto c \otimes [a \otimes b]$.

Remark 5.18. Since both the domain and range of z have projections onto $(H_1 \otimes H_1 \otimes H_1)/(q \otimes H_1 + H_1 \otimes q)$, we have the induced map

$$\mathrm{id} - z : ((H_1 \otimes H_1)/q) \otimes H_1 \to (H_1 \otimes H_1 \otimes H_1)/(q \otimes H_1 + H_1 \otimes q)$$

given by $x \mapsto x - z(x)$.

Corollary 5.19. *Let $(f, M, z) \in \mathfrak{T}_{g,*}$. On $K_3(\Omega_f)$, we have*

$$\imath\lambda\Psi_f - z(\imath\lambda\Psi_f) = \sum_{j=1}^{g} (A_j \otimes B_j - B_j \otimes A_j) \otimes (A_j + B_j).$$

Proof. We are simply shuffling the tensor components in Corollary 5.13. For $k \otimes h \in K_2 \otimes H^1$, we have

$$\begin{aligned}\langle\, \imath\lambda\Psi_f\, ,\, k \otimes h\,\rangle &= \langle\, \lambda\Psi_f\, ,\, \imath^*(k \otimes h)\,\rangle = \langle\, \lambda\Psi_f\, ,\, k \otimes (\mathrm{id} \otimes h)(q)\,\rangle \\ &= \langle\, \Psi_f(k)\, ,\, (\mathrm{id} \otimes h)(q)\,\rangle = \langle\, \Psi_f(k) \otimes h\, ,\, q\,\rangle = \langle\, (\Psi_f \otimes \mathrm{id})(k \otimes h)\, ,\, q\,\rangle.\end{aligned}$$

Similarly, for $h \otimes k \in H^1 \otimes K_2$, we have

$$\begin{aligned}\langle\, z\imath\lambda\Psi_f\, ,\, h \otimes k\,\rangle &= \langle\, \imath\lambda\Psi_f\, ,\, z^*(h \otimes k)\,\rangle = \langle\, \imath\lambda\Psi_f\, ,\, k \otimes h\,\rangle \\ &= \langle\, (\Psi_f \otimes \mathrm{id})(k \otimes h)\, ,\, q\,\rangle = -\langle\, (\mathrm{id} \otimes \Psi_f)(h \otimes k)\, ,\, q\,\rangle.\end{aligned}$$

Therefore, for all $\kappa \in K_3 = (K_2 \otimes H^1) \cap (H^1 \otimes K_2)$, we have

$$\begin{aligned}\langle\, \imath\lambda\Psi_f - z\imath\lambda\Psi_f\, ,\, \kappa\,\rangle &= \langle\, (\Psi_f \otimes \mathrm{id} + \mathrm{id} \otimes \Psi_f)(\kappa)\, ,\, q\,\rangle \\ &= \langle\, \sum_{j=1}^{g} (A_j \otimes B_j - B_j \otimes A_j) \otimes (A_j + B_j)\, ,\, \kappa\,\rangle\end{aligned}$$

by Corollary 5.13. ∎

Up until now, we have been studying the element $\lambda\Psi_f \in ((H_1 \otimes H_1)/q) \otimes H^1$, but it will be more convenient now to describe $\imath\lambda\Psi_f \in ((H_1 \otimes H_1)/q) \otimes H_1$.

Corollary 5.20. *Let $(f, M, z) \in \mathfrak{T}_{g,*}$. Let $\phi = \imath\lambda\Psi_f \in ((H_1 \otimes H_1)/q) \otimes H_1$ on A_{Ω_f}. Then ϕ satisfies the following two inhomogeneous equations.*

$$\text{(a)} \quad (\mathrm{id} + \theta)\phi = \sum_{j=1}^{g} (A_j \otimes A_j \otimes B_j - B_j \otimes B_j \otimes A_j) \quad \textit{in } ((H_1 \otimes H_1)/q) \otimes H_1.$$

$$\text{(b)} \quad (\mathrm{id} - z)\phi = \sum_{j=1}^{g} (A_j \otimes B_j - B_j \otimes A_j) \otimes (A_j + B_j) \quad \textit{in } (H_1 \otimes H_1 \otimes H_1)/(H_1 \otimes q + q \otimes H_1).$$

Proof. This repeats the conclusions of Corollaries 5.4 and 5.19. ∎

We proceed to solve these inhomogeneous equations because the image $\imath\lambda\Psi$ of the period map Ψ must lie inside their solution set. We produce a particular solution of the inhomogeneous equation over every $Z \in \mathfrak{h}_g$, and then demonstrate that the homogeneous solutions correspond to $(\bigwedge^3 H_1)_{\mathbb{R}}$. The following remark can be verified by a direct computation, and since we do not make use of this lemma, we leave the proof to the reader.

Remark 5.21. Let $Z \in \mathfrak{h}_g$, then on A_Z,

$$-\tfrac{1}{2}\eta = \tfrac{1}{2} \sum_{j=1}^{g} (-B_j \otimes B_j \otimes A_j - B_j \wedge A_j \otimes A_j + A_j \otimes A_j \otimes B_j + A_j \wedge B_j \otimes B_j)$$

satisfies the inhomogeneous equations of 5.20.

Lemma 5.22. *Let (H^1, q) be a principally polarized Hodge structure of weight one, and let H_1 be the dual of H^1. Then $\phi \in ((H_1 \otimes H_1)/q) \otimes H_1$ satisfies the homogeneous*

$$(\mathrm{id} + \theta)\phi = 0 \ \textit{in } ((H_1 \otimes H_1)/q) \otimes H_1 \ \textit{and}$$
$$(\mathrm{id} - z)\phi = 0 \ \textit{in } (H_1 \otimes H_1 \otimes H_1)/(q \otimes H_1 + H_1 \otimes q)$$

if and only if $\phi \in \bigwedge^3 H_1 \hookrightarrow ((H_1 \otimes H_1)/q) \otimes H_1$.

Proof. If $\phi \in \bigwedge^3 H_1$, then as an element of $H_1 \otimes H_1 \otimes H_1$, ϕ will change by the sign of the permutation when we apply θ and z; hence $\theta\phi = -\phi$ and $z\phi = \phi$ in $H_1 \otimes H_1 \otimes H_1$, and in its images.

On the other hand, let $\phi \in ((H_1 \otimes H_1)/q) \otimes H_1$, and suppose that $\theta\phi = -\phi$ in $((H_1 \otimes H_1)/q) \otimes H_1$ and that $z\phi = \phi$ in $(H_1 \otimes H_1 \otimes H_1)/(q \otimes H_1 + H_1 \otimes q)$. We must show that ϕ is contained in the image $\bigwedge^3 H_1$ in $((H_1 \otimes H_1)/q) \otimes H_1$. Since $(\mathrm{id} + \theta)\phi = 0$, we may choose a preimage in $(\bigwedge^2 H_1/q) \otimes H_1$ and hence a preimage $\hat{\phi} \in \bigwedge^2 H_1 \otimes H_1$. Use the diagram of Lemma 1.12 to compute $\hat{\phi} - z\hat{\phi}$ and push it into $(H_1 \otimes H_1 \otimes H_1)/(q \otimes H_1 + H_1 \otimes q)$ where it is $\phi - z\phi = 0$; therefore, $\hat{\phi} - z\hat{\phi} = q \otimes h + h' \otimes q$ for $h, h' \in H_1$. So $\hat{\phi} - q \otimes h = h' \otimes q + z\hat{\phi} \in \bigwedge^3 H_1$ because it is alternating in the first two tensor components and in the last two tensor components. Therefore, ϕ is in the image of $\bigwedge^3 H_1$ because it is an image of $\hat{\phi} - q \otimes h$. ∎

The special conditions on the image of Ψ given by Corollary 5.20 of course imply special conditions on the images of elements of the Torelli group under the map δ. We will momentarily have need of these conditions; they are in fact a theorem of D. Johnson.

Corollary 5.23 (Johnson). $\tau(\mathrm{I}_{g,*}) \subseteq \bigwedge^3 H_1(S)_{\mathbb{Z}}$.

Proof. Let $h \in \mathrm{I}_{g,*}$ and select any $(f, M, z) \in \mathfrak{T}_{g,*}$. Then Ψ_f and $\Psi_{f\circ h}$ both lie in the fiber of $\mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{R}}$ over $\Omega_f = \Omega_{f\circ h}$. By Corollary 5.20, we know that $\imath\lambda\Psi_{f\circ h}$ and $\imath\lambda\Psi_f$ each satisfy the inhomogeneous equations of 5.20. Therefore, their difference $\imath\lambda\Psi_{f\circ h} - \imath\lambda\Psi_f = \imath\jmath_{\Omega_f}\delta(h)$ satisfies the homogeneous equations and hence by Corollary 5.22 is in $\bigwedge^3 H_1(\Omega_f)$. This shows that $\tau(h) = \imath\delta(h) \in \bigwedge^3 H_1(S)$, and so we have $\tau(h) \in (\bigwedge^3 H_1(S))_{\mathbb{R}} \cap (((H_1 \otimes H_1)/q) \otimes H_1)_{\mathbb{Z}}$. ∎

We are now ready to state our main result. For the convenience of the reader, we now repeat some of the notation. The map $\Psi : \mathfrak{T}_{g,*} \to \mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{R}} \to \mathcal{E}xt(\mathcal{K}_2, \mathcal{H}^1)$ is defined by sending $(f, M, z) \in \mathfrak{T}_{g,*}$ to the abelian period matrix $\Omega_f \in \mathfrak{h}_g$ and the period map $\Psi_f \in \mathrm{Hom}(K_2(\Omega_f), H^1(\Omega_f))_{\mathbb{R}}$. Ψ_f is the image of a uniquely chosen element in $\mathrm{Hom}(K_2(M), H^1(M))_{\mathbb{R}}$ which gives the congruence class in $\mathrm{Ext}(K_2, H^1)$ of the extension of mixed Hodge structures associated to (M, z). The mapping class group $\mathcal{M}_{g,*}$ acts on $\mathfrak{T}_{g,*}$ by sending (f, M, z) to $(f\circ h, M, z)$ and on the $\mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{R}}$ bundle by the affine action of sending (Z, ϕ) to $(Z \cdot \rho_h, (\phi + \jmath_Z\lambda^{-1}\hat{\delta}(h)) \cdot \rho_h)$. The cocycle $\hat{\delta}$ is in $Z^1(\mathcal{M}_{g,*}, ((H_1 \otimes H_1)/q) \otimes H^1(S))$. The map Ψ is equivariant with these two (right) actions of $\mathcal{M}_{g,*}$, and in each case we have induced actions by $\mathcal{M}_{g,*}/\mathcal{N}_{g,*}$ because $\mathcal{N}_{g,*}$, the kernel of Johnson's homomorphism, acts trivially. By a homology involution, we mean an element $y \in \mathcal{M}_{g,*}$ whose induced map y_* on $H_1(S, \mathbb{Z})$ is $-\mathrm{id}$. We view $\lambda^{-1}\imath^{-1}\bigwedge^3\mathcal{H}_1$ as a subbundle of $\mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)$ via the injection $\bigwedge^3 H_1 \hookrightarrow ((H_1 \otimes H_1)/q) \otimes H_1$ and the identification $\mathrm{Hom}(K_2, H^1) \xrightarrow[\cong]{\lambda} ((H_1 \otimes H_1)/q) \otimes H^1 \xrightarrow[\cong]{\imath} ((H_1 \otimes H_1)/q) \otimes H_1$.

Theorem 5.24. *Let $y \in \mathcal{M}_{g,*}$ be any homology involution and $\jmath\lambda^{-1}\hat{\delta}(y)$ the corresponding global section of $\mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{Z}}$. The period map $\Psi : \mathfrak{T}_{g,*} \to \mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{R}}$ factors through the translation by $-\frac{1}{2}\jmath\lambda^{-1}\hat{\delta}(y)$ of the subbundle $\lambda^{-1}\imath^{-1}(\bigwedge^3\mathcal{H}_1)_{\mathbb{R}}$ so that we have:*

$$\Psi : \mathfrak{T}_{g,*}/\mathcal{N}_{g,*} \to (-\tfrac{1}{2}\jmath\lambda^{-1}\hat{\delta}(y) + \lambda^{-1}\imath^{-1}\textstyle\bigwedge^3\mathcal{H}_1)_{\mathbb{R}} \to \mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{R}},$$

and Ψ is equivariant with respect to the action of $\mathcal{M}_{g,*}/\mathcal{N}_{g,*}$.

We also have the factorization from Torelli space through the translation of a torus bundle:

$$\Psi : \mathfrak{T}_{g,*}/\mathrm{I}_{g,*} \to \frac{-\frac{1}{2}\jmath\lambda^{-1}\hat{\delta}(y) + \lambda^{-1}\imath^{-1}(\bigwedge^3\mathcal{H}_1)_{\mathbb{R}}}{\lambda^{-1}\imath^{-1}(\bigwedge^3\mathcal{H}_1)_{\mathbb{Z}}} \to \frac{\mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{R}}}{\lambda^{-1}\imath^{-1}(\bigwedge^3\mathcal{H}_1)_{\mathbb{Z}}},$$

and Ψ is equivariant with respect to the affine action of $\mathrm{Sp}_g(\mathbb{Z}) \cong \mathcal{M}_{g,*}/\mathrm{I}_{g,*}$. The induced map $\Psi : \mathfrak{T}_{g,*}/\mathcal{M}_{g,*} \to \mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{R}}/(\mathcal{M}_{g,*})$ factors in each fiber over Z through the affine quotient of the translated torus $(-\frac{1}{2}\jmath_Z\lambda^{-1}\hat{\delta}(y) + \lambda^{-1}\imath^{-1}(\bigwedge^3 H_1)_{\mathbb{R}})/\lambda^{-1}\imath^{-1}(\bigwedge^3 H_1)_{\mathbb{Z}}$ by all $\sigma \in \mathrm{Sp}_g(\mathbb{Z})$ that fix Z.

Proof. By Corollary 5.20, the map $\imath\lambda\Psi$ factors through the solution space of the inhomogeneous equations of 5.20. Let $(f, M, z) \in \mathfrak{T}_{g,*}$ and let $y \in \mathcal{M}_{g,*}$ be a homology involution. Then both Ψ_f and $\Psi_{f\circ y}$ are in the same fiber $\mathrm{Hom}(K_2(\Omega_f), H^1(\Omega_f))_{\mathbb{R}}$ over Ω_f so that $\imath\lambda(\Psi_f - \Psi_{f\circ y})$ satisfies the homogeneous equations and so is in $\bigwedge^3 H_1$ by Lemma 5.22. By Proposition 4.2, $\Psi_{f\circ y} = (\Psi_f + \jmath_{\Omega_f}\lambda^{-1}\hat{\delta}(y)) \cdot \rho_y = (\Psi_f + \jmath_{\Omega_f}\lambda^{-1}\hat{\delta}(y)) \cdot$

$(-I) = -\Psi_f - \jmath_{\Omega_f}\lambda^{-1}\hat{\delta}(y)$. Therefore $\Psi_f - \Psi_{f\circ y} = 2\Psi_f + \jmath_{\Omega_f}\lambda^{-1}\hat{\delta}(y)$ is in the subspace $\lambda^{-1}\imath^{-1}(\bigwedge^3 H_1(\Omega_f))_{\mathbb{R}}$. This demonstrates that for any f, $\Psi_f \in -\frac{1}{2}\jmath_{\Omega_f}\lambda^{-1}\hat{\delta}(y) + \lambda^{-1}\imath^{-1}(\bigwedge^3 H_1(\Omega_f))_{\mathbb{R}}$ as asserted. The map Ψ is known by Corollary 4.5 to be equivariant with respect to the affine action of $\mathcal{M}_{g,*}/\mathcal{N}_{g,*}$. We also need to check that this action stabilizes $-\frac{1}{2}\jmath\lambda^{-1}\hat{\delta}(y) + \lambda^{-1}\imath^{-1}\bigwedge^3\mathcal{H}_1$ over the entire $\mathfrak{h}_g$.

The stability of $-\frac{1}{2}\jmath\lambda^{-1}\hat{\delta}(y) + \lambda^{-1}\imath^{-1}\bigwedge^3\mathcal{H}_1$ under the action of $\mathcal{M}_{g,*}$ follows by:

$$
\begin{aligned}
&(-\tfrac{1}{2}\jmath_{\Omega_f}\lambda^{-1}\hat{\delta}(y) + \lambda^{-1}\imath^{-1}(\textstyle\bigwedge^3 H_1(\Omega_f))_{\mathbb{R}})\cdot h \\
&\quad = (-\tfrac{1}{2}\jmath_{\Omega_f}\lambda^{-1}\hat{\delta}(y) + \jmath_{\Omega_f}\lambda^{-1}\imath^{-1}(\textstyle\bigwedge^3 H_1(S))_{\mathbb{R}} + \jmath_{\Omega_f}\lambda^{-1}\hat{\delta}(h))\cdot\rho_h \\
&\qquad\qquad \text{(by Definition 4.4 of action)} \\
&\quad = (\jmath_{\Omega_f}\lambda^{-1}(-\tfrac{1}{2}h_*\hat{\delta}(y) + \tfrac{1}{2}\delta((h,y)) + \imath^{-1}(\textstyle\bigwedge^3 H_1(S)_{\mathbb{R}}))\cdot\rho_h \\
&\qquad\qquad \text{(by Lemma 3.8)} \\
&\quad = \jmath_{\Omega_f\cdot\rho_h}h_*^{-1}\lambda^{-1}(-\tfrac{1}{2}h_*\hat{\delta}(y) + \imath^{-1}(\textstyle\bigwedge^3 H_1(S))_{\mathbb{R}}) \\
&\qquad\qquad \text{(by diagram 1.10 and Corollary 5.23)} \\
&\quad = -\tfrac{1}{2}\jmath_{\Omega_f\cdot\rho_h}\lambda^{-1}\hat{\delta}(y) + \lambda^{-1}\imath^{-1}(\textstyle\bigwedge^3 H_1(\Omega_f\cdot\rho_h))_{\mathbb{R}}.
\end{aligned}
$$

We now quotient the domain and range of Ψ by the Torelli group $\mathrm{I}_{g,*}$. By Definition 4.4, $h \in \mathrm{I}_{g,*}$ acts on (Z,ϕ) by $Z\cdot h = Z$ and $\phi\cdot h = \phi + j_Z\lambda^{-1}\delta(h)$, so that $\mathrm{I}_{g,*}$ acts on $\mathrm{Hom}(K_2(Z), H^1(Z))_{\mathbb{R}}$ merely by translation. By Johnson's results, we have $\tau(\mathrm{I}_{g,*}) = \imath\delta(\mathrm{I}_{g,*}) = (\bigwedge^3 H_1)_{\mathbb{Z}}$, so that

$$\mathcal{H}om(\mathcal{K}_2,\mathcal{H}^1)_{\mathbb{R}}/(\mathrm{I}_{g,*}) = \frac{\mathcal{H}om(\mathcal{K}_2,\mathcal{H}^1)_{\mathbb{R}}}{\lambda^{-1}\imath^{-1}(\bigwedge^3\mathcal{H}_1)_{\mathbb{Z}}}.$$

To show that the induced Ψ is well-defined, we in fact only need Corollary 5.23 that $\imath\delta(\mathrm{I}_{g,*}) \subseteq (\bigwedge^3 H_1)_{\mathbb{Z}}$; but to call $\lambda^{-1}\imath^{-1}(\bigwedge^3 H_1)_{\mathbb{R}}/\mathrm{I}_{g,*}$ a torus, we need to know that the image of the Torelli group is all of $(\bigwedge^3 H_1)_{\mathbb{Z}}$. The normality of $\mathrm{I}_{g,*}$ in $\mathcal{M}_{g,*}$ is all that is needed to induce the affine action of $\mathcal{M}_{g,*}/\mathrm{I}_{g,*}$ on $\mathcal{H}om(\mathcal{K}_2,\mathcal{H}^1)_{\mathbb{R}}$; however, this affine action of $\mathrm{Sp}_g(\mathbb{Z})$ (as set up here) is not the natural bundle action. We may quotient by $\mathcal{M}_{g,*}$ and by the equivariance of Ψ obtain a map $\Psi : \mathfrak{T}_{g,*}/\mathcal{M}_{g,*} \to \mathcal{H}om(\mathcal{K}_2,\mathcal{H}^1)_{\mathbb{R}}/\mathcal{M}_{g,*}$. An element ϕ in the fiber over Z gets identified with $(\phi + \jmath_Z\lambda^{-1}\hat{\delta}(h))\cdot\rho_h$ for every $h \in \mathcal{M}_{g,*}$ such that $Z\cdot\rho_h = Z$. Generically, only Torelli elements and homology involutions fix Z; a homology involution stabilizes $-\frac{1}{2}\jmath_Z\lambda^{-1}\hat{\delta}(y) + \lambda^{-1}\imath^{-1}(\bigwedge^3 H_1)_{\mathbb{R}}$, acting by $-\mathrm{id}$ on $\lambda^{-1}\imath^{-1}(\bigwedge^3 H_1)_{\mathbb{R}}$, and the Torelli elements mod out by the lattice $\lambda^{-1}\imath^{-1}(\bigwedge^3 H_1)_{\mathbb{Z}}$. We may therefore ignore further action by Torelli elements and simply pick representatives $\sigma \in \mathrm{Sp}_g(\mathbb{Z})$ which fix Z. ∎

As mentioned in the introduction, the second paragraph of the above theorem provides a new proof for, and provides a global version of, some previously known results due to Harris, Hain, and Pulte. We now compare the above theorem to Theorem 4.10 in [13]. For each fixed compact Riemann surface M with Jacobian $\mathrm{Jac}(M)$, let $\mathcal{A}_0^1(\mathrm{Jac}(M))$ denote the algebraic 1-cycles homologous to 0 on $\mathrm{Jac}(M)$ modulo rational equivalence. Let

$$\mathrm{J}_2(\mathrm{Jac}(M)) = \frac{\mathrm{Hom}(F^2H^3(\mathrm{Jac}(M)),\mathbb{C})}{H_3(\mathrm{Jac}(M),\mathbb{Z})}$$

be the intermediate Jacobian and

$$\Phi : \mathcal{A}_0^1(\mathrm{Jac}(M)) \to \mathrm{J}_2(\mathrm{Jac}(M))$$

be the Abel-Jacobi map of Griffiths (see [13], p.734). For two markings of M, (f_1, M, z_1) and (f_2, M, z_2), let M_{z_i} denote the image $(w_{f_i})_*(M)$ of M in $\mathrm{Jac}(M)$, so that $M_{z_1} - M_{z_2} \in \mathcal{A}_0^1(\mathrm{Jac}(M))$ and in fact $M_{z_1} - M_{z_2}^- \in \mathcal{A}_0^1(\mathrm{Jac}(M))$ as well. Pulte showed that there is an injection of the torus $\mathrm{J}_2(\mathrm{Jac}(M))$ into the torus $\mathrm{Ext}(K_2(M), H^1(M))$ such that

$$\Phi(M_{z_1} - M_{z_2}) = [\tilde{\Psi}_{(f_1,M,z_1)}] - [\tilde{\Psi}_{(f_2,M,z_2)}], \text{ and}$$
$$\Phi(M_{z_1} - M_{z_2}^-) = [\tilde{\Psi}_{(f_1,M,z_1)}] + [\tilde{\Psi}_{(f_2,M,z_2)}] \text{ in } \mathrm{Ext}(K_2(M), H^1(M)).$$

Therefore, the difference of the images of (f_1, M, z_1) and (f_2, M, z_2) into Ext under the $\tilde{\Psi}$ map factors through $\mathrm{J}_2(\mathrm{Jac}(M))$. This should be compared with the second paragraph of the Main Theorem which says that any two points (f_2, M, z_2) and (f_1, M, z_1) in Teichmüller space which have the same abelian period matrix $\Omega_{f_1} = \Omega_{f_2}$ will have $[\Psi_{f_1}]$ and $[\Psi_{f_2}]$ that both lie in $(-\frac{1}{2}\jmath_{\Omega_{f_1}}\lambda^{-1}\delta(y) + \lambda^{-1}\imath^{-1}(\bigwedge^3 H_1(\Omega_{f_1}))_{\mathbb{R}})/\lambda^{-1}\imath^{-1}(\bigwedge^3 H_1)_{\mathbb{Z}}$. The difference $[\Psi_{f_1}] - [\Psi_{f_2}]$ therefore lies in $\lambda^{-1}\imath^{-1}(\bigwedge^3 H_1(\Omega_{f_1}))_{\mathbb{R}})/\lambda^{-1}\imath^{-1}(\bigwedge^3 H_1)_{\mathbb{Z}}$, which can be identified with $\mathrm{J}_2(\mathrm{Jac}(M))$ ([13, Lemma 3.5]). From the second part of Pulte's Theorem 4.10 (in [13]), we can also conclude that $\Phi(M_z - M_z^-) = 2[\tilde{\Psi}_{f,M,z}]$ also lies in $\mathrm{J}_2(\mathrm{Jac}(M)) \hookrightarrow \mathrm{Ext}(K_2(M), H^1(M))$, so that $\tilde{\Psi}_{(f,M,z)} \in \frac{1}{2}\mathrm{Hom}(K_2, H^1)_{\mathbb{Z}} + \mathrm{J}_2(\mathrm{Jac}(M))$. Since the torus bundle $\mathcal{J}_2$ with fibers $\mathrm{J}_2(A_Z)$ exists over $\mathfrak{h}_g$, this part of Pulte's result implies that over Torelli space, the Ψ map always lies in a translate of the $\mathrm{J}_2(A_Z)$ torus by half a lattice element in $\mathrm{Hom}(K_2, H^1)_{\mathbb{Z}}$. Furthermore, Pulte's work gives a nice reason why $\tilde{\Psi}_{(f_1,M,z_1)} - \tilde{\Psi}_{(f_2,M,z_2)}$ should factor through $\mathrm{J}_2(\mathrm{Jac}(M))$ by showing that $[\tilde{\Psi}_{f_1}] - [\tilde{\Psi}_{f_2}]$ is the Abel-Jacobi image of $M_{z_1} - M_{z_2}$. In this paper we knew that the higher bilinear relations imply that Ψ satisfies certain constraints and were interested to determine what those constraints were. We have here referred to only a few of the results in Pulte's paper [13].

§6. Holomorphic quadratic periods.

In this section we show in Lemma 6.2 that if a $\phi \in \mathrm{Hom}(K_2, H^1)_{\mathbb{R}}$ comes from an element of $\bigwedge^3 H_1$, then ϕ is completely determined by its values on F^2K_2. This observation shows that Ψ is completely determined by its purely holomorphic quadratic periods and allows us to improve several theorems in the literature [9]. Finally, in Theorem 6.8 we go further and compute the remaining mixed periods of Ψ in terms of the purely holomorphic ones.

Definition 6.1. Let (H^1, q) be a principally polarized Hodge structure of weight one and $K_2 = \mathrm{Ker}\, q \subseteq H^1 \otimes H^1$. Define

$$\pi : \mathrm{Hom}(K_2, H^1)_{\mathbb{R}} \to \mathrm{Hom}(F^2K_2, H^1)_{\mathbb{C}}$$

to be the map given by restriction of the domain K_2 to F^2K_2.

Recall the isomorphisms λ and $\imath$ of Definitions 1.11, and note that by Lemma 1.12, $\lambda^{-1}\imath^{-1}$ gives an injection $\lambda^{-1}\imath^{-1} : (\bigwedge^3 H_1)_{\mathbb{R}} \to \mathrm{Hom}(K_2, H^1)_{\mathbb{R}}$.

Lemma 6.2. *Let (H^1, q) be a principally polarized Hodge structure of weight one with $K_2 = \operatorname{Ker} q \subseteq H^1 \otimes H^1$ and $H_1 =$ dual of H^1. Then the map $\pi \circ \lambda^{-1} \circ \imath^{-1}$ is an injection:*

$$\pi \circ \lambda^{-1} \circ \imath^{-1} : (\textstyle\bigwedge^3 H_1)_{\mathbb{R}} \xrightarrow{\imath^{-1}} ((H_1 \otimes H_1/q) \otimes H^1)_{\mathbb{R}} \xrightarrow{\lambda^{-1}} \operatorname{Hom}(K_2, H^1)_{\mathbb{R}} \xrightarrow{\pi} \operatorname{Hom}(F^2K_2, H^1)_{\mathbb{C}}.$$

Proof. For this proof only, denote a standard basis of $H_1(\mathbb{Z})$ by a_j, b_j where $j = 1, \ldots, g$. A basis for $(\bigotimes^3 H_1)_{\mathbb{Z}}$ is obtained by forming all possible triple tensors of these basis elements. For an element $\phi \in \operatorname{Hom}(K_2, H^1)_{\mathbb{C}}$ we may choose a representative of $\imath\lambda\phi$ in $(\bigotimes^3 H_1)_{\mathbb{C}}$ and write

$$\imath\lambda\phi = \sum [c^{aaa}_{ijk} a_i \otimes a_j \otimes a_k + c^{aab}_{ijk} a_i \otimes a_j \otimes b_k + c^{aba}_{ijk} a_i \otimes b_j \otimes a_k + \text{etc.}]$$

where $c^{\bullet\bullet\bullet}_{ijk} \in \mathbb{C}$. The condition that $\phi \in \operatorname{Hom}(K_2, H^1)_{\mathbb{R}}$ is just that we may choose all $c^{\bullet\bullet\bullet}_{ijk} \in \mathbb{R}$, or equivalently that $\overline{\phi} = \phi$. Since we wish to say something regarding F^2K_2, we consider instead a basis of $H^1 = H^{1,0} \oplus H^{0,1}$ given by $z_i, \overline{z}_i$ for $z_i \in H^{1,0}$ and $i = 1, \ldots, g$. The dual basis in H_1 then has the form $x_i, \overline{x}_i$ for $i = 1, \ldots, g$. In this basis we have a corresponding expression for $\imath\lambda\phi$:

$$\imath\lambda\phi = \sum [c^{xxx}_{ijk} x_i \otimes x_j \otimes x_k + c^{xx\overline{x}}_{ijk} x_i \otimes x_j \otimes \overline{x}_k + c^{x\overline{x}x}_{ijk} x_i \otimes \overline{x}_j \otimes x_k + \text{etc.}]. \tag{6.3}$$

The condition that $\phi \in \operatorname{Hom}(K_2, H^1)_{\mathbb{R}}$, which is $\overline{\phi} = \phi$, becomes the equalities

$$\overline{c^{xxx}_{ijk}} = c^{\overline{x}\,\overline{x}\,\overline{x}}_{ijk}, \quad \overline{c^{xx\overline{x}}_{ijk}} = c^{\overline{x}\,\overline{x}\,x}_{ijk}, \quad \overline{c^{x\overline{x}\,\overline{x}}_{ijk}} = c^{\overline{x}\,xx}_{ijk}, \quad \text{etc.}$$

Assume now that $\pi\phi = 0$ so that for all $z_m, z_n \in H^{1,0}$, we have $z_m \otimes z_n \in F^2K_2$ and $\phi(z_m \otimes z_n) = 0$ in H^1. This implies that $\imath\lambda\phi$ annihilates $z_m \otimes z_n \otimes H^1$ because we have $\langle\, \imath\lambda\phi\,,\, z_m \otimes z_n \otimes H^1 \,\rangle = \langle\, \lambda\phi\,,\, z_m \otimes z_n \otimes H_1 \,\rangle = \langle\, \phi(z_m \otimes z_n)\,,\, H_1 \,\rangle = 0$. From 6.3 we therefore have:

$$\sum_k (c^{xxx}_{mnk} x_k + c^{xx\overline{x}}_{mnk} \overline{x}_k) = 0 \quad \text{in } H_1 \tag{6.4}$$

so that $c^{xxx}_{mnk} = c^{xx\overline{x}}_{mnk} = 0$ for all m, n, k. The condition $\phi \in \operatorname{Hom}(K_2, H^1)_{\mathbb{R}}$ then implies that $c^{\overline{x}\,\overline{x}\,\overline{x}}_{mnk} = c^{\overline{x}\,\overline{x}\,x}_{mnk} = 0$ as well. Finally, the assumption that $\imath\lambda\phi \in \bigwedge^3 H_1$ implies that the $c^{\bullet\bullet\bullet}_{ijk}$ are alternating when the three columns of indices are permuted. Hence we have that $c^{xx\overline{x}}_{ijk} = 0$ implies $c^{x\overline{x}x}_{ikj} = 0$ and $c^{\overline{x}xx}_{kij} = 0$ as well. Likewise, $c^{\overline{x}\,\overline{x}\,x}_{ijk} = 0$ implies that $c^{\overline{x}\,x\,\overline{x}}_{ikj} = 0$ and $c^{x\overline{x}\,\overline{x}}_{kij} = 0$ as well. Thus $\imath\lambda\phi = 0$. ∎

Corollary 6.5. *The period map $\Psi : \mathfrak{T}_{g,*} \to \mathcal{H}om(\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{R}}$ is determined by its image in $\mathcal{H}om(F^2\mathcal{K}_2, \mathcal{H}^1)_{\mathbb{C}}$.*

Proof. For $(f_i, M_i, z_i) \in \mathfrak{T}_{g,*}$, the values Ψ_{f_i} are in $\operatorname{Hom}(K_2(\Omega_{f_i}), H^1(\Omega_{f_i}))_{\mathbb{R}}$. If $\Omega_{f_1} = \Omega_{f_2}$ so that Ψ_{f_1} and Ψ_{f_2} are in the same fiber, then we have shown in Corollary 5.20 and Lemma 5.22 that $\imath\lambda(\Psi_{f_1} - \Psi_{f_2}) \in \bigwedge^3 H_1(\Omega_{f_i})$. By the above Lemma 6.2, then $\pi\Psi_{f_1} = \pi\Psi_{f_2}$ if and only if $\Psi_{f_1} = \Psi_{f_2}$. ∎

One way to interpret this corollary is to say that the quadratic periods $\sigma_{ij}(a_k), \sigma_{ij}(b_k)$ completely determine Ψ. This is interesting with regard to the works of Gunning [4] and Jablow [9] who studied only these quadratic periods. The τ_{ij} periods are hence determined by the σ_{ij} periods, and we actually tell how to compute the τ_{ij} periods from the σ_{ij} periods at the end of this section.

From the point of view of variation of mixed Hodge structure as discussed in Hain [6], we have the injection $\mathrm{J}_2(\mathrm{Jac}(M)) \hookrightarrow \mathrm{Ext}(K_2(M), H^1(M))$ and the isomorphism

$$\mathrm{Ext}(K_2, H^1) \cong \frac{\mathrm{Hom}(F^2(K_2 \otimes H^1), \mathbb{C})_{\mathbb{C}}}{\mathrm{Hom}(K_2 \otimes H^1, \mathbb{Z})_{\mathbb{Z}}}.$$

We have $F^2(K_2 \otimes H^1) = (F^2 K_2 \otimes H^1) \oplus (K_2^{1,1} \otimes H^{1,0})$ and the above Lemma 6.2 shows that the image of the intermediate Jacobian in Ext is in fact determined by its restriction to $(F^2 K_2) \otimes H^1$.

E. Jablow [9] proved the following proposition with the additional assumption that $\mathrm{Jac}(M)$ has no complex multiplication.

Proposition 6.6. *Let $(f, M, z) \in \mathfrak{T}_{g,*}$ be a marked compact Riemann surface of genus $g \geq 1$. Let $h \in \mathrm{I}_{g,*}$. If $\sigma_{ij}(c) \cdot h = \sigma_{ij}(c)$ for all quadratic periods σ_{ij} and for all $c \in \pi_1(M, z)$,then $\tau(h) = 0$ (where τ is Johnson's homomorphism).*

Proof. Since $\Psi_f(\omega_i \otimes \omega_j)([c_k]) = \sigma_{ij}(c_k)$ for generators c_k of $\pi(M, z)$, the assumption that h fixes all quadratic periods is the assumption that $\Psi_f = \Psi_{f\circ h}$ on $F^2 K_2(\Omega_f)$. By Corollary 6.5, this implies that $\Psi_f = \Psi_{f\circ h}$. Since $h \in \mathrm{I}_{g,*}$, we have by Corollary 4.3 that $\Psi_{f\circ h} = \Psi_f + \jmath_{\Omega_f} \lambda^{-1} \delta(h)$, which immediately implies that $\tau(h) = \imath\delta(h) = 0$. ∎

E. Jablow [9] proved the following theorem for $g = 3$ with a different generic set than $\mathfrak{E}_g$. The definition of $\mathfrak{E}_g$ was given in Definition 4.6.

Theorem 6.7. *Two marked compact Riemann surfaces from $\mathfrak{E}_g \subseteq \mathfrak{T}_{g,*}$ have equal abelian and quadratic periods if and only if they differ by an element of $\mathcal{N}_{g,*} = \mathrm{Ker}\,\tau$.*

Proof. The two marked surfaces have the same abelian periods, so we have that $\Omega_{f_1} = \Omega_{f_2}$ and that Ψ_{f_1} and Ψ_{f_2} are in the same fiber $\mathrm{Hom}(K_2(\Omega_f), H^1(\Omega_f))_{\mathbb{R}}$. By Corollary 6.5, we obtain $\Psi_{f_1} = \Psi_{f_2}$ since they have equal quadratic periods and hence are equal on $F^2 K_2$. However, by Proposition 4.7, Ψ injects on $\mathfrak{E}_g/\mathcal{N}_{g,*} \subseteq \mathfrak{T}_{g,*}/\mathcal{N}_{g,*}$. ∎

Corollary 6.5 says that the quadratic periods completely determine Ψ, and we now give explicit formulae for this phenomenon. The mixed periods $\tau_{ij}(c) = \int_c (\overline{\omega}_i \omega_j - \Lambda_{ij} \overline{\omega}_1 \omega_1 + u_{ij})$, where $u_{ij} = u(\overline{\omega}_i \otimes \omega_j - \Lambda_{ij} \overline{\omega}_1 \otimes \omega_1)$, are rather mysterious because the $1,0$-forms u_{ij} such that $du_{ij} + \overline{\omega}_i \wedge \omega_j - \Lambda_{ij} \overline{\omega}_1 \wedge \omega_1 = 0$ are not familiar. We can, however, give the following formulae for the computation of the mixed periods.

Theorem 6.8. *Let $N = (\Omega - \overline{\Omega})^{-1}$. For any i, j, k, we have*

$$\begin{aligned}\tau_{ij}(b_k) - \sum_{\ell=1}^{g} \tau_{ij}(a_\ell)\Omega_{\ell k} = {} & \sigma_{jk}(b_i) - \sum_{\ell=1}^{g} \sigma_{jk}(a_\ell)\overline{\Omega}_{\ell i} \\ & + \delta_{jk}\overline{\Omega}_{ij} - \delta_{ik}\Omega_{ij} + \overline{\Omega}_{ij}\Omega_{jk} - \Omega_{ij}\Omega_{ik} \\ & - \Lambda_{ij}[\sigma_{1k}(b_1) - \sum_{\ell=1}^{g} \sigma_{1k}(a_\ell)\overline{\Omega}_{\ell 1} \\ & \quad + \delta_{1k}\overline{\Omega}_{11} - \delta_{1k}\Omega_{11} + \overline{\Omega}_{11}\Omega_{1k} - \Omega_{11}\Omega_{1k}].\end{aligned}$$

$$\begin{aligned}\tau_{ij}(a_k) = {} & \sum_{m=1}^{g} \{-\sigma_{jm}(b_i) - \overline{\sigma}_{im}(b_j) + \sum_{\ell=1}^{g}(\sigma_{jm}(a_\ell)\overline{\Omega}_{\ell i} + \overline{\sigma}_{im}(a_\ell)\Omega_{\ell j})\} N_{mk} \\ & + \delta_{ik}\Omega_{ij} - \delta_{jk}\overline{\Omega}_{ij} + \sum_{m=1}^{g} \{\overline{\Omega}_{im}\Omega_{jm} - \delta_{ij}\overline{\Omega}_{im}\} N_{mk} \\ & + \Lambda_{ij}\left[\sum_{m=1}^{g} \{\sigma_{1m}(b_1) + \overline{\sigma}_{1m}(b_1) - \sum_{\ell=1}^{g}(\sigma_{1m}(a_\ell)\overline{\Omega}_{\ell 1} + \overline{\sigma}_{1m}(a_\ell)\Omega_{\ell 1})\} N_{mk}\right. \\ & \left. -\delta_{1k}(\Omega_{11} - \overline{\Omega}_{11}) + \sum_{m=1}^{g}\{-\overline{\Omega}_{1m}\Omega_{1m} + \overline{\Omega}_{1m}\} N_{mk}\right].\end{aligned}$$

Proof. These formulae may be derived by manipulating the formulae given in Remark 5.3, equation 5.15, and equation 5.16. ∎

References

1. L. Bers, *Fiber spaces over Teichmüller spaces*, Acta Math. **130** (1973), 89–126.
2. J. Carlson, *Extensions of mixed Hodge structures*, Journées de Géometrie Algébrique d'Angers, Sijthoff and Noordhoff, Alphen aan den Rijn (1980), 77–105.
3. K.-T. Chen, *Iterated path integrals*, Bull. Amer. Math. Soc. **83** (1977), 831–879.
4. R.C. Gunning, *Quadratic periods of hyperelliptic abelian integrals*, Problems in Analysis, Princeton Univ. Press, Princeton, N.J. (1969), 239–247.
5. R.C. Gunning, *Lectures on Riemann surfaces, Jacobi varieties. Mathematical Notes #12*, Princeton Univ. Press, Princeton, N.J., 1972.
6. R. Hain, *The geometry of the mixed Hodge structure on the fundamental group*, Algebraic Geometry, Bowdoin, Amer. Math. Soc.: Proc. Symp. Pure Math. **46, 2** (1987), 247–282.
7. B. Harris, *Harmonic volumes*, Acta Math. **150, 1-2** (1983), 91–123.
8. J.I. Igusa, *Theta Functions, Grundlehren der mathematische Wissenschaften 194*, Springer-Verlag, Berlin, Heidelberg, New York, 1972.
9. E. Jablow, *Quadratic vector classes on Riemann surfaces*, Duke Jour. Math. **53** (1986), 221–232.
10. D. Johnson, *A survey of the Torelli group*, Low Dimensional Topology, Contemporary Mathematics **20**, American Mathematical Society, Providence, R.I. (1983), 165–179.
11. S. Koizumi, *The ring of algebraic correspondences on a generic curve of genus g*, Nagoya Math Jour. **60** (1976), 173–180.
12. W. Manger, *Die Klassen von topologischen Abbildungen einer geschlossenen Fläsche auf sich*, Math. Zeit. **44** (1939).
13. M. Pulte, *The fundamental group of a Riemann surface: mixed Hodge structures and algebraic cycles*, Duke Math. Jour. **57** (1988), 721–760.

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