THE HYPERELLIPTIC LOCUS

Cris Poor

Fordham University

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Abstract. It is shown in theorem 2.6.1 that the vanishing properties of the thetanullwerte of hyperelliptic Jacobians characterize them among all irreducible principally polarized abelian varieties. An alternate proof of Mumford's theorem characterizing hyperelliptic Jacobians among all principally polarized abelian varieties by vanishing and nonvanishing properties is sketched in section 2.7.

Introduction

The Schottky problem is the problem of characterizing Jacobians among all abelian varieties. In 1888, for genus four, Schottky gave a homogeneous polynomial in the theta constants which vanishes on $\mathcal{H}_4$ precisely at the Jacobian points; a proof of this was finally published by Igusa in 1981 [11]. A solution of the Schottky problem in general, such as given by Schottky and Igusa in genus four, would be a set of polynomials in the theta constants which vanish precisely on the Jacobian locus of $\mathcal{H}_g$, the Siegel upper half space. These equations have proved elusive, whereas other interesting methods of characterizing Jacobians have met with more success; here however we restrict ourselves to the approach which requires the specification of a sufficient number of equations.

Along the same lines as the Schottky problem we may consider other Schottky–type problems; such as the characterization of hyperelliptic Jacobians among all abelian varieties, the topic of this paper. It was known to Schottky by 1880 [12,763] that a Jacobian of genus three is hyperelliptic precisely when an even theta constant vanishes. Great progress was made in 1984 when Mumford, using the methods of dynamical systems, characterized hyperelliptic Jacobians among all abelian varieties by the vanishing and nonvanishing of certain theta constants [14]. For simplicity and strength this theorem can hardly be improved; from the point of view of the original Schottky problem however, and from the desire to have an algebraic description of moduli space, it is beneficial to replace the nonvanishing conditions by further equalities. That the vanishing conditions define hyperelliptic Jacobians among all

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irreducible abelian varieties is proven in Theorem 2.6.1 and is a solution of the Schottky–type problem for hyperelliptic curves of arbitrary genus. One still wonders if the irreducibility hypothesis can be removed. The result is new for $g \geq 5$ and is encouraging because it is a result for each genus which does not demand the existence of auxiliary parameters or nonvanishing conditions.

2.6.1 Main Theorem. *Let $\eta \in \Xi_g$ and $\Omega \in \mathcal{H}_g$. The following two statements are equivalent.*

1. *Ω is irreducible and Ω satisfies the equations $V_{g,\eta}$.*
2. *There is a marked hyperelliptic Riemann surface M of genus g which has Ω as its period matrix and $\mathrm{Jac}(M) = \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g)$. Furthermore, there is a model of M, $y^2 = \prod_{i\in B}(x - a_i)$, with a_∞ as the basepoint of the Abel–Jacobi map $w : M \to \mathrm{Jac}(M)$ such that $w(a_i) = [(\Omega I)\eta_i]$ in $\mathrm{Jac}(M)$.*

The set Ξ_g is an explicit set of maps from $B = \{1, 2, \ldots, 2g + 1, \infty\}$ to $\frac{1}{2}\mathbb{Z}^{2g}$ satisfying certain conditions given in definition 1.4.11. Attached to any map $\eta \in \Xi_g$ is a certain $U \subseteq B$ and a function ϵ_U on B taking the values ± 1. If an $\Omega \in \mathcal{H}_g$ is the period matrix of a marked hyperelliptic curve then there is change of basepoint which does not change Ω and an $\eta \in \Xi_g$ such that $w(a_i) = (\Omega I)\eta_i$ and Ω satisfies the vanishing equations for η, $V_{g,\eta}$. The content of the main theorem 2.6.1 is the converse: that if an irreducible Ω satisfies the vanishing equations $V_{g,\eta}$ for some η then Ω is hyperelliptic. The vanishing equations $V_{g,\eta}$ only depend upon the class of $\eta \in \Xi_g$ as a map into $\frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ and the classes of Ξ_g are a finite set canonically bijective with the azygetic bases of $\frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ and hence in noncanonical bijection with Γ/Γ_2. An irreducible $\Omega \in \mathcal{H}_g$ is then hyperelliptic if and only if one of its $|Sp_g(\mathbb{Z}/2\mathbb{Z})|$ representatives in $\mathcal{H}_g/\Gamma_2$ satisfies a fixed choice of vanishing equations. Certain sets of equations play an important role in this paper so we list these now for easy reference.

1.4.18 Definition. *Let $\eta \in \Xi_g$. The set of equations $V_{g,\eta}$ called the* vanishing equations *is defined by*

$$\theta[\eta_S](0, \Omega) = 0, \qquad \forall S \subseteq B : |S| \equiv 0 \mod 2 \text{ and } |U \circ S| \neq g + 1.$$

1.4.21 Definition. *Let $\eta \in \Xi_g$. Define $\xi_{ijkl} = \frac{1}{2}(\Omega\, I)(\eta_i + \eta_j - \eta_k - \eta_l)$. The set of equations $F_{g,\eta}$ called* Fay's trisecant formula *is defined by all the 3×3 minors which express the rank condition:*

$$\mathrm{rank}\{\vec{\theta}_2(\xi_{ijkl}, \Omega), \vec{\theta}_2(\xi_{iklj}, \Omega), \vec{\theta}_2(\xi_{iljk}, \Omega)\} \leq 2.$$

1.6.1 Definition. *Let $\Omega \in \mathcal{H}_g$ and $\eta \in \Xi_g$. The following equation is called the generalized Frobenius theta formula,* $\mathrm{Frob}_{g,\eta}$.

$$\forall\ a_i, z_i \in \mathbb{C}^g : \sum_{i=1}^{4} a_i = \sum_{i=1}^{4} z_i = 0, \quad \sum_{J\in B} \epsilon_U(J) \prod_{i=1}^{4} \theta[a_i + \eta_J](z_i, \Omega) = 0$$

This paper uses many results of Mumford [14]; however the methods used here are classical compared with the dynamical systems approach, and Mumford's theorem characterizing hyperelliptic curves is not used, so that the most powerful techniques are quite different than those in Tata II. The motivation for the proof is that the moduli of the hyperelliptic curve may be recovered in terms of the theta expressions $(\frac{\theta[\eta_1](0,\Omega)}{\theta[\eta_3](0,\Omega)}\frac{\theta[\eta_2](0,\Omega)}{\theta[\eta_4](0,\Omega)})^2$ using the crossratio identities of section 1.5 in the hyperelliptic case. Then using these hyperelliptic curve moduli a hyperelliptic Ω' may be constructed such that for any $\Omega \in \mathcal{H}_g$ which satisfies the crossratio identities we have

$$(*) \qquad (\frac{\theta[\eta_1](0,\Omega)}{\theta[\eta_3](0,\Omega)}\frac{\theta[\eta_2](0,\Omega)}{\theta[\eta_4](0,\Omega)})^2 = (\frac{\theta[\eta_1](0,\Omega')}{\theta[\eta_3](0,\Omega')}\frac{\theta[\eta_2](0,\Omega')}{\theta[\eta_4](0,\Omega')})^2.$$

The existence of these theta expressions, however, requires that the theta constants be nonzero. This requirement is equivalent to the nonvanishing conditions in Mumford's theorem, here seen to behave as a nondegeneracy condition. The first main obstacle then was to translate this nondegeneracy condition into a different form. Using Fay's trisecant formula as presented in Gunning [6] we translate another nondegeneracy condition, that the rank is two in Fay's trisecant formula above, into an irreducibility condition on Ω, which is much easier to work with. A version of the multisecant formula follows from the trisecant formula and implies the needed nonvanishing of the above theta constants. The second obstacle is that the equality (*) of the theta expressions is not immediately sufficient to conclude that Ω is Γ–equivalent to Ω'. A number of invariant theory calculations are used to take appropriate square roots of these expressions which are then sufficient to show that Ω is Γ_2–equivalent to Ω'. As further questions, we can ask whether the vanishing conditions alone define the hyperelliptic locus, whether the vanishing conditions define the hyperelliptic locus in some closure of moduli space, and how far is this characterization from giving all modular forms of a certain level which vanish on the hyperelliptic locus?

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1. Chapter One

§1.1 Abelian varieties and theta functions.

We review the well–known ideas and formulas we will need from the theory of abelian varieties and theta functions. Here the citations after the proposition headings are references for the reader's convenience and not necessarily attributions. The *Siegel upper half–space of rank* g is the set of all symmetric $g \times g$ complex matrices with positive definite imaginary part and is denoted by $\mathcal{H}_g$. For any $\Omega \in \mathcal{H}_g$ we construct a lattice $\mathcal{L} \subset \mathbb{C}^g$ by setting $\mathcal{L} = \mathbb{Z}^g + \Omega\mathbb{Z}^g$. Any $w \in \mathbb{C}^g$ can be uniquely written as $w = (\Omega\, I)\zeta = \Omega\zeta' + I\zeta''$ for $\zeta = [\zeta'|\zeta''] \in \mathbb{R}^{2g}$. A canonical $\mathbb{R}$–valued alternating form on $\mathbb{C}^g$ which is $\mathbb{Z}$–valued on $\mathcal{L}$ may be defined by $E((\Omega\, I)\zeta, (\Omega\, I)\xi) = {}^t\zeta\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\xi = {}^t\zeta J\xi$. Since we will not make reference to more intrinsic definitions the complex torus $A = \mathbb{C}^g/\mathcal{L}$ along with the above alternating form E will be referred to as a *principally polarized abelian variety* (p.p.a.v.).

Two of these principally polarized abelian varieties, (A_1, E_1) and (A_2, E_2) given by $\Omega_1, \Omega_2 \in \mathcal{H}_g$, will be called *equivalent* when there is an analytic group isomorphism $\phi : A_1 \to A_2$ whose lift $\hat{\phi} : \mathbb{C}^g \to \mathbb{C}^g$ maps E_1 to E_2. This amounts to the requirement that there exists a symplectic matrix $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_g(\mathbb{Z})$ such that $\Omega_1 = (d + \Omega_2 b)^{-1}(c + \Omega_2 a)$; here $\mathrm{Sp}_g(\mathbb{Z}) = \{\sigma \in \mathrm{Gl}_{2g}(\mathbb{Z}) : \sigma J {}^t\sigma = J\}$ will be denoted by Γ. Letting Γ act on $\mathcal{H}_g$ in this way we see that the equivalence classes of principally polarized abelian varieties are in bijection with $\mathcal{H}_g/\Gamma$, called the *moduli space* of p.p.a.v.s. Actually, in order for the above action of Γ to agree with Igusa's [10, 24] we must compose it with the antiautomorphism of Γ given by $\sigma \mapsto {}^t\sigma$; this agreed upon, $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ acts on $\Omega \in \mathcal{H}_g$ as $\sigma \cdot \Omega = (a\Omega + b)(c\Omega + d)^{-1}$.

1.1.1 Definition (First order theta function). *Let $\Omega \in \mathcal{H}_g$ and $\zeta = [\zeta'|\zeta''] \in \mathbb{R}^{2g}$. The* first order theta function with characteristic ζ, $\theta[\zeta](w, \Omega)$, *is defined for all $w \in \mathbb{C}^g$ by*

$$\theta[\zeta](w, \Omega) = \sum_{n \in \mathbb{Z}^g} \boldsymbol{e}^{2\pi i \{ \frac{1}{2} {}^t(n+\zeta')\Omega(n+\zeta') + (n+\zeta')\cdot(w+\zeta'')\}}.$$

The positive definiteness of $\mathrm{Im}\Omega$ implies that $\theta[\zeta](w, \Omega)$ converges and is analytic for all $w \in \mathbb{C}^g$. The standard abbreviations for $\theta[0](w, \Omega)$ are $\theta(w, \Omega)$ and $\theta(w)$, and in place of $\theta[\zeta](0, \Omega)$ one often writes $\theta[\zeta]$. Hereafter we take the liberty of dropping the transpose ${}^t(n+\zeta')\Omega(n+\zeta') = (n+\zeta')\Omega(n+\zeta')$ when it occurs in an exponent. The function $\theta[\zeta](w, \Omega)$ is not periodic with respect to $\mathcal{L}$ but instead transforms by a certain factor of automorphy; indeed the first order theta function is characterized up to a constant factor by its transformation property.

1.1.2 Definition. *The factor of automorphy $\xi : \mathcal{L} \times \mathbb{C}^g \to \mathbb{C}^{\cdot}$ for $l = (\Omega I)\lambda = \Omega\lambda' + I\lambda'' \in \mathcal{L}$ and $w \in \mathbb{C}^g$ is given by*

$$\xi(l, w) = \boldsymbol{e}^{-2\pi i \{ \frac{1}{2}\lambda'\Omega\lambda' + \lambda'\cdot w\}}.$$

The map $e : \mathbb{R}^{2g} \times \mathbb{R}^{2g} \to \mathbb{C}^{\cdot}$ for all $\zeta, \eta \in \mathbb{R}^{2g}$ is given by

$$e(\zeta, \eta) = \boldsymbol{e}^{2\pi i \{\zeta'\cdot\eta'' - \eta'\cdot\zeta''\}} = \boldsymbol{e}^{2\pi i \zeta J \eta}.$$

1.1.3 Proposition. [13, 121-123] *For all $l = (\Omega I)\lambda \in \mathcal{L}$ we have*

$$\theta[\zeta](w + l, \Omega) = e(\zeta, \lambda)\xi(l, w)\theta[\zeta](w, \Omega).$$

And conversely, for any holomorphic $f : \mathbb{C}^g \to \mathbb{C}$ such that for all $l \in \mathcal{L}$, $f(w + l) = e(\zeta, \lambda)\xi(l, w)f(w)$, there is a unique $c \in \mathbb{C}$ such that $f(w) = c\,\theta[\zeta](w, \Omega)$.

Under the action of the lattice $\mathcal{L}$ the theta function $\theta(w, \Omega)$ transforms by the factor of automorphy ξ and so the zero divisor of $\theta(w, \Omega)$ is well–defined in A. We call this divisor the *theta locus* and write $\Theta = \{ [w] \in A : \theta(w, \Omega) = 0\}$ although we occasionally speak as if $\Theta \subset \mathbb{C}^g$. Translations of the theta locus by an element a of A are conveniently written $\Theta_a = \{w \in A : w - a \in \Theta\}$.

1.1.4 Lemma. [10, 186] [13, 164] $\theta(w,\Omega)$ *vanishes simply on* Θ*, and for* $a \in A$ *we have* $\Theta_a = \Theta \iff a = 0$ *in* A*.*

We call $\Omega \in \mathcal{H}_g$ *symplectically reducible* if there exists a $\sigma \in \mathrm{Sp}_g(\mathbb{Z})$ such that $\sigma\cdot\Omega$ is in block diagonal form $\begin{pmatrix}\Omega_1 & 0\\ 0 & \Omega_2\end{pmatrix}$ for some $\Omega_i \in \mathcal{H}_{g_i}$, $g_1+g_2=g$. Otherwise we call Ω *symplectically irreducible*. The theta locus Θ is irreducible as an analytic divisor in A if and only if Ω is symplectically irreducible [11, 539-540]. This last fact will not be used in any forceful way but only to allow us to speak unambiguously of an element Ω as irreducible.

A special role is played by the theta functions $\theta[\zeta](w,\Omega)$ with $\zeta \in \frac{1}{2}\mathbb{Z}^{2g}$. For purposes such as vanishing orders the class of ζ in $\frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ is the relevant notion because for $n \in \mathbb{Z}^{2g}$ we have:

$$\theta[\zeta+n](w,\Omega) = \boldsymbol{e}^{2\pi i\zeta'\cdot n''}\theta[\zeta](w,\Omega). \tag{1.1.5 }$$

These 2^{2g} theta functions with half–integral characteristics, one choice of ζ from each coset of $\frac{1}{2}\mathbb{Z}^{2g} \mod \mathbb{Z}^{2g}$, arise because Θ is not intrinsically distinguished as a symmetric divisor with first Chern class E from among its translates Θ_a, $a \in \frac{1}{2}\mathcal{L}$, when the *equivalence* of abelian varieties is considered. Therefore any symmetrical account will treat the 2^{2g} functions $\theta[\zeta](w,\Omega)$ together; of these $2^{g-1}(2^g+1)$ are even functions of w and $2^{g-1}(2^g-1)$ are odd.

1.1.6 Lemma. $\forall\zeta \in \frac{1}{2}\mathbb{Z}^{2g}$, $\theta[\zeta](-w,\Omega) = \boldsymbol{e}^{4\pi i\zeta'\cdot\zeta''}\theta[\zeta](w,\Omega)$.

For $\zeta \in \mathbb{R}^{2g}$ the unit $\boldsymbol{e}^{4\pi i\zeta'\cdot\zeta''}$ is denoted by $e_*(\zeta)$ and due to 1.1.6 when $\zeta \in \frac{1}{2}\mathbb{Z}^{2g}$ we have $e_*(\zeta) = \pm 1$ accordingly as $\theta[\zeta](w,\Omega)$ is even or odd. The characteristics $\zeta \in \frac{1}{2}\mathbb{Z}^{2g}$ are then also refered to as *even* or *odd*. When $w=0$ the $\theta[\zeta](0,\Omega)$ for $\zeta \in \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ are functions of only Ω and are called *thetanullwerte*. For odd ζ we have $\theta[\zeta](0,\Omega) = 0$ but for even ζ the thetanullwerte play an important role. In this connection the *principal congruence subgroups* $\Gamma_n = \{\sigma \in \Gamma : \sigma \equiv I_{2g} \mod n\}$ are relevant as well as Igusa's *intermediate normal subgroups*, $\Gamma_{n,2n}$, for which $\Gamma_{2n} \lhd \Gamma_{n,2n} \lhd \Gamma_n$. For even n, $\Gamma_{n,2n}$ consists of the $\sigma = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \Gamma_n$ such that $2n$ divides the diagonals of b and c.

1.1.7 Theorem (Igusa). [10, 189] *The quotient space* $\mathcal{H}_g/\Gamma_{4,8}$ *is a complex manifold with* $\mathcal{H}_g$ *as its universal cover. The map* $I : \mathcal{H}_g/\Gamma_{4,8} \to \mathbb{P}^N$ *given by* $\Omega \mapsto \{\theta[\zeta](0,\Omega)\}_{\zeta\in\frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}}$ *is holomorphic, locally biholomorphic with its image, and injective.*

The above result of Igusa shows that for $\Omega \in \mathcal{H}_g$ the equivalence class of A as a p.p.a.v. is completely determined by the thetanullwerte $\theta[\zeta](0,\Omega)$, $\zeta \in \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$. Actually they separate the points of $\mathcal{H}_g/\Gamma_{4,8}$ which is a finite cover of $\mathcal{H}_g/\Gamma$; if we let the group Γ_2 act on $\mathcal{H}_g/\Gamma_{4,8}$ then a certain Γ_2–invariant subfield of $\mathbb{C}(\theta[\zeta])_{\zeta\in\frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}}$ determines $\mathcal{H}_g/\Gamma_2$. We will write $F_0(\mathbb{C}[\theta[\zeta]]_\zeta)$ for the subfield of $\mathbb{C}(\theta[\zeta])$ generated by quotients of homogeneous polynomials of the same degree.

1.1.8 Lemma. [13, 190,207] [10, 175-176] Γ_2 *is generated by elements of the form*

$$\begin{pmatrix} I & 2B \\ 0 & I \end{pmatrix}, \begin{pmatrix} I & 0 \\ 2C & I \end{pmatrix}, \text{ and } \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}$$

where $A = I + 2\bar{A} \in Gl_g(\mathbb{Z})$, $B = {}^tB$, $C = {}^tC$, *and* $\bar{A}$, B, C *are integral* $g \times g$ *matrices. The projective action of these generators on the thetanullwerte for* $\zeta \in \frac{1}{2}\mathbb{Z}^{2g}$ *is:*

$$\theta[\zeta](0, \sigma \cdot \Omega) = \begin{pmatrix} \boldsymbol{e}^{+2\pi i \zeta' B \zeta'} \\ \boldsymbol{e}^{\pm 4\pi i \zeta' \bar{A} \zeta''} \\ \boldsymbol{e}^{-2\pi i \zeta'' C \zeta''} \end{pmatrix} \theta[\zeta](0, \Omega).$$

For any integral $\bar{A}$ there is an integral D such that $I + 2\bar{A} + 4D \in \mathrm{Gl}_g(\mathbb{Z})$ so that $\bar{A}$ may be selected arbitrarily in the induced action of Γ_2. By the projective action of Γ_2 on the thetanullwerte we mean the action on $\frac{\theta[\zeta](0,\Omega)}{\theta[0](0,\Omega)}$. This projective action is determined by the values ${}^t\zeta \begin{pmatrix} B & \bar{A} \\ {}^t\bar{A} & C \end{pmatrix} \zeta = (\zeta \otimes \zeta) \begin{pmatrix} B & \bar{A} \\ {}^t\bar{A} & C \end{pmatrix}$ and so for a product $P = \prod_{i=1}^k \theta[\zeta_i](0, \Omega)$ the action is determined by the values $(\sum_{i=1}^k \zeta_i \otimes \zeta_i) \begin{pmatrix} B & \bar{A} \\ {}^t\bar{A} & C \end{pmatrix}$. The condition for P to be projectively invariant is then that $\sum_{i=1}^k \zeta_i \otimes \zeta_i$ send all symmetric integral $2g \times 2g$ matrices into $\mathbb{Z}$.

1.1.9 Lemma. *Let* Ω_1, $\Omega_2 \in \mathcal{H}_g$. *There exists a* $\sigma \in \Gamma_2$ *such that* $\sigma \cdot \Omega_1 = \Omega_2$ *if and only if the following conditions are satisfied:*

(1) $\forall \zeta \in \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}, \qquad \theta[\zeta](0, \Omega_1) = 0 \iff \theta[\zeta](0, \Omega_2) = 0$
(2) $\forall\, \Gamma_2$*–invariant* $\psi \in F_0(\mathbb{C}[\theta[\zeta](0, \Omega)]_\zeta)$ *such that* Ω_1, $\Omega_2 \in$ *Domain*(ψ) *we have* $\psi(\Omega_1) = \psi(\Omega_2)$.

Proof. Suppose that $\sigma \cdot \Omega_1 = \Omega_2$ for $\sigma \in \Gamma_2$. We have $\theta[\zeta](0, \Omega_2) = \theta[\zeta](0, \sigma \cdot \Omega_1) = (\text{unit})\theta[\zeta](0, \Omega_1)$ which implies (1). If ψ is Γ_2–invariant and defined at Ω_2 then $\psi(\Omega_2) = \psi(\sigma \cdot \Omega_1) = \psi(\Omega_1)$ and this verifies (2).

Now suppose that (1) and (2) hold. Since all the thetanullwerte of Ω_1 do not vanish we deduce from (1) that there exists a $\delta \in \frac{1}{2}\mathbb{Z}^{2g}$ such that $\theta[\delta](0, \Omega_1) \neq 0$ and $\theta[\delta](0, \Omega_2) \neq 0$. In what follows δ is fixed. Consider the field $E = \mathbb{C}(x_\cdot, \xi_\cdot)$ and the polynomial domain $E[z]$ where z, x_ζ, and ξ_ζ are indeterminants for all even $\zeta \in \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$. We let Γ_2 act on $E[z]$ by fixing z and the x_ζ and using the projective action of Γ_2 on the thetanullwerte to move the ξ_ζ. So for $\sigma \in \Gamma_2$ we have $(\xi_\zeta)^\sigma = u_\zeta \xi_\zeta$ if and only if we have $\frac{\theta[\zeta](0,\sigma\cdot\Omega)}{\theta[\delta](0,\sigma\cdot\Omega)} = u_\zeta \frac{\theta[\zeta](0,\Omega)}{\theta[\delta](0,\Omega)}$. Define an element $r = \sum_\zeta x_\zeta \xi_\zeta \in E$ and the polynomial P of degree $d = [\Gamma_2 : \Gamma_{4,8}]$ by

$$P(z; \vec{x}; \vec{\xi}) = \prod_{\sigma \in \Gamma_{4,8} \diagdown \Gamma_2} (z - r^\sigma).$$

$P(z; \vec{x}; \vec{\xi})$ is Γ_2–invariant because the ξ_ζ are $\Gamma_{4,8}$–invariant by the existence of Igusa's map I in 1.1.7 and because Γ_2 just permutes the cosets of $\Gamma_{4,8} \diagdown \Gamma_2$. If

P is written as a polynomial in z and the x_ζ with coefficients in $\mathbb{C}[\xi.]$, $P(z;\vec{x};\vec{\xi}) = \sum_i \sum_{\text{multi-index }\alpha} q_{i;\alpha}(\xi.)z^i(x.)^\alpha$, then we conclude that the polynomials $q_{i;\alpha}(\xi.)$ must be Γ_2–invariant. Hence $q_{i;\alpha}(\frac{\theta[\zeta](0,\Omega)}{\theta[\delta](0,\Omega)})$ is a Γ_2–invariant element of $F_0(\mathbb{C}[\theta[\zeta]]_\zeta)$ which is defined for both Ω_1 and Ω_2. By (2) we have $q_{i;\alpha}(\frac{\theta[\zeta](0,\Omega_1)}{\theta[\delta](0,\Omega_1)}) = q_{i;\alpha}(\frac{\theta[\zeta](0,\Omega_2)}{\theta[\delta](0,\Omega_2)})$ or $P(z;\vec{x};\frac{\theta[\zeta](0,\Omega_1)}{\theta[\delta](0,\Omega_1)}) = P(z;\vec{x};\frac{\theta[\zeta](0,\Omega_2)}{\theta[\delta](0,\Omega_2)})$. All d roots of these two polynomials are known to us and must coincide; hence there exists a $\sigma_1 \in \Gamma_2$ such that in $\mathbb{C}(x.)$ we have:

$$\sum_\zeta x_\zeta \frac{\theta[\zeta](0,\Omega_1)}{\theta[\delta](0,\Omega_1)} = \sum_\zeta x_\zeta \frac{\theta[\zeta](0,\sigma_1\cdot\Omega_2)}{\theta[\delta](0,\sigma_1\cdot\Omega_2)}.$$

This implies for all ζ that $\frac{\theta[\zeta](0,\Omega_1)}{\theta[\delta](0,\Omega_1)} = \frac{\theta[\zeta](0,\sigma_1\cdot\Omega_2)}{\theta[\delta](0,\sigma_1\cdot\Omega_2)}$ and that $I(\Omega_1) = I(\sigma_1\cdot\Omega_2)$ in $\mathbb{P}^N$. By Igusa's theorem 1.1.7 there exists a $\sigma_2 \in \Gamma_{4,8}$ such that $\Omega_1 = \sigma_2\cdot(\sigma_1\cdot\Omega_2)$ and this concludes the proof.

The second order theta functions have a theory parallel to that of the first order theta functions. They transform by $e(\zeta,\cdot)\,\xi^2$ and this characterizes them. The vector $\vec{\theta}_2(w,\Omega)$ is a convenient basis for the vector space of analytic functions which transform by ξ^2, each member of which is an even function. The second order theta functions have no common zeros. The two parallel theories are intertwined in the addition formula of Weierstraß and its inversion which are given below.

1.1.10 Definition (Second order theta functions). *Let $\Omega \in \mathcal{H}_g$ and $\zeta \in \mathbb{R}^{2g}$. A* second order *theta function with* characteristic *ζ is defined $\forall\, w \in \mathbb{C}^g$ by:*

$$\theta_2[\zeta](w,\Omega) = \theta[\tfrac{\zeta'}{2}|\zeta''](2w,2\Omega).$$

1.1.11 Proposition. [13, 124] *For all $l = (\Omega\ I)\lambda \in \mathcal{L}$ we have $\theta_2[\zeta](w+l,\Omega) = e(\zeta,\lambda)\xi(l,w)^2\theta_2[\zeta](w,\Omega)$.*

1.1.12 Definition. *The* vector *of second order theta functions $\vec{\theta}_2(w,\Omega)$ is defined as $\{\theta_2[\nu|0](w,\Omega),\quad \nu \in \mathbb{Z}^g/2\mathbb{Z}^g\}$.*

1.1.13 Proposition. [13, 124] *Let $f : \mathbb{C}^g \to \mathbb{C}$ be analytic and satisfy: $\forall l = (\Omega\ I)\lambda \in \mathcal{L}$, $f(w+l) = \xi(l,w)^2 f(w)$; then $\exists_1\, \vec{c} \in \mathbb{C}^{2^g} : f(w) = \vec{c}\cdot\vec{\theta}_2(w,\Omega)$.*

1.1.14 Lemma. [10, 168] *Let $\Omega \in \mathcal{H}_g$. Then for all $w \in \mathbb{C}^g$, $\vec{\theta}_2(w,\Omega) \neq 0$.*

1.1.15 Proposition (Weierstraß's addition formula). [4, 3] *Let $\Omega \in \mathcal{H}_g$.*

$$\forall\ x,y \in \mathbb{C}^g, \qquad \vec{\theta}_2(x,\Omega)\cdot\vec{\theta}_2(y,\Omega) = \theta(x+y,\Omega)\,\theta(x-y,\Omega).$$

1.1.16 Proposition (Inversion). [4, 3] *Let $\Omega \in \mathcal{H}_g$. $\forall x,y \in \mathbb{C}^g, \forall \nu_1,\nu_2 \in \mathbb{R}^g$,*

$$\theta_2[\nu_1|0](x,\Omega)\,\theta_2[\nu_2|0](y,\Omega) = 2^{-g}\sum_{p\in\mathbb{Z}^g/2\mathbb{Z}^g} \boldsymbol{e}^{-2\pi i p\cdot\nu_1}\theta[\tfrac{\nu_1+\nu_2}{2}|\tfrac{p}{2}](x+y,\Omega)\theta[\tfrac{\nu_1-\nu_2}{2}|\tfrac{p}{2}](x-y,\Omega).$$

Finally, a few words about göpel systems. Their main significance lies in the fact that for any $\Omega \in \mathcal{H}_g$ there will be at least one thetanullwerte in every göpel system which does not vanish.

1.1.17 Definition. *A göpel system $\Sigma \subset \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ is a set of 2^g even characteristics $\zeta \in \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ such that $\Sigma + \Sigma + \Sigma = \Sigma$.*

1.1.18 Lemma. [10, 219-220] *Let $\Omega \in \mathcal{H}_g$ and let $\Sigma \subset \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ be a göpel system. Then $\exists \zeta \in \Sigma : \theta[\zeta](0, \Omega) \neq 0$.*

The typical göpel system is $\Sigma = \{\, [\nu|0] : \nu \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g\}$, and the addition formula 1.1.15 can be seen to imply 1.1.18 in this case. I have tried to string the formulas together with some prose but here are two that didn't fit in.

1.1.19 Lemma. $\forall \Omega \in \mathcal{H}_g$, $w \in \mathbb{C}^g$, $\zeta, \eta \in \mathbb{R}^{2g}$,

$$\theta[\zeta + \eta](w, \Omega) = \boldsymbol{e}^{2\pi i \eta' \cdot \{w + \zeta'' + \eta'' + \frac{1}{2}\Omega\eta'\}} \theta[\zeta](w + (\Omega\, I)\eta, \Omega).$$

1.1.20 Lemma. $\forall x, a, b, c, d \in \mathbb{R}^{2g}$,

$$\frac{\theta((\Omega\, I)(x + a - c), \Omega)}{\theta((\Omega\, I)(x + a - d), \Omega)} \frac{\theta((\Omega\, I)(x + b - d), \Omega)}{\theta((\Omega\, I)(x + b - c), \Omega)} =$$
$$\frac{\theta[x]((\Omega\, I)(a - c), \Omega)}{\theta[x]((\Omega\, I)(a - d), \Omega)} \frac{\theta[x]((\Omega\, I)(b - d), \Omega)}{\theta[x]((\Omega\, I)(b - c), \Omega)} = \boldsymbol{e}^{2\pi i (a-b)'\Omega(c-d)'}$$
$$\boldsymbol{e}^{2\pi i \{(a-b)' \cdot (c-d)'' + (c-d)' \cdot (a-b)''\}} \frac{\theta[x + a - c](0, \Omega)}{\theta[x + a - d](0, \Omega)} \frac{\theta[x + b - d](0, \Omega)}{\theta[x + b - c](0, \Omega)}.$$

§1.2 Riemann Surfaces, Jacobians, and Riemann's Vanishing Theorem.

Let M be a compact Riemann surface of genus $g \geq 1$. Let $A_1, \ldots, A_g, B_1, \ldots, B_g$ be "canonical" generators for $\pi_1(M, p_0)$; so that we have $\prod_{i=1}^{g}(A_i B_i {A_i}^{-1} {B_i}^{-1}) = 1$. We also take $[A_i]$, $[B_i]$, as a basis for $H_1(M, \mathbb{Z})$ so that the intersection pairing in this basis is $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We immediately go up to the universal cover $\widehat{M}$ with $\pi : \widehat{M} \to M$ and make our constructions on this "marked" surface. A *marking* of M is a choice of $z_0 \in \widehat{M}$ such that $\pi(z_0) = p_0$ and a choice of canonical generators for $\pi_1(M, p_0)$. We select a *base point* $z_0 \in \widehat{M}$ in order to identify the fundamental group $\pi_1(M, p_0)$ with $G = \mathrm{Deck}(\widehat{M}/M)$ in the following manner: given $T \in G = \mathrm{Deck}(\widehat{M}/M)$ any path δ_T from z_0 to Tz_0 defines $\pi\delta_T \in \pi_1(M, p_0)$.

Standard existence theorems produce certain meromorphic 1–forms on M and meromorphic functions on copies of $\widehat{M}$. The 1–forms when pulled back to $\widehat{M}$ will be G–invariant but the functions on $(\widehat{M})^r$ that we consider will transform by certain factors of automorphy with respect to G. We mention only those analytic objects we will need later.

1.2.1 Proposition. *The complex vector space of holomorphic 1–forms on M, $H^0(M, \mathcal{O}^{1,0})$, has dimension g. Elements of this space are called* abelian *differentials.*

1.2.2 Lemma. *Given a marked Riemann surface M with $g \geq 1$ there exists a unique basis $\omega_1, \dots, \omega_g$ of abelian differentials called a* canonical *basis such that* $\int_{A_j} \omega_i = \delta_{ij}$.

1.2.3 Definition. *For all $t \in \mathbb{C}^g$ define the character $\rho_t : G \to \mathbb{C}^{\cdot}$ by $\rho_t(A_i) = 1$, and $\rho_t(B_i) = \mathbf{e}^{2\pi i t_i}$.*

1.2.4 Proposition (Gunning). [5, 25] *Given a marked Riemann surface M there exists a unique meromorphic function p on $(\widehat{M})^4$ such that:*

(1) *The only zeros and poles of $p(z_1, z_2, a_1, a_2)$ are simple zeros when $z_1 = a_1$ or $z_2 = a_2 \mod G$, and simple poles when $z_1 = a_2$ or $z_2 = a_1 \mod G$. The function $p(z_1, z_2, a_1, a_2)$ is identically 1 when $z_1 = z_2$ or $a_1 = a_2$.*
(2) $p(z_1, z_2, a_1, a_2) = p(a_1, a_2, z_1, z_2) = p(z_2, z_1, a_1, a_2)^{-1} = p(z_1, z_2, a_2, a_1)^{-1}$
(3) *For $T \in G$, $p(Tz_1, z_2, a_1, a_2) = \rho_t(T) p(z_1, z_2, a_1, a_2)$ where $t_j = \int_{a_2}^{a_1} \omega_j$.*

1.2.5 Proposition (Gunning). [8, 52-53] *Given a marked Riemann surface M there exists a holomorphic function q on $(\widehat{M})^2$ such that:*

(1) *The only zeros of $q(z_1, z_2)$ are simple zeros when $z_1 = z_2 \mod G$.*
(2) $q(z_1, z_2) = -q(z_2, z_1)$
(3) $p(z_1, z_2, a_1, a_2) = \frac{q(z_1,a_1)}{q(z_1,a_2)} \frac{q(z_2,a_2)}{q(z_2,a_1)}$

The function $p(z_1, z_2, a_1, a_2)$ is *Gunning's crossratio function.* It is the generalization of the usual crossratio $\langle z_1, z_2, a_1, a_2 \rangle = \frac{z_1 - a_1}{z_1 - a_2} \frac{z_2 - a_2}{z_2 - a_1}$ on $\mathbb{P}^1$ and will figure prominently in all that follows. The function q is Gunning's *prime function* and is closely realted to the prime function of Klein and Fay. We have not listed its most important properties and merely want to point out that the crossratio function can be factorized as in (3) by a skew function q. From the existence of such a factorization we immediately verify that, just like $\langle z_1, z_2, a_1, a_2 \rangle$, $p(z_1, z_2, a_1, a_2)$ satisfies the following symmetries which will henceforth be termed the *crossratio symmetries.* Notice that $\langle z_1, z_2, a_1, a_2 \rangle$ satisfies the further relation $\langle z_1, z_2, a_1, a_2 \rangle + \langle z_1, a_1, z_2, a_2 \rangle = 1$; Fay's trisecant formula is the generalization of this, see [15].

1.2.6 Lemma (The crossratio symmetries). *Let M be a marked Riemann surface. Gunning's crossratio function p satisfies: $\forall$ distinct $I, J, K, L, N \in \widehat{M}$,*

(1) $p(I, J, K, L) = p(J, I, L, K) = p(K, L, I, J) = p(L, K, J, I)$
(2) $p(I, J, K, L) p(J, I, K, L) = 1$
(3) $p(I, J, K, L) p(I, K, L, J) p(I, L, J, K) = -1$
(4) $p(I, J, K, L) p(I, J, L, N) = p(I, J, K, N)$.

We now construct the Jacobian of M, $\mathrm{Jac}(M)$. Use the vector $\vec{\omega} = (\omega_1, \dots, \omega_g)$ formed by the canonical basis of abelian differentials to construct the Abel–Jacobi map $w : \widehat{M} \to \mathbb{C}^g$, $w(z) = \int_{z_0}^{z} \vec{\omega}$. The deck group G acts on w via translations in $\mathbb{C}^g$; for $T \in G$ we have $w(Tz) = w(z) + \int_T \vec{\omega}$. These translations form a lattice in $\mathbb{C}^g$ given by $\mathcal{L} = \{\int_T \vec{\omega}, T \in G\}$ and a $\mathbb{Z}$–basis for $\mathcal{L}$ is $(\int_{A_j} \omega_i, \int_{B_j} \omega_i) \stackrel{\text{def}}{=} (I\,\Omega)$. We can now view w as a map from M to $\mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g)$ which is the Jacobian of M, $\mathrm{Jac}(M)$.

1.2.7 Proposition. [13, 142] *Let M be a marked Riemann surface, then $\Omega_{ij} = \int_{B_j} \omega_i \in \mathcal{H}_g$ and Jac(M) is a principally polarized abelian variety.*

If the marking of M is changed then a different period matrix Ω results. The period matrix Ω is independent of the base point and dependent only on the homology classes $[A_i]$ and $[B_i]$. A different homology basis $\tilde{A}, \tilde{B}$ which preserves the intersection matrix J will be related to A, B by $\begin{pmatrix} \tilde{A} \\ \tilde{B} \end{pmatrix} = \sigma \begin{pmatrix} A \\ B \end{pmatrix}$ for $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_g(\mathbb{Z})$. This will change Ω to $\tilde{\Omega} = (c + d\Omega)(a + b\Omega)^{-1}$ and consequently the equivalence class of the p.p.a.v. $\mathrm{Jac}(M)$ depends only upon M and not upon the marking. The Abel–Jacobi map will become $\tilde{w}(z) = ({}^t a + \Omega^t b)^{-1} w(z)$.

Since $\mathrm{Jac}(M)$ is an abelian group we can extend the Abel–Jacobi map w to a homomorphism $w : \mathrm{Div}(M) \to \mathrm{Jac}(M)$ where $\mathrm{Div}(M)$ is the free abelian group on points of M. The map from divisors of degree zero is independent of the base point and by Abel's theorem has kernel equal to the principal divisors. For $D \in \mathrm{Div}(M)$ elements $[D] \in \mathrm{Pic}(M) = \mathrm{Div}(M)/(\text{principal divisors})$ are called *divisor classes.*

1.2.8 Proposition. [2, 92] *$w : Pic^0(M) \to Jac(M)$ is a group isomorphism.*

When we are considering a Jacobian the theta functions on it are often called Jacobian instead of or as well as abelian theta functions. There is an important theorem due to Riemann which relates the vanishing of the Jacobian theta function to the dimensions of sections of divisor classes on M. Recall that the *canonical divisor class* $K_M \in \mathrm{Pic}^{2g-2}(M)$ is the divisor class of any meromorphic differential.

1.2.9 Definition. *For $D \in Div(M)$ let $\mathcal{L}(D) = \{f$ meromorphic on $M : div(f) + D \geq 0\}$.*

1.2.10 Theorem (Riemann Vanishing). [2, 298] *Let M be a marked compact Riemann surface of genus $g \geq 1$ and let J be its Jacobian. $\exists r \in \mathbb{C}^g : [2r] = w(K_M)$ in J and for all $D \in Div^{g-1}(M)$*

$$ord_{z=0}\theta(r - w(D) + z, \Omega) = dim_{\mathbb{C}}\mathcal{L}(D)$$

We follow [7] in calling r the riemann point. The classical terminology [2, 290] is the vector of Riemann constants.

1.2.11 Definition (Theta characteristic). *A divisor class $\chi \in Pic^{g-1}(M)$ such that $2\chi = K_M$ in $Pic^{2g-2}(M)$ is called a theta characteristic.*

These theta characteristics exist and hence there are 2^{2g} of them. One of these theta characteristics is mapped onto $[r]$ by w and we can rewrite Riemann's vanishing theorem in terms of it.

1.2.12 Corollary. *Let $\chi \in Pic^{g-1}(M)$ such that $w(\chi) = [r]$ then for all $E \in Pic^0(M)$ we have $ord_0\theta(w(E) + z, \Omega) = dim_{\mathbb{C}}\mathcal{L}(\chi - E)$.*

We may describe the theta locus Θ of $\mathrm{Jac}(M)$ quite explicitly in these terms.

1.2.13 Corollary. [13, 164] *Let J be the Jacobian of a marked Riemann surface, M, of genus $g \geq 1$. Then the theta locus $\Theta = \{[z] \in J : \theta(z, \Omega) = 0\}$ is irreducible and equal to $\{[r] - w(D) \in J : D \in Div^{g-1}(M)$, and $\mathcal{L}(D) \neq 0\}$.*

§1.3 Addition Theorems.

Powerful identities special to Jacobian theta functions center around the trisecant formula of Fay [4, 34]. Identities which can be formulated solely in terms of the period matrix Ω are then necessary conditions for an abelian variety to be a Jacobian.

1.3.1 Definition. [8, 53] *Let M be a marked Riemann surface. Let $K = \{k_i\}_{i=1}^n$, $L = \{l_i\}_{i=1}^n$ for $a, b, k_i, l_i \in \widehat{M}$. The* higher crossratio functions *$p(a, b; K; L)$ are defined by*

$$p(a, b; K; L) = \prod_{i=1}^{n} p(a, b, k_i, l_i).$$

For $n = 1$ we simply have Gunning's crossratio function from §1.2 because $p(a, b; \{k_1\}; \{l_1\}) = p(a, b, k_1, l_1)$. The higher crossratio functions occur naturally in the inhomogeneous forms of the multisecant identity. In the homogeneous forms the expression $c(K; L)$ occurs and is of course related to the higher crossratio functions.

1.3.2 Definition. *Let M be a marked Riemann surface. Let $K = \{k_i\}_{i=1}^N$, $L = \{l_i\}_{i=1}^N$ for $k_i, l_i \in \widehat{M}$. Define $c(K; L)$ by 1.3.2 or inductively by 1.3.3 .*

$$c(K; L) = \frac{\prod\limits_{1 \leq m < n \leq N} q(k_m, k_n) q(l_n, l_m)}{\prod\limits_{1 \leq m, n \leq N} q(k_m, l_n)} \tag{1.3.2 }$$

(1.3.3)

$$c(K; L) p(k_i, l_j; L \backslash l_j; K \backslash k_i) = (-1)^{i+j} c(K \backslash k_i; L \backslash l_j) / q(k_i, l_j)$$

1.3.4 Proposition (Fay). [4, 33] *Let $J = \mathbb{C}^g / (\mathbb{Z}^g + \Omega \mathbb{Z}^g)$ be the Jacobian of a marked Riemann surface M with genus $g \geq 1$ and universal cover $\widehat{M}$, $\pi : \widehat{M} \to M$. Let $w : \widehat{M} \to \mathbb{C}^g$ be the Abel–Jacobi map. Let $\pi K \cap \pi L = \emptyset$ for $K = \{k_i\}_{i=1}^N$, $L = \{l_i\}_{i=1}^N$ where $k_i, l_i \in \widehat{M}$. Then for all $y \in \mathbb{C}^g$ the following is true.*

$$\det_{1 \leq m, n \leq N} \{ \frac{\theta(y + w(k_m - l_n), \Omega)}{q(k_m, l_n)} \} = c(K; L) \, \theta(y, \Omega)^{N-1} \theta(y + w(K - L), \Omega)$$

When we write $w(K - L)$ we treat K and L as positive divisors. We may expand the determinant in 1.3.4 by minors of the i^{th} row and use 1.3.4 again to evaluate these minors. We then use 1.3.3 to write the coefficients in terms of the higher crossratio functions p.

1.3.5 Proposition (Multisecant formula). *Let the notation be as in proposition 1.3.4. For any i, j we have:*

$$\theta(y,\Omega)\,\theta(y+w(K-L),\Omega) = \sum_{j=1}^{N} p(k_i,l_j;L\backslash l_j;K\backslash k_i)\,\theta(y+w(k_i-l_j),\Omega)\,\theta(y+w(K\backslash k_i - L\backslash l_j),\Omega).$$

This may be rewritten by applying Weierstraß's addition formula 1.1.15.

1.3.6 Proposition. [8, 47] *Let the notation be as in proposition 1.3.4. For $t_i = \frac{1}{2}w(K\backslash k_i - L - k_i)$ we have:*

$$\vec{\theta}_2(t_i+w(k_i),\Omega) = \sum_{j=1}^{N} p(k_i,l_j;L\backslash l_j;K\backslash k_i)\vec{\theta}_2(t_i+w(l_j),\Omega).$$

1.3.7 Corollary. *For $t_i = \frac{1}{2}w(K\backslash k_i - L - k_i)$ we have:*

$$\mathrm{rank}\{\vec{\theta}_2(t_i+w(k_i),\Omega),\vec{\theta}_2(t_i+w(l_1),\Omega),\ldots,\vec{\theta}_2(t_i+w(l_N),\Omega)\} \le N.$$

This rank condition is useful in studying general abelian varieties, at least when the $w(k_i)$, $w(l_i)$ may be written in terms of Ω, because it does not involve the higher crossratio functions. R. Gunning has proven [7] [8, 43] a remarkable converse to the above corollary which says that if $t_i \in \mathbb{C}^g$ satisfies the above rank condition for a Jacobian then $2t_i$ is of the required form. Fay's trisecant formula is the case $N = 2$ of 1.3.4–1.3.7.

1.3.8 Proposition (Fay's trisecant formula). [4, 34] *Let $J = \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g)$ be the Jacobian of a marked Riemann surface M with genus $g \ge 1$. Let $w: \widehat{M} \to \mathbb{C}^g$ be the Abel–Jacobi map. For $y \in \mathbb{C}^g$ and $z_1, z_2, a_1, a_2 \in \widehat{M}: \{\pi z_1, \pi a_2\} \cap \{\pi z_2, \pi a_1\} = \emptyset$ we have:*

(1.3.9)

$$\begin{vmatrix} \dfrac{\theta(y+w(z_1-z_2),\Omega)}{q(z_1,z_2)} & \dfrac{\theta(y+w(a_2-z_2),\Omega)}{q(a_2,z_2)} \\ \dfrac{\theta(y+w(z_1-a_1),\Omega)}{q(z_1,a_1)} & \dfrac{\theta(y+w(a_2-a_1),\Omega)}{q(a_2,a_1)} \end{vmatrix} = \frac{q(z_1,a_2)q(a_1,z_2)}{q(z_1,z_2)q(z_1,a_1)q(a_2,z_2)q(a_2,a_1)}\theta(y,\Omega)\theta(y+w(z_1+a_2-z_2-a_1),\Omega)$$

(1.3.10)

$$\theta(y,\Omega)\,\theta(y+w(z_1+a_2-z_2-a_1),\Omega) = p(z_1,z_2,a_1,a_2)\,\theta(y+w(z_1-z_2),\Omega)\,\theta(y+w(a_2-a_1),\Omega) + p(z_1,a_1,z_2,a_2)\,\theta(y+w(z_1-a_1),\Omega)\,\theta(y+w(a_2-z_2),\Omega)$$

(1.3.11)

$$\vec\theta_2(\tfrac{1}{2}w(z_1+a_2-z_2-a_1),\Omega) =$$
$$p(z_1,z_2,a_1,a_2)\,\vec\theta_2(\tfrac{1}{2}w(z_1+a_1-z_2-a_2),\Omega) + p(z_1,a_1,z_2,a_2)\,\vec\theta_2(\tfrac{1}{2}w(z_1+z_2-a_1-a_2),\Omega)$$

(1.3.12)

$$\operatorname{rank}\{\vec\theta_2(\tfrac{1}{2}w(z_1+a_2-z_2-a_1)),\vec\theta_2(\tfrac{1}{2}w(z_1+a_1-z_2-a_2)),\vec\theta_2(\tfrac{1}{2}w(z_1+z_2-a_1-a_2))\} \le 2.$$

We can obtain a nice expression for Gunning's crossratio function p by substituting $y = \alpha + w(a_1 - z_1)$ in 1.3.10. If $\alpha \in \Theta$ and α is not in $\Theta^{(1)}$, the singular locus of Θ, then $\theta(\alpha+w(p-q)) \not\equiv 0$, see [2, 298]. The term $\theta(y+w(z_1-a_1),\Omega)$ in 1.3.10 vanishes and we have $\theta(\alpha+w(a_1-z_1),\Omega)\,\theta(\alpha+w(a_2-z_2)),\Omega) = p(z_1,z_2,a_1,a_2)\,\theta(\alpha+w(a_1-z_2),\Omega)\,\theta(\alpha+w(a_2-z_1),\Omega)$ which gives the following corollary.

1.3.13 Corollary (Gunning). [7, 155] *For all $\alpha \in \Theta \sim \Theta^{(1)}$ p is given as a meromorphic function on $\widehat{M}^4$ via:*

$$p(a_1,a_2,z_1,z_2) = \frac{\theta(\alpha+w(a_1-z_1),\Omega)}{\theta(\alpha+w(a_1-z_2),\Omega)}\frac{\theta(\alpha+w(a_2-z_2),\Omega)}{\theta(\alpha+w(a_2-z_1),\Omega)}.$$

We can obtain a similar expression for $p(a,b;A;B)$ when $|A| = |B| = g$ by substituting $y = r - w(K - k_i - l_i)$ for $N = g+1$ into 1.3.5. Again all but two of the terms vanish and we have:

$$\theta(r-w(K-k_i-l_i),\Omega)\theta(r-w(L-k_i-l_i),\Omega) =$$
$$p(k_i,l_i;L\backslash l_i;K\backslash k_i)\theta(r+w(k_i-K\backslash k_i),\Omega)\theta(r+w(l_i-L\backslash l_i),\Omega).$$

We rewrite this to obtain the next corollary by making the denotations $B = K\backslash k_i$, $A = L\backslash l_i$, $k = k_i$, and $l = l_i$.

1.3.14 Corollary. *Let r be the riemann point. The higher crossratio function p for $|A| = |B| = g$ is given as a meromorphic function on $\widehat{M}^{2g+2}$ via:*

$$p(k,l;A;B) = \frac{\theta(r+w(k)-w(A),\Omega)}{\theta(r+w(k)-w(B),\Omega)}\frac{\theta(r+w(l)-w(B),\Omega)}{\theta(r+w(l)-w(A),\Omega)}.$$

§1.4 Hyperelliptic Review.

We review the standard results on hyperelliptic curves as given by D. Mumford in [14] and follow this notation (almost) entirely. It is important to point out that all of the constructions of this section are well–defined once $2g+2$ distinct points of $\mathbb{P}^1$ are given. We will use this fact in §2.4. Let $g \ge 1$ and let $a_1, a_2, \ldots a_{2g+1}, a_\infty$ be $2g+2$ distinct points of $\mathbb{P}^1$. If none of the a_i are ∞ then we construct the Riemann surface M associated to the plane curve $y^2 = \prod_{i\in B}(x-a_i)$ where $B = \{1,2,\ldots 2g+1,\infty\}$. If one of the a_i is ∞ we permute them so that $a_\infty = \infty$ and let M be given by $y^2 = \prod_{i\in B\backslash\infty}(x-a_i)$. The Riemann surface is *hyperelliptic* because x has degree 2 as a function from M to $\mathbb{P}^1$; all hyperelliptic Riemann surfaces are known to be given in this way. In the first case y has degree $2g+2$ and in the second case $2g+1$.

The *hyperelliptic involution* is the automorphism, $\mathcal{I}$, of M which sends (x, y) to $(x, -y)$. For $g \geq 2$ the function x is the unique function of degree 2 on M up to linear fractional transformations, and so the unordered crossratios of the branch points of x, $\langle a_i, a_j, a_k, a_l \rangle = \frac{(a_i - a_k)}{(a_i - a_l)} \cdot \frac{(a_j - a_l)}{(a_j - a_k)}$, give true invariants of M. Conversely, a choice of a_i and hence M can be recovered from these crossratios so that there are $2g + 2 - 3 = 2g - 1$ complex parameters in the family of hyperelliptic curves of genus g. For $g = 1$ the crossratios of the four a_i also specify M [9, 318].

The $(a_i, 0)$ are the ramification points of x on M and we denote these by their images in $\mathbb{P}^1$, a_i, as is traditional. If we denote the divisor at infinity by L_∞ then L_∞ is of the form $\infty_1 + \infty_2$ in the first case and 2∞ in the second. We use $L_\infty = \infty + \mathcal{I}\infty$ for a notation which includes both cases. The divisor of y is $a_1 + \cdots + a_{2g+1} + a_\infty - (g+1)L_\infty$ and the divisor of $x - x(p)$ is $p + \mathcal{I}p - L_\infty$; hence $L = [L_\infty] \in \mathrm{Pic}^2(M)$ is a divisor class given by $p + \mathcal{I}p$ for any $p \in M$ and is called the *hyperelliptic point*. An unnormalized basis of abelian differentials is given by $dx/y, x^1 dx/y, \ldots, x^{g-1} dx/y$, and since $\mathrm{div}(dx/y) = (g-1)L_\infty$, the canonical class $K_M \in \mathrm{Pic}^{2g-2}(M)$ is given by $(g-1)L$. The integral domain of meromorphic functions with poles only on multiples of L_∞ is $\mathbb{C}[x, y]$, and this knowledge can be used to compute dimensions of sections of divisor classes. We now give the parameterization of the two–torsion and the theta characteristics in $\mathrm{Pic}(M)$ as presented in [14].

1.4.1 Definition. $\forall S \subseteq B : |S| \equiv 0 \mod 2$, *define* $e_S \in \mathit{Pic}^0(M)$ *by*

$$e_S = \sum_{i \in S} a_i - \frac{|S|}{2} L.$$

1.4.2 Definition. $\forall T \subseteq B : |T| \equiv g - 1 \mod 2$, *define* $f_T \in \mathit{Pic}^{g-1}(M)$ *by*

$$f_T = \sum_{i \in T} a_i + \frac{g - 1 - |T|}{2} L.$$

1.4.3 Lemma. [14, 3.32] *In* $\mathit{Pic}^0(M)$ *we have for all* $S, S_1, S_2 \subseteq B$ *such that* $|S| \equiv |S_1| \equiv |S_2| \equiv 0 \mod 2$,

1. $2e_S = 0$
2. $e_{S_1} + e_{S_2} = e_{S_1 \circ S_2}$ *(where* $S_1 \circ S_2 = S_1 \backslash S_2 \amalg S_2 \backslash S_1$*)*
3. $e_{S_1} = e_{S_2} \iff S_1 = S_2$ *or* $S_1 = {S_2}^c$
4. *Hence the set of* e_S *in* $\mathit{Pic}^0(M)$ *forms a group isomorphic to* $(\mathbb{Z}/2\mathbb{Z})^{2g}$.

1.4.4 Lemma. [14, 3.95] *In* $\mathit{Pic}^{g-1}(M)$ *we have for all* $S, T, T_1, T_2 \subseteq B$ *such that* $|T| \equiv |T_1| \equiv |T_2| \equiv g - 1 \mod 2$, $|S| \equiv 0 \mod 2$,

1. $2f_T = K_M = (g-1)L$ *in* $\mathit{Pic}^{2g-2}(M)$
2. $f_T + e_S = f_{T \circ S}$
3. $f_{T_1} = f_{T_2} \iff T_1 = T_2$ *or* $T_1 = {T_2}^c$
4. *The set of* f_T *in* $\mathit{Pic}^{g-1}(M)$ *gives all theta characteristics for* M.
5. $\dim_{\mathbb{C}} \mathcal{L}(f_T) = \lfloor \frac{g+1-|T|}{2} \rfloor$.

The set $B = \{1, 2, \dots, 2g+1, \infty\}$ forms a group under the symmetric difference, $\circ$, and lemma 1.4.3 says that $\{S \subseteq B : |S| \equiv 0 \mod 2, \circ\}/\{\emptyset, B\}$ is isomorphic to the two–torsion in $\mathrm{Pic}^0(M)$. Lemma 1.4.4 says that $\{T \subseteq B : |T| \equiv g-1 \mod 2\}/\{\emptyset, B\}$ is isomorphic to the set of theta characteristics in $\mathrm{Pic}^{g-1}(M)$ as a homogeneous space. Item (5) of lemma 1.4.4 is remarkable and important but we do not repeat the proof here.

Mark the Riemann surface M as in §1.2 so that we have an Abel–Jacobi map $w : \mathrm{Div}(\widehat{M}) \to \mathbb{C}^g$, and $w : \mathrm{Pic}^0(M) \to \mathbb{C}^g/\mathcal{L}$, where $\mathcal{L} = \mathbb{Z}^g + \Omega\mathbb{Z}^g$. We choose lifts $\hat{a}_i \in \widehat{M}$ such that $\pi\hat{a}_i = a_i \in M$ and *we choose the base point of our marking to be* $\hat{a}_\infty$. Since $a_i - a_\infty$ is two–torsion in $\mathrm{Pic}^0(M)$ we conclude that $w(\hat{a}_i) \in \frac{1}{2}\mathcal{L}$.

1.4.5 Definition. *For all $i \in B$, define $\eta_i \in \frac{1}{2}\mathbb{Z}^{2g}$ by $w(\hat{a}_i) = (\Omega\, I)\eta_i$. For all $T \subseteq B$, define $\eta_T = \sum_{i\in T} \eta_i$ so that we have $w(\sum_T \hat{a}_i) = (\Omega\, I)\eta_T$.*

The existence of point sections in the hyperelliptic case allows us to define these η_i and to formulate Jacobian theta identities for general abelian theta functions. We now describe the possible maps η abstractly. The properties of the e_S in lemma 1.4.3 become properties of the η_S.

1.4.6 Lemma. *In $\frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ we have for all $S, S_1, S_2 \subseteq B$ such that $|S| \equiv |S_1| \equiv |S_2| \equiv 0 \mod 2$,*

(1) $2\eta_S = 0$
(2) $\eta_{S_1} + \eta_{S_2} = \eta_{S_1 \circ S_2}$
(3) $\eta_{S_1} = \eta_{S_2} \iff S_1 = S_2$ *or* $S_1 = {S_2}^c$
(4) *A group isomorphism of $\{S \subseteq B : |S| \equiv 0 \mod 2, \circ\}/\{\emptyset, B\}$ to $\frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ is given by the map $S \mapsto \eta_S$.*

1.4.7 Definition. *For all $\zeta, \xi \in \mathbb{R}^{2g}$, define e_2 by $e_2(\zeta, \xi) = \mathbf{e}^{4\pi i\, {}^t\zeta J \xi}$.*

1.4.8 Lemma. *For all $\zeta, \xi \in \mathbb{R}^{2g}$, we have* $e_2(\zeta, \xi) = \dfrac{e_*(\zeta + \xi)}{e_*(\zeta)\, e_*(\xi)}$.

This e_* was defined after lemma 1.1.6.

1.4.9 Proposition. [14] *Let M be a marked hyperelliptic curve given by $y^2 = \prod_{i\in B}(x - a_i)$ with $\hat{a}_\infty \in \widehat{M}$ as the base point. There exists a map $\eta : B \to \frac{1}{2}\mathbb{Z}^{2g}$ and a $U \subseteq B$ such that $\eta_S = \sum_{i\in S} \eta_i$, $\eta_\infty = \vec{0}$, and $|U| \equiv g+1 \mod 4$, and such that the following conditions hold.*

(1) $\eta : \{S \subseteq B : |S| \equiv 0 \mod 2, \circ\}/\{\emptyset, B\} \cong \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$.
(2) $e_*(\eta_S) = (-1)^{\frac{g+1-|S\circ U|}{2}}$, $\quad \forall\, |S| \equiv 0 \mod 2$.
(3) $e_2(\eta_{S_1}, \eta_{S_2}) = (-1)^{|S_1 \cap S_2|}$, $\quad \forall\, |S_1|, |S_2| \equiv 0 \mod 2$.
(4) $w(e_S) = (\Omega I)\eta_S$ *in* $Jac(M)$, $\quad \forall\, |S| \equiv 0 \mod 2$.
(5) $w(f_T) = (\Omega I)\eta_T$ *in* $Jac(M)$, $\quad \forall\, |T| \equiv g-1 \mod 2$.
(6) $w(f_U) = (\Omega I)\eta_U = r =$ *the riemann point in* $Jac(M)$.

Proof. Item (1) is item (4) of lemma 1.4.6 and $\eta_\infty = \vec{0}$ because $\hat{a}_\infty$ is the base point of our marking. For item (4) we have $w(e_S) = w(\sum_{i\in S} a_i - \frac{|S|}{2}L) = \sum_{i\in S} w(a_i) -$

$\frac{|S|}{2}w(2\hat{a}_\infty) = \sum_{i\in S}(\Omega I)\eta_i = (\Omega I)\eta_S$. For item (5) we have $w(f_T) = w(\sum_{i\in T} a_i + \frac{g-1-|T|}{2}L) = \sum_{i\in T}(\Omega I)\eta_i + \frac{g-1-|T|}{2}2(\Omega I)\eta_\infty = (\Omega I)\eta_T$. For item (6) we know by item (4) of lemma 1.4.4 that the riemann point r is the image of a half–canonical divisor class and so there exists a $U \subseteq B$ such that $|U| \equiv g-1 \mod 2$, and $w(f_U) = r$ in $\mathrm{Jac}(M)$. Corollary 1.2.12 of the Riemann vanishing theorem tells us that $\mathrm{ord}_0\, \theta((\Omega I)\eta_S + z, \Omega) = \mathrm{ord}_0\, \theta(w(e_S)+z,\Omega) = \dim \mathcal{L}(f_U - e_S) = \dim \mathcal{L}(f_{U\circ S})$. Using item (5) of lemma 1.4.4 we put this more succinctly as:

$$\mathrm{ord}_0\, \theta[\eta_S](z,\Omega) = |\frac{g+1-|U\circ S|}{2}|. \tag{1.4.10}$$

The order of vanishing of $\theta[\eta_S](z,\Omega)$ must have the same parity as $\theta[\eta_S](z,\Omega)$ does as a function of z; therefore by lemma 1.1.6 we have $e_*(\eta_S) = (-1)^{|\frac{g+1-|S\circ U|}{2}|} = (-1)^{\frac{g+1-|S\circ U|}{2}}$ which is item (2) . If we let $S = \emptyset$ we see that $1 = e_*(\eta_\emptyset) = (-1)^{\frac{g+1-|U|}{2}}$ implies that $|U| \equiv g+1 \mod 4$. Item (3) follows from item (2) using lemma 1.4.8 and a purely combinatorial argument. Consider the equation,

$$e_2(\eta_{S_1},\eta_{S_2}) = \frac{e_*(\eta_{S_1\circ S_2})}{e_*(\eta_{S_1})\, e_*(\eta_{S_2})} = (-1)^{\frac{g+1-|U\circ S_1\circ S_2|}{2} - \frac{g+1-|U\circ S_1|}{2} - \frac{g+1-|U\circ S_2|}{2}}.$$

The combinatorial identity $\frac{g+1-|U\circ S_1\circ S_2|}{2} - \frac{g+1-|U\circ S_1|}{2} - \frac{g+1-|U\circ S_2|}{2} = |S_1\cap S_2| - 2|U\cap S_1\cap S_2| + (|U| - (g+1))/2$ shows that $e_2(\eta_{S_1},\eta_{S_2}) = (-1)^{|S_1\cap S_2|}$ because $|U| \equiv g+1 \mod 4$. $\square$

1.4.11 Definition. *Ξ_g is the set of maps $\eta : B \to \frac{1}{2}\mathbb{Z}^{2g}$ such that:*

(1) $\forall S \subseteq B, \quad \eta_S = \sum_{i\in S}\eta_i, \quad \eta_\infty = \vec{0}$
(2) $\eta : \{S \subseteq B : |S| \equiv 0 \mod 2, \circ\}/\{\emptyset, B\} \cong \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ *is a group isomorphism.*
(3) $\forall\, |S_1|, |S_2| \equiv 0 \mod 2, \quad e_2(\eta_{S_1},\eta_{S_2}) = (-1)^{|S_1\cap S_2|}$.
(4) $\exists_1\, U \subseteq B : |U| \equiv g+1 \mod 4$*:* $\forall\, |S| \equiv 0 \mod 2$, $e_*(\eta_S) = (-1)^{\frac{g+1-|S\circ U|}{2}}$.

Two elements of Ξ_g which are equal as maps into $\frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ will be said to have the same *class*. Proposition 1.4.9 asserts that for a marked hyperelliptic curve with $\hat{a}_\infty$ as the base point there is some $\eta \in \Xi_g$ such that $w(\hat{a}_i) = (\Omega I)\eta_i$ and $w(f_U) = [r]$. If we change our lifts $\hat{a}_i$ then the map η will change but its class in Ξ_g will not. If we keep the base point $\hat{a}_\infty$ fixed but let $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_g(\mathbb{Z})$ change the canonical homology basis as in §1.2 then the new Abel–Jacobi map is $\tilde{w}(z) = ({}^t a + \Omega\, {}^t b)^{-1} w(z)$, and the new period matrix is $\tilde{\Omega} = ({}^t a + \Omega\, {}^t b)^{-1}({}^t c + \Omega\, {}^t d) = (c + d\Omega)(a + b\Omega)^{-1}$. These imply that $\tilde{\eta} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}\eta$ as is easily verified. So $\sigma \in \mathrm{Sp}_g(\mathbb{Z})$ acts linearly on Ξ_g in composition with the automorphism of $\Gamma = \mathrm{Sp}_g(\mathbb{Z})$ given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$. However, Γ_2 does not alter the class of η in Ξ_g. The action of Γ/Γ_2 is actually free and transitive on the classes of Ξ_g. This gives us an explicit description of the classes in Ξ_g because an individual member is easy to display.

1.4.12 Definition. *An* azygetic base *for* $V = \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ *is an ordered set of* $2g+1$ *elements* $\alpha_1, \dots \alpha_{2g+1}$ *such that* $\sum_{i=1}^{2g+1} \alpha_i = 0$, $V = \operatorname{span}(\alpha_i)$, *and* $e_2(\alpha_i, \alpha_j) = -1$ *for* $i \neq j$.

1.4.13 Lemma. *The classes in* Ξ_g *are in bijection with the azygetic bases of* $V = \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ *and* $\Gamma = Sp_g(\mathbb{Z})$ *acts equivariantly and transitively on both.*

Proof. Given $\eta \in \Xi_g$ we verify that $\eta_1, \dots \eta_{2g+1}$ is an azygetic base. We have $V = \operatorname{span}(\eta_i) = \operatorname{Im}(\eta)$ because η is an isomorphism, $\sum_{i=1}^{2g+1} \eta_i = \eta_{(B\backslash\infty)} \equiv \eta_{(B\backslash\infty)^{\mathbf{c}}} = \eta_\infty = 0$, and $e_2(\eta_i, \eta_j) = e_2(\eta_{(i,\infty)}, \eta_{(j,\infty)}) = (-1)^{|\{i,\infty\}\cap\{j,\infty\}|} = (-1)^1 = -1$ for $i \neq j$. The equivariance is clear.

Conversely, suppose that $\alpha_1, \dots \alpha_{2g+1}$ is an azygetic base for V. Since Γ/Γ_2 is transitive [10, 212] on azygetic bases there is a $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ such that $\sigma\alpha_i = \eta_i$, where the η_i are given for some hyperelliptic M. If we change the canonical homology basis of M via $\bar{\sigma}^{-1}$, where $\bar{\sigma} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$, then η changes to $\tilde{\eta} = \sigma^{-1}\eta$ and so $\tilde{\eta}_i = \alpha_i$. Hence every azygetic base arises from an element of Ξ_g. Finally, for completeness, we point out the uniqueness of $U \amalg U^c$ once the η_i are given. Since $e_*(\eta_i) = (-1)^{\frac{g+1-|\{i,\infty\}\circ U|}{2}} = -1$ if $i \in U, \infty \in U$; 1 if $i \notin U, \infty \in U$; 1 if $i \in U, \infty \notin U$, and -1 if $i \notin U, \infty \notin U$, we calculate that $U \amalg U^c = \{i \in B\backslash\infty : e_*(\eta_i) = -1\} \cup \{\infty\} \amalg \{i \in B\backslash\infty : e_*(\eta_i) = +1\}$. $\square$

We shall frequently deal with the set Ξ_g, and point out that it is explicitly given. The finite group $\Gamma/\Gamma_2 \cong \operatorname{Sp}_g(F_2)$ is transitive on Ξ_g and an azygetic base is easily given. Here are $2g$ elements of one that Mumford constructs from an explicit marking.

$$\tfrac{1}{2}\begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix}, \tfrac{1}{2}\begin{pmatrix} 0 & 1 & \dots & 0 \\ 1 & 1 & \dots & 0 \end{pmatrix}, \dots, \tfrac{1}{2}\begin{pmatrix} 0 & 0 & \dots & 1 \\ 1 & 1 & \dots & 1 \end{pmatrix}$$
$$\tfrac{1}{2}\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix}, \tfrac{1}{2}\begin{pmatrix} 0 & 1 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix}, \dots, \tfrac{1}{2}\begin{pmatrix} 0 & 0 & \dots & 1 \\ 1 & 1 & \dots & 0 \end{pmatrix}$$

There are a number of combinatorial identities we shall need later on and these are given in the next lemma.

1.4.14 Definition. *For* $S \subseteq B$ *let* $\epsilon_S(j) = \begin{cases} +1, & \text{for } j \in S \\ -1, & \text{for } j \notin S. \end{cases}$

1.4.15 Lemma. *Let* $\eta \in \Xi_g$.

(1) $\forall\, |S| \equiv 0 \mod 2$, $j \in B$, *we have* $\quad e_2(\eta_S, \eta_j) = \epsilon_S(j)\epsilon_S(\infty)$
(2) $\forall\, S, T$, *we have* $\quad \epsilon_S(j)\epsilon_T(j)\epsilon_{S\circ T}(j) = -1$
(3) $\forall\, |S| \equiv 0 \mod 2$, *we have* $\quad \sum_{j\in B} \epsilon_U(j) e_2(\eta_S, \eta_j) = \pm 4(\frac{g+1-|S\circ U|}{2})$

Proof. For item (1) we have $e_2(\eta_S, \eta_j) = e_2(\eta_S, \eta_{(j,\infty)}) = (-1)^{|S\cap\{j,\infty\}|}$ by the definition 1.4.11 (3). The four cases of $\epsilon_S(j)\epsilon_S(\infty) = (-1)^{|S\cap\{j,\infty\}|}$ are easily checked. For item (2) note that j is in either none or two of S, T, $S\circ T$, so that we have $\epsilon_S(j)\epsilon_T(j)\epsilon_{S\circ T}(j) = -1$. For item (3) we calculate $\sum_{j\in B}\epsilon_U(j)e_2(\eta_S,\eta_j) = \sum_{j\in B}\epsilon_U(j)\epsilon_S(j)\epsilon_S(\infty) = -\epsilon_S(\infty)\sum_{j\in B}\epsilon_{U\circ S}(j) = -\epsilon_S(\infty)(|U\circ S| - |(U\circ S)^{\mathbf{c}}|)$ $= -\epsilon_S(\infty)(|U\circ S| - (2g+2-|U\circ S|)) = \pm 2(g+1-|S\circ U|)$. □

In the proof of proposition 1.4.9 we observed in equation 1.4.10 which thetanullwerte vanished for a hyperelliptic curve and indeed even the order of this vanishing. The next two definitions are motivated by the result 1.4.10 that for $|S|\equiv 0 \mod 2$ we have $\mathrm{ord}_0\,\theta[\eta_S](z,\Omega) = |\frac{g+1-|U\circ S|}{2}|$.

1.4.16 Definition. *Let $\zeta\in\frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ and $\eta\in\Xi_g$. We say ζ has* hyperelliptic η–order ν, *if $\zeta=\eta_S$ for some $S\subseteq B$ with $|S|$ even and $\nu = |\frac{g+1-|U\circ S|}{2}|$.*

1.4.17 Proposition (Vanishing and nonvanishing). [14, 3.103] *Let Ω be the period matrix of a marked hyperelliptic Riemann surface M with $\hat{a}_\infty$ as the base point. Let $w(\hat{a}_i) = (\Omega I)\eta_i$ with $\eta\in\Xi_g$. We have that $\theta[\zeta](0,\Omega)$ does not vanish if and only if ζ has hyperelliptic η–order zero,*

$$\theta[\zeta](0,\Omega)\neq 0 \iff \exists S\subseteq B : |S|\equiv 0 \mod 2 : \zeta=\eta_S \text{ and } |U\circ S| = g+1.$$

Proof. This follows from equation 1.4.10 .

1.4.18 Definition. *Let $\eta\in\Xi_g$. The set of equations $V_{g,\eta}$, called the* vanishing equations, *is defined by*

$$(1.4.19\ )\qquad \forall S\subseteq B : |S|\equiv 0 \mod 2, \text{ and } |U\circ S|\neq g+1, \qquad \theta[\eta_S](0,\Omega) = 0.$$

Clearly the equations $V_{g,\eta}$ are unaltered if Ω is replaced by $\sigma\cdot\Omega$ for $\sigma\in\Gamma_2$; this is because Γ_2 does not permute the thetanullwerte. By proposition 1.4.17 we see that if Ω is the period matrix of a marked hyperelliptic curve then there exists an $\eta\in\Xi_g$ such that Ω satisfies the equations $V_{g,\eta}$. We write $\Omega\in V'_{g,\eta}$ for brevity. Notice that if we have $\zeta\in\frac{1}{2}\mathbb{Z}^{2g}$ then in view of equation 1.4.10 the hyperelliptic η–order of ζ is provided directly by lemma 1.4.15 (3) as $\pm\frac{1}{4}\sum_{j\in B}\epsilon_U(j)e_2(\zeta,\eta_j)$. We now consider the further identities available to us because the point sections $w(\hat{a}_i) = (\Omega I)\eta_i$ are known.

1.4.20 Definition. *Let $\eta\in\Xi_g$, $\Omega\in\mathcal{H}_g$. Define $\xi_{ijkl} = \frac{1}{2}(\Omega\, I)(\eta_i+\eta_j-\eta_k-\eta_l)$.*

1.4.21 Definition. *Let $\eta\in\Xi_g$. The set of equations $F_{g,\eta}$, called* Fay's trisecant formula, *is defined by all the 3×3 minors which express the rank condition:*

$$\mathrm{rank}\{\vec{\theta}_2(\xi_{ijkl},\Omega), \vec{\theta}_2(\xi_{iklj},\Omega), \vec{\theta}_2(\xi_{iljk},\Omega)\}\leq 2.$$

1.4.22 Proposition (Fay's trisecant formula). *Let Ω be the period matrix of a marked hyperelliptic Riemann surface M; then there exists an $\eta \in \Xi_g$ such that Ω satisfies $F_{g,\eta}$. We write $\Omega \in F'_{g,\eta}$ for brevity.*

Proof. We consider Fay's trisecant formula as given in equation 1.3.12 of proposition 1.3.8. That equation gives us a rank condition on the $\vec{\theta}_2(\frac{1}{2}w(\hat{a}_i + \hat{a}_j - \hat{a}_k - \hat{a}_l), \Omega)$. Since $w(\hat{a}_i) = (\Omega I)\eta_i$ for some $\eta \in \Xi_g$, this is the rank condition on the $\vec{\theta}_2(\xi_{ijkl}, \Omega)$ which we desire.

We may give formulae for Gunning's crossratio function.

1.4.23 Proposition. *Let M be a marked hyperelliptic Riemann surface with $w(\hat{a}_i) = (\Omega I)\eta_i$ and $\eta \in \Xi_g$. Let p be Gunning's crossratio function. For all V such that $|V| = g - 1$, and for all distinct $i, j, k, l \notin V$ we have:*

$$p(\hat{a}_i, \hat{a}_j, \hat{a}_k, \hat{a}_l) = \frac{\theta((\Omega I)(\eta_U - \eta_V + \eta_i - \eta_k), \Omega)}{\theta((\Omega I)(\eta_U - \eta_V + \eta_i - \eta_l), \Omega)} \frac{\theta((\Omega I)(\eta_U - \eta_V + \eta_j - \eta_l), \Omega)}{\theta((\Omega I)(\eta_U - \eta_V + \eta_j - \eta_k), \Omega)} = \frac{\theta[\eta_U - \eta_V]((\Omega I)(\eta_i - \eta_k), \Omega)}{\theta[\eta_U - \eta_V]((\Omega I)(\eta_i - \eta_l), \Omega)} \frac{\theta[\eta_U - \eta_V]((\Omega I)(\eta_j - \eta_l), \Omega)}{\theta[\eta_U - \eta_V]((\Omega I)(\eta_j - \eta_k), \Omega)} = \boldsymbol{e}^{2\pi i(\eta_i - \eta_j)'\Omega(\eta_k - \eta_l)'} \boldsymbol{e}^{2\pi i\{(\eta_i - \eta_j)'\cdot(\eta_k - \eta_l)'' + (\eta_k - \eta_l)'\cdot(\eta_i - \eta_j)''\}} \frac{\theta[\eta_U - \eta_V + \eta_i - \eta_k](0, \Omega)}{\theta[\eta_U - \eta_V + \eta_i - \eta_l](0, \Omega)} \frac{\theta[\eta_U - \eta_V + \eta_j - \eta_l](0, \Omega)}{\theta[\eta_U - \eta_V + \eta_j - \eta_k](0, \Omega)}.$$

Proof. Since $|V| = g - 1$ the hyperelliptic η–order of $\eta_U - \eta_V$ is 1, and so $\alpha = (\Omega I)(\eta_U - \eta_V)$ is in the theta locus Θ. The hyperelliptic η–order of $\eta_U - \eta_V + \eta_i - \eta_k$ is 0 as long as $i, k \notin V$ so by corollary 1.3.13 we have the first equality of the proposition.

$$p(\hat{a}_i, \hat{a}_j, \hat{a}_k, \hat{a}_l) = \frac{\theta(\alpha + w(\hat{a}_i) - w(\hat{a}_k), \Omega)}{\theta(\alpha + w(\hat{a}_i) - w(\hat{a}_l), \Omega)} \frac{\theta(\alpha + w(\hat{a}_j) - w(\hat{a}_l), \Omega)}{\theta(\alpha + w(\hat{a}_j) - w(\hat{a}_l), \Omega)}$$

The remaining equalities follow from lemma 1.1.20.

1.4.24 Proposition. *Let M be a marked hyperelliptic Riemann surface with $w(\hat{a}_i) = (\Omega I)\eta_i$ and $\eta \in \Xi_g$. Let p be Gunning's higher crossratio function. For all $K, L \subseteq B$ such that $|K| = |L| = g$, and for all distinct $i, j \notin K \cup L$ we have:*

$$p(\hat{a}_i, \hat{a}_j; \{\hat{a}_m\}_{m \in K}; \{\hat{a}_n\}_{n \in L}) = \frac{\theta((\Omega I)(\eta_U + \eta_i - \eta_K), \Omega)}{\theta((\Omega I)(\eta_U + \eta_i - \eta_L), \Omega)} \frac{\theta((\Omega I)(\eta_U + \eta_j - \eta_L), \Omega)}{\theta((\Omega I)(\eta_U + \eta_j - \eta_K), \Omega)} = \frac{\theta[\eta_U]((\Omega I)(\eta_i - \eta_K), \Omega)}{\theta[\eta_U]((\Omega I)(\eta_i - \eta_L), \Omega)} \frac{\theta[\eta_U]((\Omega I)(\eta_j - \eta_L), \Omega)}{\theta[\eta_U]((\Omega I)(\eta_j - \eta_K), \Omega)} = \boldsymbol{e}^{2\pi i(\eta_i - \eta_j)'\Omega(\eta_K - \eta_L)'} \boldsymbol{e}^{2\pi i\{(\eta_i - \eta_j)'\cdot(\eta_K - \eta_L)'' + (\eta_K - \eta_L)'\cdot(\eta_i - \eta_j)''\}} \frac{\theta[\eta_U + \eta_i - \eta_K](0, \Omega)}{\theta[\eta_U + \eta_i - \eta_L](0, \Omega)} \frac{\theta[\eta_U + \eta_j - \eta_L](0, \Omega)}{\theta[\eta_U + \eta_j - \eta_K](0, \Omega)}.$$

Proof. This follows from corollary 1.3.14 and lemma 1.1.20 in the manner of the proof of proposition 1.4.23.

1.4.25 Corollary. *Let $K = \{k_m\}_1^g$, $L = \{l_m\}_1^g$ and let there be given distinct $i, j, k_m, l_m \notin V_m$ such that $|V_m| = g - 1$. We have:*

$$\frac{\theta[\eta_U + \eta_i - \eta_K](0,\Omega)}{\theta[\eta_U + \eta_i - \eta_L](0,\Omega)} \frac{\theta[\eta_U + \eta_j - \eta_L](0,\Omega)}{\theta[\eta_U + \eta_j - \eta_K](0,\Omega)} = \prod_{m=1}^{g} \frac{\theta[\eta_U - \eta_{V_m} + \eta_i - \eta_{k_m}](0,\Omega)}{\theta[\eta_U - \eta_{V_m} + \eta_i - \eta_{l_m}](0,\Omega)} \frac{\theta[\eta_U - \eta_{V_m} + \eta_j - \eta_{l_m}](0,\Omega)}{\theta[\eta_U - \eta_{V_m} + \eta_j - \eta_{k_m}](0,\Omega)}.$$

Proof. If we note that

$$p(\hat{a}_i, \hat{a}_j; \{\hat{a}_m\}_{m\in K}; \{\hat{a}_n\}_{n\in L}) = \prod_{m=1}^{g} p(\hat{a}_i, \hat{a}_j, \hat{a}_{k_m}, \hat{a}_{l_m})$$

and choose appropriate V_m so that $|V_m| = g - 1$ and $i, j, k_m, l_m \notin V_m$, then we may use the previous two corollaries. The exponential factors cancel and we are left with the assertion that was to be shown.

A useful remark about 1.4.25 is that if Ω satifies the equation of 1.4.25 then so does $\sigma \cdot \Omega$ for $\sigma \in \Gamma_2$. The discussion after 1.1.8 can be used to show that both sides of the equation in corollary 1.4.25 transform in the same way with respect to Γ_2. Neither this remark nor corollary 1.4.25 is actually used in this paper but they provide motivation for lemma 2.5.9 in chapter 2. As a final topic we show the existence of certain göpel systems of hyperelliptic η–order zero which we will use later.

1.4.26 Lemma. *Let $\eta \in \Xi_g$. There is a göpel system $\Sigma \subseteq \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ all of whose elements have hyperelliptic η–order zero. In fact there are two göpel systems Σ_1 and Σ_2, each of whose elements have hyperelliptic η–order zero and which also satisfy: $\Sigma_1 \cap \Sigma_2 = \{\delta\}$, and $(\delta + \Sigma_1) \oplus (\delta + \Sigma_2) = \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$.*

Proof. Let (B, η, U) be given. Partition B arbitrarily in half so that we have the disjoint union $B = \{m_1, m_2, \ldots m_{g+1}\} \amalg \{n_1, n_2, \ldots n_{g+1}\} = M \amalg N$. Define $P_k = (m_k n_k)$ for $1 \leq k \leq g+1$, and $Q_1 = (m_{g+1} n_1)$, $Q_k = (m_{k-1} n_k)$ for $2 \leq k \leq g+1$. Let E_1 be the subgroup of $\{S \subseteq B : |S| \text{ even}\}/\{\emptyset, B\}$ generated by the classes of $P_1, \ldots P_{g+1}$, and E_2 by those of $Q_1, \ldots Q_{g+1}$. The E_i are clearly subgroups of order 2^g. Let $\Sigma_i = \eta_{\{M \circ U + E_i\}}$; then $\Sigma_i + \Sigma_i + \Sigma_i = \eta_{\{3M \circ U + E_i + E_i + E_i\}} = \eta_{\{M \circ U + E_i\}} = \Sigma_i$ so that the Σ_i have the triple sum property of göpel systems.

Let $\eta_S \in \Sigma_1$ so that $S = M \circ U \circ \{P_i\}_{i\in T}$ for $T \subseteq \{1, \ldots, g+1\}$ and $U \circ S = M \circ \{P_i\}_{i\in T} = \{m_i n_j\}_{i\notin T, j\in T}$. Therefore we have $|U \circ S| = |T \cup T^{\mathbf{c}}| = g + 1$ and η_S has hyperelliptic η–order zero. This shows that every element of Σ_1 has hyperelliptic η–order zero and the same clearly holds for Σ_2. Since these elements have hyperelliptic η–order zero they are even and this shows that the Σ_i are göpel systems. To complete the proof we need to show that $E_1 + E_2 = \{S \subseteq B : |S| \text{ even}\}/\{\emptyset, B\}$. We have the following equalities.

$$
\begin{aligned}
(m_{g+1}\, n_1) &= Q_1 & (m_{g+1}\, m_1) &= Q_1 + P_1\\
(m_{g+1}\, n_2) &= Q_1 + P_1 + Q_2 & (m_{g+1}\, m_2) &= Q_1 + P_1 + Q_2 + P_2\\
&\vdots & &\vdots\\
(m_{g+1}\, n_k) &= Q_k + \sum_{i=1}^{k-1} Q_i + P_i & (m_{g+1}\, m_k) &= \sum_{i=1}^{k} Q_i + P_i\\
&\vdots & &\vdots\\
(m_{g+1}\, n_{g+1}) &= Q_{g+1} + \sum_{i=1}^{g} Q_i + P_i & (m_{g+1}\, m_{g+1}) &= \emptyset
\end{aligned}
$$

The elements $(m_{g+1}\, n_k)$ and $(m_{g+1}\, m_k)$ generate $\{S \subseteq B : |S| \text{ even}\}/\{\emptyset, B\}$ and so by counting we have $E_1 \oplus E_2 = \{S \subseteq B : |S| \text{ even}\}/\{\emptyset, B\}$. We then have $\{U \circ M + E_1\} \cap \{U \circ M + E_2\} = \{U \circ M\}$ and so we let $\delta = \eta_{U\circ M}$. Hence we conclude that $\Sigma_1 \cap \Sigma_2 = \{\delta\}$ and $(\delta + \Sigma_1) \oplus (\delta + \Sigma_2) = \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$.

§1.5 Crossratio identities for hyperelliptic theta functions.

In this section the projective crossratios of the branch points of a hyperelliptic Riemann surface M are expressed in terms of its thetanullwerte. All of the identities among projective crossratios become identities among the hyperelliptic thetanullwerte. The main point is that these identities among the thetanullwerte given in proposition 1.5.4 are necessary *and sufficient* conditions for certain expressions in the thetanullwerte to be the values of projective crossratios.

1.5.1 Proposition. *Let M be a marked hyperelliptic Riemann surface given by $y^2 = \prod_{i\in B}(x - a_i)$. Let $\widehat{M}$ be the universal cover with $\pi : \widehat{M} \to M$. Let $\hat{a}_\infty$ be the base point and assume that $w(\hat{a}_i) = (\Omega I)\eta_i$ for $\eta \in \Xi_g$. As functions of z_1, z_2 on $\widehat{M} \times \widehat{M}$ we have:*

$$\frac{x(\pi z_1) - a_k}{x(\pi z_1) - a_l}\,\frac{x(\pi z_2) - a_l}{x(\pi z_2) - a_k} = \boldsymbol{e}^{-4\pi i(\eta_k - \eta_l)'\cdot w(z_1 - z_2)}(p(z_1, z_2, \hat{a}_k, \hat{a}_l))^2.$$

Proof. From proposition 1.2.4 recall that $p(z_1, z_2, \hat{a}_k, \hat{a}_l)$ transforms, as a function of z_1, by the character ρ_t, where $t = w(\hat{a}_k - \hat{a}_l) = (\Omega I)(\eta_k - \eta_l)$. Therefore $(p(z_1, z_2, \hat{a}_k, \hat{a}_l))^2$ transforms by ${\rho_t}^2 = \rho_{2t}$. Since $2t = (\Omega I)(2\eta_k - 2\eta_l)$ is a member of the lattice $\mathcal{L} = \mathbb{Z}^g + \Omega\mathbb{Z}^g$, we know that ρ_{2t} may be trivialized by $f(z) = \boldsymbol{e}^{2\pi i(2\eta_k - 2\eta_l)'\cdot w(z)}$; that is, we have $f(Tz) = \rho_{2t}(T)f(z)$. Hence we see that $g(z_1) = \boldsymbol{e}^{-2\pi i(2\eta_k - 2\eta_l)'\cdot w(z_1 - z_2)}(p(z_1, z_2, \hat{a}_k, \hat{a}_l))^2$ is meromorphic for $z_1 \in \widehat{M}$ and induces a function on M because $g(z_1)$ is invariant under the deck group G. As a function of z_1, g has the same divisor on $\widehat{M}$ as $\frac{x(\pi z_1) - a_k}{x(\pi z_1) - a_l}$; namely, zeros at $2\hat{a}_k \mod G$, and poles at $2\hat{a}_l \mod G$. Nontrivial meromorphic functions on M with the same zeros and poles must differ by a nonzero multiplicative constant c and so we conclude that

$$\frac{x(\pi z_1)-a_k}{x(\pi z_1)-a_l}\frac{x(\pi z_2)-a_l}{x(\pi z_2)-a_k} = c(z_2,\hat{a}_k,\hat{a}_l)\,\boldsymbol{e}^{-4\pi i(\eta_k-\eta_l)'\cdot w(z_1-z_2)}(p(z_1,z_2,\hat{a}_k,\hat{a}_l))^2. \tag{1.5.2}$$

Both sides of equation 1.5.2 have the crossratio symmetry $p^{z_1 z_2 a_1 a_2}p^{z_2 z_1 a_1 a_2}=1$, so switching z_1 with z_2 we conclude that $c(z_2,\hat{a}_k,\hat{a}_l)=c(z_1,\hat{a}_k,\hat{a}_l)^{-1}$. This means that $c(z_2,\hat{a}_k,\hat{a}_l)=c(\hat{a}_k,\hat{a}_l)$ is really independent of z_2, and setting $z_1=z_2$ we conclude from 1.5.2 that $c(\hat{a}_k,\hat{a}_l)=1$.

1.5.3 Corollary (Schottky). [2, 328] *Let M be a hyperelliptic Riemann surface as given in proposition 1.5.1. Let $V\subseteq B$ such that $|V|=g-1$ and let distinct $i,j,k,l\in B$ be given such that $i,j,k,l\notin V$. The crossratios of the branch points a_i are given by*

$$\begin{aligned}\langle a_i,a_j,a_k,a_l\rangle &= \boldsymbol{e}^{-4\pi i(\eta_k-\eta_l)'(\Omega I)(\eta_i-\eta_j)}[p(\hat{a}_i,\hat{a}_j,\hat{a}_k,\hat{a}_l)]^2\\ &= \boldsymbol{e}^{4\pi i(\eta_i-\eta_j)'(\eta_k-\eta_l)''}\Big(\frac{\theta[\eta_U-\eta_V+\eta_i-\eta_k]}{\theta[\eta_U-\eta_V+\eta_i-\eta_l]}\frac{\theta[\eta_U-\eta_V+\eta_j-\eta_l]}{\theta[\eta_U-\eta_V+\eta_j-\eta_k]}\Big)^2.\end{aligned}$$

Proof. The first equality is simply proposition 1.5.1 with $z_1=\hat{a}_i$ and $z_2=\hat{a}_j$. If we use the proposition 1.4.23 to express $p(\hat{a}_i,\hat{a}_j,\hat{a}_k,\hat{a}_l)$ as

$$\boldsymbol{e}^{2\pi i(\eta_i-\eta_j)'\Omega(\eta_k-\eta_l)'}\,\boldsymbol{e}^{2\pi i\{(\eta_i-\eta_j)'\cdot(\eta_k-\eta_l)''+(\eta_k-\eta_l)'\cdot(\eta_i-\eta_j)''\}}\frac{\theta[\eta_U-\eta_V+\eta_i-\eta_k](0,\Omega)}{\theta[\eta_U-\eta_V+\eta_i-\eta_l](0,\Omega)}\frac{\theta[\eta_U-\eta_V+\eta_j-\eta_l](0,\Omega)}{\theta[\eta_U-\eta_V+\eta_j-\eta_k](0,\Omega)},$$

then we obtain the second equality.

1.5.4 Proposition. *Let $\Omega\in\mathcal{H}_g$ and $\eta\in\Xi_g$. Suppose the numbers*

$$c^{ijkl}=\boldsymbol{e}^{4\pi i(\eta_i-\eta_j)'\cdot(\eta_k-\eta_l)''}\Big(\frac{\theta[\eta_U-\eta_V+\eta_i-\eta_k]}{\theta[\eta_U-\eta_V+\eta_i-\eta_l]}\frac{\theta[\eta_U-\eta_V+\eta_j-\eta_l]}{\theta[\eta_U-\eta_V+\eta_j-\eta_k]}(0,\Omega)\Big)^2$$

are defined and nonzero for distinct $i,j,k,l\notin V$, $|V|=g-1$, and are independent of V. Then there exist $2g+2$ distinct $a_i\in\mathbb{P}^1$ such that $c^{ijkl}=\langle a_i,a_j,a_k,a_l\rangle$, if and only if the hyperelliptic crossratio identity *(1.5.5) holds.*

$$\begin{aligned}&\boldsymbol{e}^{4\pi i(\eta_i-\eta_j)'\cdot(\eta_k-\eta_l)''}\theta[\eta_U-\eta_V+\eta_i-\eta_k](0,\Omega)^2\theta[\eta_U-\eta_V+\eta_j-\eta_l](0,\Omega)^2\\ &+\boldsymbol{e}^{4\pi i(\eta_i-\eta_k)'\cdot(\eta_j-\eta_l)''}\theta[\eta_U-\eta_V+\eta_i-\eta_j](0,\Omega)^2\theta[\eta_U-\eta_V+\eta_k-\eta_l](0,\Omega)^2\\ &=\theta[\eta_U-\eta_V+\eta_i-\eta_l](0,\Omega)^2\theta[\eta_U-\eta_V+\eta_j-\eta_k](0,\Omega)^2\end{aligned}\tag{1.5.5}$$

Proof. According to proposition 3.2.1 we must simply verify that the c^{ijkl} satisfy the crossratio symmetries along with $c^{ijkl} + c^{ikjl} = 1$. The thetanullwerte cancel out in the crossratio symmetries and we merely need to check the correctness of the remaining signs due to the exponential factors $\boldsymbol{e}^{4\pi i(\eta_i - \eta_j)' \cdot (\eta_k - \eta_l)''}$. These signs must be correct however, since they are correct for hyperelliptic Ω when the c^{ijkl} really are projective crossratios. Proceeding by direct calculation would use the facts: $e_2(\eta_i + \eta_j, \eta_k + \eta_l) = 1$, and $e_2(\eta_k, \eta_j)e_2(\eta_j, \eta_k)e_2(\eta_l, \eta_k) = -1$. The equation $c^{ijkl} + c^{ikjl} = 1$ clearly becomes equation 1.5.5. □

The derivation of thetanullwerte identities from crossratios in the hyperelliptic case was first done by H. Farkas in [3, 298]. Although the crossratio identity does not appear explicitly in [3] the formulas there certainly imply it. The formulas of Farkas are in fact more delicate, giving information about appropriate square roots. The crossratio identity also follows from Fay's trisecant formula. In equation 1.3.10 let $y = (\Omega I)(\eta_U - \eta_V + \eta_j + \eta_k)$ and let a_2, z_2, a_1, z_1 be $\hat{a}_i$, $\hat{a}_j$, $\hat{a}_k$, $\hat{a}_l$, and use proposition 1.4.23. The new contribution made here is pointing out the meaning of the crossratio identity 1.5.5; it means that certain nullwerte quotients are projective crossratios.

§1.6 Frobenius Theta Formula.

In this section we show that the vanishing equations are equivalent to a formula developed by Mumford in [14] and called *the generalized Frobenius theta formula*. We will also show that either of these equations implies Fay's trisecant formula in the form $F_{g,\eta}$. That the vanishing equations imply the Frobenius theta formula is due to Mumford.

1.6.1 Lemma. *Let $\Omega \in \mathcal{H}_g$ and $\eta \in \Xi_g$. The following four equations are equivalent and (1.6.3) is usually called the Frobenius theta formula.*

$$\forall\ a_i, z_i \in \mathbb{C}^g; \quad \forall\ \nu, M \in \mathbb{Z}^{2g} : \sum_{i=1}^{4} a_i = \nu, \text{ and } \sum_{i=1}^{4} z_i = (\Omega I)M \in \mathcal{L},$$

(1.6.2)

$$\sum_{J \in B} \epsilon_U(J) \boldsymbol{e}^{4\pi i \eta'_J \Omega \eta'_J} \boldsymbol{e}^{4\pi i M'(\Omega I)\eta_J} \prod_{i=1}^{4} \theta[a_i](z_i + (\Omega I)\eta_J, \Omega) = 0.$$

(1.6.3)

$$\forall\ a_i, z_i \in \mathbb{C}^g : \sum_{i=1}^{4} a_i = \sum_{i=1}^{4} z_i = 0, \quad \sum_{J \in B} \epsilon_U(J) \prod_{i=1}^{4} \theta[a_i + \eta_J](z_i, \Omega) = 0.$$

(1.6.4)

$$\forall\ z_i \in \mathbb{C}^g, p \in \mathbb{Z}^g : \sum_{i=1}^{4} z_i = 0, \quad \sum_{J \in B} \epsilon_U(J) \prod_{i=1}^{4} \theta[\begin{smallmatrix} \frac{p}{2} \\ 0 \end{smallmatrix} + \eta_J](z_i, \Omega) = 0.$$

(1.6.5)

$$\forall\ z_i \in \mathbb{C}^g : \sum_{i=1}^{4} z_i = 0, \quad \sum_{J \in B} \epsilon_U(J) \prod_{i=1}^{4} \theta[\eta_J](z_i, \Omega) = 0.$$

Proof. We will show 1.6.2 $\iff$ 1.6.3, and then 1.6.3 $\implies$ 1.6.4 $\implies$ 1.6.5 $\implies$ 1.6.3. The equation 1.6.3 is just 1.6.2 with $\nu = 0$ and $M = 0$, if we use $\theta[a_i + \eta_J](z_i, \Omega) = \boldsymbol{e}^{2\pi i \eta'_J \cdot \{z_i + a''_i + \eta''_J + \frac{1}{2}\Omega \eta'_J\}} \theta[a_i](z_i + (\Omega I)\eta_J, \Omega)$ from lemma 1.1.19. Conversely, assume that $\sum_{i=1}^4 a_i = \nu \in \mathbb{Z}^{2g}$ and that $\sum_{i=1}^4 z_i = (\Omega I)M \in \mathcal{L}$. The Frobenius formula 1.6.3 then gives us

$$(1.6.6\ \ ) \qquad \sum_{j\in B} \epsilon_U(j) \{\prod_{i=1}^{3} \theta[a_i + \eta_j](z_i, \Omega)\} \theta[a_4 - \nu + \eta_j](z_4 - (\Omega I)M, \Omega) = 0.$$

We use the following transformation to simplify 1.6.6.

$$\theta[a_4 - \nu + \eta_j](z_4 - (\Omega I)M, \Omega) = \theta[a_4 + \eta_j](z_4, \Omega) \boldsymbol{e}^{-2\pi i a'_4 \nu''} \boldsymbol{e}^{-2\pi i \eta'_j \nu''}$$
$$\boldsymbol{e}^{-2\pi i \{(a_4-\nu)' M'' - M'(a_4-\nu)''\}} \boldsymbol{e}^{-2\pi i (\eta'_j M'' - M' \eta''_j)} \boldsymbol{e}^{-2\pi i (\frac{1}{2} M' \Omega M' - M' z_4)}.$$

We retain only that factor of the unit which depends upon j,

$$(1.6.7\ \ ) \qquad \sum_{j\in B} \epsilon_U(j) \boldsymbol{e}^{-2\pi i \eta'_j \nu''} \boldsymbol{e}^{-2\pi i (\eta'_j M'' - M' \eta''_j)} \prod_{i=1}^{4} \theta[a_i + \eta_j](z_i, \Omega) = 0.$$

We now use lemma 1.1.19 to put the η_j term in the argument:

$$\sum_{j\in B} \{\epsilon_U(j) \boldsymbol{e}^{-2\pi i \eta'_j \nu''} \boldsymbol{e}^{-2\pi i (\eta'_j M'' - M' \eta''_j)} \prod_{i=1}^{4} \boldsymbol{e}^{2\pi i \eta'_j \cdot \{z_i + a''_i + \eta''_j + \frac{1}{2}\Omega \eta'_j\}}$$
$$\cdot \prod_{i=1}^{4} \theta[a_i](z_i + (\Omega I)\eta_j, \Omega)\} = 0, \qquad \text{multiplying we obtain}$$
$$\sum_{j\in B} \{\epsilon_U(j) \boldsymbol{e}^{-2\pi i \eta'_j \nu''} \boldsymbol{e}^{-2\pi i (\eta'_j M'' - M' \eta''_j)} \boldsymbol{e}^{2\pi i \eta'_j \cdot \{(\Omega I)M + \nu'' + 4\eta''_j + 2\Omega \eta'_j\}}$$
$$\cdot \prod_{i=1}^{4} \theta[a_i](z_i + (\Omega I)\eta_j, \Omega)\} = 0, \qquad \text{and finally we obtain}$$
$$\sum_{j\in B} \epsilon_U(j) \boldsymbol{e}^{2\pi i \{M' \eta''_j + \eta'_j \Omega M' + 2\eta'_j \Omega \eta'_j\}} \prod_{i=1}^{4} \theta[a_i](z_i + (\Omega I)\eta_j, \Omega) = 0.$$

This shows that 1.6.3 implies 1.6.2. We now show that 1.6.3 implies 1.6.4. In 1.6.3 let $a_1 = a_2 = \frac{p}{2}$ and $a_3 = a_4 = -\frac{p}{2}$, then note that $\theta[\begin{smallmatrix} -\frac{p}{2} \\ 0 \end{smallmatrix} + \eta_j](z_i, \Omega) = \theta[\begin{smallmatrix} \frac{p}{2} \\ 0 \end{smallmatrix} + \eta_J](z_i, \Omega)$ because $p \in \mathbb{Z}^g$. This demonstrates 1.6.4. The equation 1.6.5 is simply 1.6.4 with $p = 0$. To demonstrate that 1.6.5 implies 1.6.3 it is simplest to reduce 1.6.3 to 1.6.5.

Use the equality $\theta[\eta_j + a_i](z_i, \Omega) = \boldsymbol{e}^{2\pi i a'_i \cdot \{z_i + \eta''_j + a''_i + \frac{1}{2}\Omega a'_i\}} \theta[\eta_j](z_i + (\Omega I)a_i, \Omega)$ to rewrite $\prod_{i=1}^4 \theta[a_i + \eta_j](z_i, \Omega)$ as

$$\prod_{i=1}^{4} \boldsymbol{e}^{2\pi i a'_i \cdot \{z_i + a''_i + \frac{1}{2}\Omega a'_i\}} \boldsymbol{e}^{2\pi i (\Sigma_1^4 a'_i) \cdot \eta''_j} \prod_{i=1}^{4} \theta[\eta_j](z_i + (\Omega I)a_i, \Omega).$$

Now the first non–zero factor does not depend upon j, and the second factor is 1 since $\sum_1^4 a_i' = 0$. So we have

$$\sum_{j\in B} \epsilon_U(j) \prod_{i=1}^{4} \theta[a_i+\eta_j](z_i,\Omega) =$$
$$\{\prod_{i=1}^{4} \boldsymbol{e}^{2\pi i a_i'\cdot\{z_i+a_i''+\frac{1}{2}\Omega a_i'\}}\}\{\sum_{j\in B}\epsilon_U(j)\prod_{i=1}^{4}\theta[\eta_j](z_i+(\Omega I)a_i,\Omega)\}.$$

When the $z_i+(\Omega I)a_i$ are replaced by z_i' we see that 1.6.3 holds if and only if 1.6.5 holds. □

1.6.8 Lemma. *Let $\Omega\in\mathcal{H}_g$ and $\eta\in\Xi_g$. The Frobenius theta formula:*

$$\forall\ z_i\in\mathbb{C}^g, p\in\mathbb{Z}^g : \sum_{i=1}^{4} z_i = 0,\quad \sum_{j\in B}\epsilon_U(j)\prod_{i=1}^{4}\theta[\begin{smallmatrix}\frac{p}{2}\\0\end{smallmatrix}+\eta_j](z_i,\Omega)=0$$

is equivalent to

$$\forall k\in\mathbb{Z}^g,\quad \forall p\in\mathbb{Z}^g,\qquad \sum_{l\in\mathbb{Z}^g/2\mathbb{Z}^g}\sum_{j\in B}\epsilon_U(j)\boldsymbol{e}^{2\pi i(k\cdot\eta_j''-(\eta_j'-\frac{p}{2})\cdot l)}\theta[2\eta_j](\frac{l}{2}+\Omega\frac{k}{2},\Omega)=0.$$

Proof. The Frobenius theta formula is equivalent to the vanishing of the Fourier coefficients in z_1, z_2, and z_3 of the following function.

$$\sum_{j\in B}\{\epsilon_U(j)\theta[\eta_j-\begin{smallmatrix}\frac{p}{2}\\0\end{smallmatrix}](z_1,\Omega)\theta[\eta_j-\begin{smallmatrix}\frac{p}{2}\\0\end{smallmatrix}](z_2,\Omega)$$
$$\cdot\theta[\eta_j-\begin{smallmatrix}\frac{p}{2}\\0\end{smallmatrix}](z_3,\Omega)\theta[\eta_j-\begin{smallmatrix}\frac{p}{2}\\0\end{smallmatrix}](-z_1-z_2-z_3,\Omega)\}$$
$$=\sum_{j\in B}\epsilon_U(j)\sum_{n_1,n_2,n_3,n_4\in\mathbb{Z}^g}\{\boldsymbol{e}^{2\pi i\{\frac{1}{2}(n_1+\eta_j'-\frac{p}{2})\Omega(n_1+\eta_j'-\frac{p}{2})+(n_1+\eta_j'-\frac{p}{2})(z_1+\eta_j'')\}}$$
$$\cdot\boldsymbol{e}^{2\pi i\{\frac{1}{2}(n_2+\eta_j'-\frac{p}{2})\Omega(n_2+\eta_j'-\frac{p}{2})+(n_2+\eta_j'-\frac{p}{2})(z_2+\eta_j'')\}}$$
$$\cdot\boldsymbol{e}^{2\pi i\{\frac{1}{2}(n_3+\eta_j'-\frac{p}{2})\Omega(n_3+\eta_j'-\frac{p}{2})+(n_3+\eta_j'-\frac{p}{2})(z_3+\eta_j'')\}}$$
$$\cdot\boldsymbol{e}^{2\pi i\{\frac{1}{2}(n_4+\eta_j'-\frac{p}{2})\Omega(n_4+\eta_j'-\frac{p}{2})+(n_4+\eta_j'-\frac{p}{2})(-z_1-z_2-z_3+\eta_j'')\}}\}$$
$$=\sum_{j\in B}\epsilon_U(j)\sum_{n_1,n_2,n_3,n_4\in\mathbb{Z}^g}\{(\prod_{i=1}^{4}\boldsymbol{e}^{2\pi i\{\frac{1}{2}(n_i+\eta_j'-\frac{p}{2})\Omega(n_i+\eta_j'-\frac{p}{2})\}})$$
$$\cdot\boldsymbol{e}^{2\pi i(n_1+n_2+n_3+n_4+4\eta_j'-2p)\cdot\eta_j''}\boldsymbol{e}^{2\pi i(n_1-n_4)\cdot z_1}\boldsymbol{e}^{2\pi i(n_2-n_4)\cdot z_2}\boldsymbol{e}^{2\pi i(n_3-n_4)\cdot z_3}\}$$
$$=\sum_{j\in B}\epsilon_U(j)\sum_{k_1,k_2,k_3\in\mathbb{Z}^g}\sum_{n\in\mathbb{Z}^g}\{(\prod_{i=1}^{3}\boldsymbol{e}^{2\pi i\{\frac{1}{2}(k_i+n+\eta_j'-\frac{p}{2})\Omega(k_i+n+\eta_j'-\frac{p}{2})\}})$$

$$\cdot \boldsymbol{e}^{2\pi i\{\frac{1}{2}(n+\eta_j'-\frac{p}{2})\Omega(n+\eta_j'-\frac{p}{2})\}}\,\boldsymbol{e}^{2\pi i\{k_1+k_2+k_3+4n+4\eta_j'-2p\}\cdot\eta_j''}\,\boldsymbol{e}^{2\pi i\{k_1\cdot z_1+k_2\cdot z_2+k_3\cdot z_3\}}\}$$

The last equality was obtained by a unimodular change of summation indices: $k_1 = n_1 - n_4$, $k_2 = n_2 - n_4$, $k_3 = n_3 - n_4$, and $n = n_4$. The above function in z_1, z_2, and z_3 vanishes identically precisely when its Fourier coefficients are zero. This is equivalent to the following equations:

$$\begin{aligned}
&\forall\ p, k_1, k_2, k_3 \in \mathbb{Z}^g\\
0 = &\sum_{j\in B}\epsilon_U(j)\sum_{n\in\mathbb{Z}^g}\{\boldsymbol{e}^{2\pi i\{\frac{1}{2}(n+\eta_j'-\frac{p}{2})\Omega(n+\eta_j'-\frac{p}{2})+k_1\Omega(n+\eta_j'-\frac{p}{2})+\frac{1}{2}k_1\Omega k_1\}}\\
&\cdot\boldsymbol{e}^{2\pi i\{\frac{1}{2}(n+\eta_j'-\frac{p}{2})\Omega(n+\eta_j'-\frac{p}{2})+k_2\Omega(n+\eta_j'-\frac{p}{2})+\frac{1}{2}k_2\Omega k_2\}}\\
&\cdot\boldsymbol{e}^{2\pi i\{\frac{1}{2}(n+\eta_j'-\frac{p}{2})\Omega(n+\eta_j'-\frac{p}{2})+k_3\Omega(n+\eta_j'-\frac{p}{2})+\frac{1}{2}k_3\Omega k_3\}}\\
&\cdot\boldsymbol{e}^{2\pi i\{\frac{1}{2}(n+\eta_j'-\frac{p}{2})\Omega(n+\eta_j'-\frac{p}{2})\}}\,\boldsymbol{e}^{2\pi i\{k_1+k_2+k_3\}\cdot\eta_j''}\,\boldsymbol{e}^{2\pi i\{n+\eta_j'-\frac{p}{2}\}\cdot 4\eta_j''}\}.
\end{aligned}$$

If we take out the factor $\boldsymbol{e}^{\pi i\{k_1\Omega k_1+k_2\Omega k_2+k_3\Omega k_3\}}$ which is independent of the summation indices we obtain:

$$\begin{aligned}
\forall\ p,k_1,k_2,k_3\in\mathbb{Z}^g,\quad 0 = &\sum_{j\in B}\epsilon_U(j)\sum_{n\in\mathbb{Z}^g}\{\boldsymbol{e}^{2\pi i\{k_1+k_2+k_3\}\cdot\eta_j''}\\
&\cdot\boldsymbol{e}^{2\pi i\{\frac{1}{2}(n+\eta_j'-\frac{p}{2})4\Omega(n+\eta_j'-\frac{p}{2})+(n+\eta_j'-\frac{p}{2})\cdot(\Omega(k_1+k_2+k_3)+4\eta_j'')\}}\}.
\end{aligned}$$

Since the k_i only occur in the sum $k = k_1 + k_2 + k_3$, this is equivalent to:

$$\begin{aligned}
\forall\ p,k\in\mathbb{Z}^g,\quad 0 = &\sum_{j\in B}\epsilon_U(j)\sum_{n\in\mathbb{Z}^g}\{\boldsymbol{e}^{2\pi i k\cdot\eta_j''}\\
&\cdot\boldsymbol{e}^{2\pi i\{\frac{1}{2}(2n+2\eta_j'-p)\Omega(2n+2\eta_j'-p)+(2n+2\eta_j'-p)\cdot(\Omega\frac{k}{2}+2\eta_j'')\}}\}.
\end{aligned}\tag{1.6.9}$$

Instead of summing $2n-p$ over $n\in\mathbb{Z}^g$ in 1.6.9, we sum n over $n\in\mathbb{Z}^g$ and just multiply the summand by $\chi_{\{2\mathbb{Z}^g-p\}}(n)$. Here $\chi_S(n)$ is the characteristic function for S, and we have the convenient expression $\chi_{\{2\mathbb{Z}^g-p\}}(n) = 2^{-g}\sum_{l\in\mathbb{Z}^g/2\mathbb{Z}^g}\boldsymbol{e}^{2\pi i\frac{1}{2}l\cdot(n+p)}$. We have, for all $p,k\in\mathbb{Z}^g$,

$$\begin{aligned}
&0 = \sum_{j\in B}\epsilon_U(j)\sum_{n\in\mathbb{Z}^g}2^{-g}\sum_{l\in\mathbb{Z}^g/2\mathbb{Z}^g}\{\boldsymbol{e}^{2\pi i\frac{1}{2}l\cdot(n+p)}\,\boldsymbol{e}^{2\pi i k\cdot\eta_j''}\\
&\quad\cdot\boldsymbol{e}^{2\pi i\{\frac{1}{2}(n+2\eta_j')\Omega(n+2\eta_j')+(n+2\eta_j')\cdot(\Omega\frac{k}{2}+2\eta_j'')\}}\},\\
\text{and}\quad &0 = \sum_{j\in B}\epsilon_U(j)\sum_{n\in\mathbb{Z}^g}2^{-g}\sum_{l\in\mathbb{Z}^g/2\mathbb{Z}^g}\{\boldsymbol{e}^{2\pi i(k\cdot\eta_j''+\frac{1}{2}p\cdot l-\eta_j'\cdot l)}\\
&\quad\cdot\boldsymbol{e}^{2\pi i\{\frac{1}{2}(n+2\eta_j')\Omega(n+2\eta_j')+(n+2\eta_j')\cdot(\frac{l}{2}+\Omega\frac{k}{2}+2\eta_j'')\}}\},\\
\text{and}\quad &0 = \sum_{j\in B}\epsilon_U(j)\sum_{l\in\mathbb{Z}^g/2\mathbb{Z}^g}\boldsymbol{e}^{2\pi i(k\cdot\eta_j''-\eta_j'\cdot l)}\,\boldsymbol{e}^{i\pi p\cdot l}\theta[2\eta_j](\frac{l}{2}+\Omega\frac{k}{2},\Omega).
\end{aligned}$$

In the last step the definition of the theta function was used. This last equation is equivalent to the vanishing of the Fourier coefficients of the Frobenius theta formula and that is the assertion of 1.6.8. □

We are now ready to prove the equivalence of the vanishing equations to the Frobenius theta formula.

1.6.10 Proposition. *Let $\Omega \in \mathcal{H}_g$, and $\eta \in \Xi_g$. Then the vanishing equations, $\Omega \in V'_{g,\eta}$, hold,*

$$|S| \text{ even},\ |S \circ U| \neq g+1 \implies \theta[\eta_S](0,\Omega) = 0,$$

if and only if the Frobenius theta formula holds,

$$\forall\ a_i, z_i \in \mathbb{C}^g : \sum_{i=1}^4 a_i = \sum_{i=1}^4 z_i = 0, \quad \sum_{j\in B} \epsilon_U(j) \prod_{i=1}^4 \theta[a_i + \eta_j](z_i, \Omega) = 0.$$

Proof. We see from lemma 1.6.1 that the four versions of the Frobenius theta formula are equivalent. Lemma 1.6.8 then shows that the version 1.6.4 is equivalent to

$$\forall p, k \in \mathbb{Z}^g, \quad \sum_{j\in B} \epsilon_U(j) \sum_{l\in\mathbb{Z}^g/2\mathbb{Z}^g} \boldsymbol{e}^{2\pi i(k\cdot\eta''_j - \eta'_j\cdot l)}\, \boldsymbol{e}^{i\pi p\cdot l}\theta[2\eta_j](\frac{l}{2} + \Omega\frac{k}{2}, \Omega) = 0.$$

This equation may be rewritten by noting that $\boldsymbol{e}^{2\pi i(k\cdot\eta''_j - \eta'_j\cdot l)} = e_2(\begin{pmatrix} k/2 \\ l/2 \end{pmatrix}, \eta_j)$, where e_2 is the alternating form of definition 1.4.7, and that $\theta[2\eta_j](\frac{l}{2} + \Omega\frac{k}{2}, \Omega) = \theta(\frac{l}{2} + \Omega\frac{k}{2}, \Omega)$ by equation 1.1.5. Then we have for all $p, k \in \mathbb{Z}^g$:

$$\sum_{j\in B} \epsilon_U(j) \sum_{l\in\mathbb{Z}^g/2\mathbb{Z}^g} \boldsymbol{e}^{i\pi p\cdot l} e_2(\begin{pmatrix} k/2 \\ l/2 \end{pmatrix}, \eta_j)\theta(\frac{l}{2} + \Omega\frac{k}{2}, \Omega) = 0, \text{ or}$$

(1.6.11)

$$\forall\ p, k \in \mathbb{Z}^g, \quad \sum_{l\in\mathbb{Z}^g/2\mathbb{Z}^g} \boldsymbol{e}^{i\pi p\cdot l}\theta(\frac{l}{2} + \Omega\frac{k}{2}, \Omega)\{\sum_{j\in B} \epsilon_U(j) e_2(\begin{pmatrix} k/2 \\ l/2 \end{pmatrix}, \eta_j)\} = 0.$$

The terms of equation 1.6.11 factor nicely into a product, and recalling item (3) of lemma 1.4.15, the second factor is a multiple of the hyperelliptic η-order of $\begin{pmatrix} k/2 \\ l/2 \end{pmatrix}$. Hence the vanishing equations for Ω imply 1.6.11 because the thetanullwerte $\theta(\frac{l}{2} + \Omega\frac{k}{2}, \Omega)$ vanishes for $\begin{pmatrix} k/2 \\ l/2 \end{pmatrix}$ with hyperelliptic η-order greater than zero, and the second factor vanishes for $\begin{pmatrix} k/2 \\ l/2 \end{pmatrix}$ with hyperelliptic η-order equal to zero.

Conversely, the equation 1.6.11 also implies the vanishing equations; multiply 1.6.11 by $\boldsymbol{e}^{-i\pi p\cdot l_0}$ and sum over $p \in \mathbb{Z}^{2g}/2\mathbb{Z}^g$ to obtain:

$$\forall \quad k, l_0 \in \mathbb{Z}^g, \quad 0 = 2^g\theta(\frac{l_0}{2} + \Omega\frac{k}{2}, \Omega)\sum_{j\in B}\epsilon_U(j)e_2(\begin{pmatrix} k/2 \\ l_0/2 \end{pmatrix}, \eta_j).$$

This yields the vanishing equations, for again the theta constant must vanish whenever the hyperelliptic η-order is nonzero. This completes the proof of proposition 1.6.10.

1.6.12 Lemma. *Let $\Omega \in \mathcal{H}_g$, $\eta \in \Xi_g$, and assume that Ω satisfies the the Frobenius theta formula,* $\mathrm{Frob}_{g,\eta}$. *Then for all $V \subseteq B$ such that $|V| = g-1$ and for all distinct $i, j, k, l \notin V$ with $\bar\alpha = \eta_U - \eta_V$ we have:*

$$\begin{aligned} 0 =&\theta[\bar\alpha]((\Omega I)(\eta_i - \eta_j), \Omega)\theta[\bar\alpha]((\Omega I)(\eta_k - \eta_l), \Omega)\vec\theta_2(\xi_{ijkl}, \Omega) \\ &+\theta[\bar\alpha]((\Omega I)(\eta_i - \eta_k), \Omega)\theta[\bar\alpha]((\Omega I)(\eta_l - \eta_j), \Omega)\vec\theta_2(\xi_{iklj}, \Omega) \\ &+\theta[\bar\alpha]((\Omega I)(\eta_i - \eta_l), \Omega)\theta[\bar\alpha]((\Omega I)(\eta_j - \eta_k), \Omega)\vec\theta_2(\xi_{iljk}, \Omega). \end{aligned}$$

Proof. For $|V| = g-1$ and distinct $i, j, k, l \notin V$, let $a_1 = a_2 = \eta_U - \eta_V$, $a_3 = a_4 = 0$, $z_1 = -(\Omega I)\eta_i$, $z_2 = -(\Omega I)(\eta_j + \eta_k + \eta_l)$, $z_3 = -z + \frac{1}{2}(\Omega I)(\eta_i - \eta_j - \eta_k - \eta_l)$, and $z_4 = z + \frac{1}{2}(\Omega I)(\eta_i - \eta_j - \eta_k - \eta_l)$. Then we note that $\sum a_i = 2(\eta_U - \eta_V) \in \mathbb{Z}^{2g}$, $\sum z_i = -2(\Omega I)(\eta_j + \eta_k + \eta_l) \in \mathcal{L}$, and we apply the Frobenius theta formula 1.6.2.

$$\begin{aligned} 0 = &\sum_{J\in B}\{\epsilon_U(J)\boldsymbol{e}^{2\pi i(-2\eta_j - 2\eta_k - 2\eta_l)'(\Omega I)\eta_J}\,\boldsymbol{e}^{4\pi i\eta_J'\Omega\eta_J'} \\ &\cdot\theta[\eta_U - \eta_V]((\Omega I)(\eta_J - \eta_i), \Omega)\theta[\eta_U - \eta_V]((\Omega I)(\eta_J - \eta_j - \eta_k - \eta_l), \Omega) \\ &\cdot\theta(-z + \frac{1}{2}(\Omega I)(\eta_i - \eta_j - \eta_k - \eta_l) + (\Omega I)\eta_J, \Omega) \\ &\cdot\theta(z + \frac{1}{2}(\Omega I)(\eta_i - \eta_j - \eta_k - \eta_l) + (\Omega I)\eta_J, \Omega)\} \end{aligned}$$

We apply the addition theorem to the last two theta factors.

$$\begin{aligned} 0 = &\sum_{J\in B}\{\epsilon_U(J)\boldsymbol{e}^{2\pi i(-2\eta_j - 2\eta_k - 2\eta_l)'(\Omega I)\eta_J}\,\boldsymbol{e}^{4\pi i\eta_J'\Omega\eta_J'} \\ &\cdot\theta[\eta_U - \eta_V]((\Omega I)(\eta_J - \eta_i), \Omega)\theta[\eta_U - \eta_V]((\Omega I)(\eta_J - \eta_j - \eta_k - \eta_l), \Omega) \\ &\cdot\vec\theta_2(\tfrac{1}{2}(\Omega I)(2\eta_J + \eta_i - \eta_j - \eta_k - \eta_l), \Omega)\} \end{aligned} \tag{1.6.13}$$

Now let $B = V \amalg \{i, j, k, l\} \amalg V'$ be a disjoint partition of B. We see that $\theta[\eta_U - \eta_V]((\Omega I)(\eta_J - \eta_i), \Omega)$ differs by a unit factor from $\theta[\eta_U - \eta_i - \eta_V + \eta_J](0, \Omega)$, and so by the *vanishing equations* is going to be zero whenever $J \in V \amalg \{i\}$. We may use the vanishing equations since proposition 1.6.10 implies that these are equivalent to the

Frobenius theta formula. On the other hand, $\theta[\eta_U - \eta_V]((\Omega I)(\eta_J - \eta_j - \eta_k - \eta_l), \Omega)$ differs by a unit factor from $\theta[\eta_U - \eta_V + \eta_J - \eta_j - \eta_k - \eta_l](0, \Omega)$ and hence from $\theta[\eta_U - \eta_i - \eta_{V'} + \eta_J](0, \Omega)$. This vanishes for $J \in V' \amalg \{i\}$, leaving only three terms in the summation, namely $J = j, k, l$. Hence we may write equation 1.6.13 as follows.

$$
\begin{aligned}
0 = &\{\epsilon_U(j)\boldsymbol{e}^{2\pi i(-2\eta_j - 2\eta_k - 2\eta_l)'(\Omega I)\eta_j}\,\boldsymbol{e}^{4\pi i\eta_j'\Omega\eta_j'} \\
&\quad\cdot\theta[\eta_U - \eta_V]((\Omega I)(\eta_j - \eta_i), \Omega)\theta[\eta_U - \eta_V]((\Omega I)(-\eta_k - \eta_l), \Omega) \\
&\quad\cdot\vec{\theta}_2(\tfrac{1}{2}(\Omega I)(\eta_i + \eta_j - \eta_k - \eta_l), \Omega)\} \\
&+\{\epsilon_U(k)\boldsymbol{e}^{2\pi i(-2\eta_j - 2\eta_k - 2\eta_l)'(\Omega I)\eta_k}\,\boldsymbol{e}^{4\pi i\eta_k'\Omega\eta_k'} \\
&\quad\cdot\theta[\eta_U - \eta_V]((\Omega I)(\eta_k - \eta_i), \Omega)\theta[\eta_U - \eta_V]((\Omega I)(-\eta_j - \eta_l), \Omega) \\
&\quad\cdot\vec{\theta}_2(\tfrac{1}{2}(\Omega I)(\eta_i + \eta_k - \eta_j - \eta_l), \Omega)\} \\
&+\{\epsilon_U(l)\boldsymbol{e}^{2\pi i(-2\eta_j - 2\eta_k - 2\eta_l)'(\Omega I)\eta_l}\,\boldsymbol{e}^{4\pi i\eta_l'\Omega\eta_l'} \\
&\quad\cdot\theta[\eta_U - \eta_V]((\Omega I)(\eta_l - \eta_i), \Omega)\theta[\eta_U - \eta_V]((\Omega I)(-\eta_j - \eta_k), \Omega) \\
&\quad\cdot\vec{\theta}_2(\tfrac{1}{2}(\Omega I)(\eta_i + \eta_l - \eta_j - \eta_k), \Omega)\}
\end{aligned} \tag{1.6.14}
$$

Notice that 1.6.14 is a cyclic sum, each term becoming the next under a (jkl)-cycle; so it suffices to simplify the first term. We use $\theta[\eta_U - \eta_V]((\Omega I)(\eta_j - \eta_i), \Omega)$ $= -\theta[\eta_U - \eta_V]((\Omega I)(\eta_i - \eta_j), \Omega)$, this is true because $\eta_U - \eta_V$ is an odd characteristic. We also may rewrite $\theta[\eta_U - \eta_V]((\Omega I)(-\eta_k - \eta_l), \Omega)$ as follows,

$$
\begin{aligned}
&\theta[\eta_U - \eta_V]((\Omega I)(-\eta_k - \eta_l), \Omega) = \theta[\eta_U - \eta_V]((\Omega I)(\eta_k - \eta_l) - 2(\Omega I)\eta_k, \Omega) \\
=&\theta[\eta_U - \eta_V]((\Omega I)(\eta_k - \eta_l), \Omega)\{\boldsymbol{e}^{2\pi i\{(\eta_U - \eta_V)'(-2\eta_k'') - (-2\eta_k')(\eta_U - \eta_V)''\}} \\
&\cdot\boldsymbol{e}^{-2\pi i\{\frac{1}{2}(-2\eta_k)'\Omega(-2\eta_k)' + (-2\eta_k')(\Omega I)(\eta_k - \eta_l)\}}\}.
\end{aligned}
$$

We thus rewrite the first term of 1.6.14 as follows,

$$
\begin{aligned}
&\{\theta[\eta_U - \eta_V]((\Omega I)(\eta_i - \eta_j), \Omega)\theta[\eta_U - \eta_V]((\Omega I)(\eta_k - \eta_l), \Omega) \\
&\vec{\theta}_2(\tfrac{1}{2}(\Omega I)(\eta_i + \eta_j - \eta_k - \eta_l), \Omega)\} \\
(\text{ times })&\{(-1)\epsilon_U(j)\boldsymbol{e}^{2\pi i\{-2(\eta_j + \eta_k + \eta_l)'(\Omega I)\eta_j\}}\,\boldsymbol{e}^{4\pi i\{\eta_j'\Omega\eta_j'\}} \\
&\boldsymbol{e}^{2\pi i\{(\eta_U - \eta_V)'(-2\eta_k'') - (-2\eta_k')(\eta_U - \eta_V)''\}} \\
&\boldsymbol{e}^{-2\pi i\{\frac{1}{2}(-2\eta_k')\Omega(-2\eta_k') + (-2\eta_k')(\Omega I)(\eta_k - \eta_l)\}}\}.
\end{aligned}
$$

After a little work the unit following the "(times)" is seen to be:

$$
\begin{aligned}
&\boldsymbol{e}^{2\pi i\{-2\eta_k'\Omega\eta_j' - 2\eta_l'\Omega\eta_j' - 2\eta_k'\Omega\eta_l'\}}\,\boldsymbol{e}^{2\pi i\{-2\eta_k''\cdot\eta_j' - 2\eta_l'\cdot\eta_j'' - 2\eta_k'\cdot\eta_l''\}}(\epsilon_{U\circ V}(\infty)) \\
=&\epsilon_{U\circ V}(\infty)\boldsymbol{e}^{-4\pi i\{\eta_k'(\Omega I)\eta_l + \eta_l'(\Omega I)\eta_j + \eta_j'(\Omega I)\eta_k\}},
\end{aligned}
$$

which is cyclic in (jkl) and hence identical for each term in 1.6.14. Cancelling this cyclic unit from the summation in 1.6.14 we obtain our conclusion,

$$\begin{aligned}0 =&\theta[\eta_U-\eta_V]((\Omega I)(\eta_i-\eta_j),\Omega)\theta[\eta_U-\eta_V]((\Omega I)(\eta_k-\eta_l),\Omega)\vec{\theta}_2(\xi_{ijkl},\Omega)\\ &+\theta[\eta_U-\eta_V]((\Omega I)(\eta_i-\eta_k),\Omega)\theta[\eta_U-\eta_V]((\Omega I)(\eta_l-\eta_j),\Omega)\vec{\theta}_2(\xi_{iklj},\Omega)\\ &+\theta[\eta_U-\eta_V]((\Omega I)(\eta_i-\eta_l),\Omega)\theta[\eta_U-\eta_V]((\Omega I)(\eta_j-\eta_k),\Omega)\vec{\theta}_2(\xi_{iljk},\Omega).\end{aligned}$$

1.6.15 Lemma. *Let $\Omega\in\mathcal{H}_g$, $\eta\in\Xi_g$. For all $\nu_2,\nu_3\in\mathbb{Z}^g$, we have*

$$\begin{vmatrix}\theta_2[\nu_2](\xi_{ikjl},\Omega) & \theta_2[\nu_2](\xi_{iljk},\Omega)\\ \theta_2[\nu_3](\xi_{ikjl},\Omega) & \theta_2[\nu_3](\xi_{iljk},\Omega)\end{vmatrix} = 2^{-g}\sum_{p\in\mathbb{Z}^g/2\mathbb{Z}^g}\{(\boldsymbol{e}^{-i\pi p\cdot\nu_2}-\boldsymbol{e}^{-i\pi p\cdot\nu_3})$$
$$\cdot\theta[\tfrac{\nu_2+\nu_3}{2}|\tfrac{p}{2}]((\Omega I)(\eta_i-\eta_j),\Omega)\theta[\tfrac{\nu_2+\nu_3}{2}|\tfrac{p}{2}]((\Omega I)(\eta_k-\eta_l),\Omega)\}.$$

Proof. Here is the well–known formula of proposition 1.1.16,

$$\theta_2[\nu_1|0](x,\Omega)\,\theta_2[\nu_2|0](y,\Omega) = 2^{-g}\sum_{p\in\mathbb{Z}^g/2\mathbb{Z}^g}\boldsymbol{e}^{-2\pi i p\cdot\nu_1}\theta[\tfrac{\nu_1+\nu_2}{2}|\tfrac{p}{2}](x+y,\Omega)\theta[\tfrac{\nu_1-\nu_2}{2}|\tfrac{p}{2}](x-y,\Omega).$$

Now for $\nu_1,\nu_2\in\mathbb{Z}^g$, we have that $\theta[\frac{\nu_1-\nu_2}{2}|\frac{p}{2}](x-y,\Omega)=\theta[\frac{\nu_1+\nu_2}{2}|\frac{p}{2}](x-y,\Omega)$, so that the only factor in the summation not symmetric in ν_1, ν_2, is $\boldsymbol{e}^{-2\pi i p\cdot\nu_1}$. Hence:

$$\begin{vmatrix}\theta_2[\nu_2](z_1,\Omega) & \theta_2[\nu_2](z_2,\Omega)\\ \theta_2[\nu_3](z_1,\Omega) & \theta_2[\nu_3](z_2,\Omega)\end{vmatrix} = \theta_2[\nu_2](z_1,\Omega)\theta_2[\nu_3](z_2,\Omega)-\theta_2[\nu_3](z_1,\Omega)\theta_2[\nu_2](z_2,\Omega)$$
$$=2^{-g}\sum_{p\in\mathbb{Z}^g/2\mathbb{Z}^g}(\boldsymbol{e}^{-i\pi p\cdot\nu_2}-\boldsymbol{e}^{-i\pi p\cdot\nu_3})\theta[\tfrac{\nu_2+\nu_3}{2}|\tfrac{p}{2}](z_1+z_2,\Omega)\theta[\tfrac{\nu_2+\nu_3}{2}|\tfrac{p}{2}](z_1-z_2,\Omega).$$

If we now let $z_1=\xi_{ikjl}$ and $z_2=\xi_{iljk}$, so that $z_1+z_2=(\Omega I)(\eta_i-\eta_j)$ and $z_1-z_2=(\Omega I)(\eta_k-\eta_l)$, then we have the assertion of lemma 1.6.15.

1.6.16 Theorem. *Let $\Omega\in\mathcal{H}_g$ and $\eta\in\Xi_g$. We have $\Omega\in V'_{g,\eta}\implies\Omega\in F'_{g,\eta}$.*

Proof. We have seen in proposition 1.6.10 that equations $V_{g,\eta}$ imply the Frobenius theta formula, and we will use both sets of equations to derive the equations $F_{g,\eta}$. The equations, $F_{g,\eta}$, defined by

$$\operatorname{rank}\{\vec{\theta}_2(\xi_{ijkl},\Omega),\vec{\theta}_2(\xi_{iklj},\Omega),\vec{\theta}_2(\xi_{iljk},\Omega)\}\le 2,$$

are given by the vanishing of 3×3 minors for all $i,j,k,l\in B$ and $\nu_1,\nu_2,\nu_3\in\mathbb{Z}^g$,

$$0=\begin{vmatrix}\theta_2[\nu_1](\xi_{ijkl},\Omega) & \theta_2[\nu_1](\xi_{iklj},\Omega) & \theta_2[\nu_1](\xi_{iljk},\Omega)\\ \theta_2[\nu_2](\xi_{ijkl},\Omega) & \theta_2[\nu_2](\xi_{iklj},\Omega) & \theta_2[\nu_2](\xi_{iljk},\Omega)\\ \theta_2[\nu_3](\xi_{ijkl},\Omega) & \theta_2[\nu_3](\xi_{iklj},\Omega) & \theta_2[\nu_3](\xi_{iljk},\Omega)\end{vmatrix}.$$

We can expand this determinant out by the first row,

$$\begin{aligned}0 =&\theta_2[\nu_1](\xi_{ijkl},\Omega)\begin{vmatrix}\theta_2[\nu_2](\xi_{iklj},\Omega) & \theta_2[\nu_2](\xi_{iljk},\Omega)\\ \theta_2[\nu_3](\xi_{iklj},\Omega) & \theta_2[\nu_3](\xi_{iljk},\Omega)\end{vmatrix}\\ &+\theta_2[\nu_1](\xi_{iklj},\Omega)\begin{vmatrix}\theta_2[\nu_2](\xi_{iljk},\Omega) & \theta_2[\nu_2](\xi_{ijkl},\Omega)\\ \theta_2[\nu_3](\xi_{iljk},\Omega) & \theta_2[\nu_3](\xi_{ijkl},\Omega)\end{vmatrix}\\ &+\theta_2[\nu_1](\xi_{iljk},\Omega)\begin{vmatrix}\theta_2[\nu_2](\xi_{ijkl},\Omega) & \theta_2[\nu_2](\xi_{iklj},\Omega)\\ \theta_2[\nu_3](\xi_{ijkl},\Omega) & \theta_2[\nu_3](\xi_{iklj},\Omega)\end{vmatrix}.\end{aligned}$$

The equations $F_{g,\eta}$ can then be expressed as: $\forall\ i,j,k,l\in B$, $\forall\ \nu_2,\nu_3\in\mathbb{Z}^g$,

$$\begin{aligned}0 =&\begin{vmatrix}\theta_2[\nu_2](\xi_{iklj},\Omega) & \theta_2[\nu_2](\xi_{iljk},\Omega)\\ \theta_2[\nu_3](\xi_{iklj},\Omega) & \theta_2[\nu_3](\xi_{iljk},\Omega)\end{vmatrix}\vec{\theta}_2(\xi_{ijkl},\Omega)\\ &+\begin{vmatrix}\theta_2[\nu_2](\xi_{iljk},\Omega) & \theta_2[\nu_2](\xi_{ijkl},\Omega)\\ \theta_2[\nu_3](\xi_{iljk},\Omega) & \theta_2[\nu_3](\xi_{ijkl},\Omega)\end{vmatrix}\vec{\theta}_2(\xi_{iklj},\Omega)\\ &+\begin{vmatrix}\theta_2[\nu_2](\xi_{ijkl},\Omega) & \theta_2[\nu_2](\xi_{iklj},\Omega)\\ \theta_2[\nu_3](\xi_{ijkl},\Omega) & \theta_2[\nu_3](\xi_{iklj},\Omega)\end{vmatrix}\vec{\theta}_2(\xi_{iljk},\Omega).\end{aligned}$$

We use lemma 1.6.15 to substitute in for the determinant in the first term. Since the sum is cyclic in (jkl) we may easily do this for the last two terms as well,

$$\begin{aligned}0 = 2^{-g}&\sum_{p\in\mathbb{Z}^g/2\mathbb{Z}^g}\{(\boldsymbol{e}^{-i\pi p\cdot\nu_2}-\boldsymbol{e}^{-i\pi p\cdot\nu_3})\\ &\cdot\{\theta[\frac{1}{2}(\nu_2+\nu_3)|\frac{p}{2}]((\Omega I)(\eta_i-\eta_j),\Omega)\theta[\frac{1}{2}(\nu_2+\nu_3)|\frac{p}{2}]((\Omega I)(\eta_l-\eta_k),\Omega)\vec{\theta}_2(\xi_{ijkl},\Omega)\\ &+\theta[\frac{1}{2}(\nu_2+\nu_3)|\frac{p}{2}]((\Omega I)(\eta_i-\eta_k),\Omega)\theta[\frac{1}{2}(\nu_2+\nu_3)|\frac{p}{2}]((\Omega I)(\eta_j-\eta_l),\Omega)\vec{\theta}_2(\xi_{iklj},\Omega)\\ &+\theta[\frac{1}{2}(\nu_2+\nu_3)|\frac{p}{2}]((\Omega I)(\eta_i-\eta_l),\Omega)\theta[\frac{1}{2}(\nu_2+\nu_3)|\frac{p}{2}]((\Omega I)(\eta_k-\eta_j),\Omega)\vec{\theta}_2(\xi_{iljk},\Omega)\}\}.\end{aligned} \tag{1.6.17}$$

To show that the equations $V_{g,\eta}$ imply the equations $F_{g,\eta}$ we will need to show that $V_{g,\eta}$ implies that the sum in 1.6.17 vanishes. The common factor $\boldsymbol{e}^{-i\pi p\cdot\nu_2}-\boldsymbol{e}^{-i\pi p\cdot\nu_3}$ vanishes if and only if the characteristic $[\frac{1}{2}(\nu_2+\nu_3)|\frac{p}{2}]$ is even. For p such that the characteristic $[\frac{1}{2}(\nu_2+\nu_3)|\frac{p}{2}]$ is odd we will show that the adjacent factor in the braces {-} of 1.6.17 is zero. We may assume that the odd characteristic $[\frac{1}{2}(\nu_2+\nu_3)|\frac{p}{2}]\equiv\eta_U-\eta_V \mod \mathbb{Z}^{2g}$, where $|V|=g-1,g-5,g-9,\dots$. If however, $|V|\le g-5$ then by the vanishing equations every theta constant in the braces vanishes since, for example, $\frac{1}{2}||V\circ\{i,j\}|-(g+1)|\ge 2$. Thence we may assume that $[\frac{1}{2}(\nu_2+\nu_3)|\frac{p}{2}]=\eta_U-\eta_V$, where $|V|=g-1$.

Now we use lemma 1.6.12 to see that since $\Omega\in V'_{g,\eta}$ then $\Omega\in\mathrm{Frob}'_{g,\eta}$ and so the three term sum in the braces of 1.6.17 vanishes whenever $i,j,k,l\notin V$. To conclude the proof we need to show that the three term sum also vanishes when one of the

$i,j,k,l \in V$. This is actually a degenerate case, for if $j \in V$ then by the vanishing equations any theta constant with a $(\Omega I)\eta_j$ in its argument vanishes.

2. Chapter Two

§2.1 Construction of the p^{ijkl}.

In this section we begin with an irreducible $\Omega \in \mathcal{H}_g$ satisfying the equations $F_{g,\eta}$ of 1.4.21; these equations are consequences of Fay's trisecant formula in the hyperelliptic case. The existence and uniqueness of certain nonzero constants p^{ijkl} follow from these assumptions. If Ω were the period matrix of an appropriately marked hyperelliptic curve then these constants p^{ijkl} would be the values of Gunning's crossratio function at certain liftings of the branch points of the hyperelliptic curve. The values of Gunning's crossratio function here correspond to square roots of projective crossratios, and the "recovery" of these "crossratios" p^{ijkl} is given in theorem 2.1.1 which is the main result of this section.

Recall that Ξ_g as defined by 1.4.11 is a special set of maps from B to $\frac{1}{2}\mathbb{Z}^{2g}$, where $B = \{1, 2, \dots 2g+1, \infty\}$. We write ξ_{ijkl} for $\frac{1}{2}(\Omega I)(\eta_i + \eta_j - \eta_k - \eta_l)$. The invariance of ξ_{ijkl} up to $\pm$ signs under the obvious 4-group of permutations on $\{i,j,k,l\}$ will be used without further mention.

2.1.1 Theorem. *Let $\Omega \in \mathcal{H}_g$ be irreducible and let $\eta \in \Xi_g$. If $\Omega \in F'_{g,\eta}$ then there exist unique $p^{ijkl} \in \mathbb{C}^*$ such that for all distinct $i,j,k,l,m \in B$ we have (1)-(5).*

(1) $p^{ijkl} = p^{jilk} = p^{klij} = p^{lkji}$
(2) $p^{ijkl}\,p^{jikl} = 1$
(3) $p^{ijkl}\,p^{iljk}\,p^{iklj} = -1$
(4) $p^{ijkl}\,p^{ijlm} = p^{ijkm}$
(5) $p^{ijkl}\vec{\theta}_2(\xi_{ikjl},\Omega) + p^{ikjl}\vec{\theta}_2(\xi_{ijkl},\Omega) = \vec{\theta}_2(\xi_{iljk},\Omega)$

2.1.2 Corollary. *As a meromorphic function of α on Θ we have*

$$p^{ijkl} = \frac{\theta(\alpha + (\Omega I)(\eta_i - \eta_k),\Omega)}{\theta(\alpha + (\Omega I)(\eta_i - \eta_l),\Omega)}\frac{\theta(\alpha + (\Omega I)(\eta_j - \eta_l),\Omega)}{\theta(\alpha + (\Omega I)(\eta_j - \eta_k),\Omega)}.$$

2.1.3 Lemma (Gunning). [6] *Let $\Omega \in \mathcal{H}_g$ be irreducible. For $x, y \in \mathbb{C}^g$ we have*

$$\operatorname{rank}\{\vec{\theta}_2(x,\Omega), \vec{\theta}_2(y,\Omega)\} < 2 \iff x = \pm y \text{ in } A = \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g).$$

Proof of lemma 2.1.3. This lemma is really a generalization of lemma 1.1.14 in §1.1. One direction is clear since $\vec{\theta}_2(x,\Omega)$ is an even function of x and we have that $\vec{\theta}_2(x,\Omega) - \vec{\theta}_2(-x,\Omega) = 0$. Suppose on the other hand that

$$(2.1.4\ \ )\qquad \exists\, c_1, c_2 \in \mathbb{C}^2 : (c_1,c_2) \neq (0,0) \text{ and } c_1\vec{\theta}_2(x,\Omega) + c_2\vec{\theta}_2(y,\Omega) = 0.$$

Note that the c_i are actually each nonzero by lemma 1.1.14. An application of the addition formula 1.1.15 to the equation 2.1.4 produces

$$c_1\vec{\theta}_2(x,\Omega)\cdot\vec{\theta}_2(w,\Omega) = -c_2\vec{\theta}_2(y,\Omega)\cdot\vec{\theta}_2(w,\Omega)$$

(2.1.5)

$$c_1\theta(w+x,\Omega)\theta(w-x,\Omega) = -c_2\theta(w+y,\Omega)\theta(w-y,\Omega).$$

Take divisors as functions of w in A in the second equality of 2.1.5. In the notation $\Theta_a = \{z : z = \zeta + a \text{ for } \zeta \in \Theta\}$ we conclude: $\Theta_x \cup \Theta_{-x} = \Theta_y \cup \Theta_{-y}$. Recall from the discussion after lemma 1.1.4 that the symplectic irreducibility of Ω implies the analytic irreducibility of Θ. The uniqueness up to order of the decomposition of analytic subvarieties into irreducible components affords the following consequence: $\Theta_x = \Theta_y$ or $\Theta_x = \Theta_{-y}$. From lemma 1.1.4 this implies $x = y$ in A or $x = -y$ in A, which is the conclusion of lemma 2.1.3. □

2.1.6 Proposition. *Let $\Omega \in \mathcal{H}_g$ be irreducible and let $\eta \in \Xi_g$.*

For all $i, j, k, l \in B$ such that $i \neq l, j \neq k$, $\operatorname{rank}\{\vec{\theta}_2(\xi_{ijkl}, \Omega), \vec{\theta}_2(\xi_{iklj}, \Omega)\} > 1$.

Hence, for all distinct $i, j, k, l \in B$, $\operatorname{rank}\{\vec{\theta}_2(\xi_{ijkl}, \Omega), \vec{\theta}_2(\xi_{iklj}, \Omega), \vec{\theta}_2(\xi_{iljk}, \Omega)\} > 1$.

Proof of proposition 2.1.6. It suffices to prove the first statement. Suppose that the $\operatorname{rank}\{\vec{\theta}_2(\xi_{ijkl}, \Omega), \vec{\theta}_2(\xi_{iklj}, \Omega)\} \leq 1$; then noting the irreducibility of Ω and using lemma 2.1.3 we have $\xi_{ijkl} = \pm\xi_{iklj} \mod \mathbb{Z}^g + \Omega\mathbb{Z}^g$. Recalling the definition of the ξ_{ijkl} this implies that $\eta_j - \eta_k \in \mathbb{Z}^{2g}$ or $\eta_i - \eta_l \in \mathbb{Z}^{2g}$. For distinct i, l, however, the η_i, η_l, are distinct as half–integers in $\frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ and this contradiction shows that the relevant rank must be greater than one.

Proof of theorem 2.1.1 and corollary 2.1.2. Since $\Omega \in F'_{g,\eta}$, we have

$$\operatorname{rank}\{\vec{\theta}_2(\xi_{iklj}, \Omega), \vec{\theta}_2(\xi_{ijkl}, \Omega), \vec{\theta}_2(\xi_{iljk}, \Omega)\} \leq 2.$$

By proposition 2.1.6, the irreducibility of Ω implies that for distinct i, j, k, l, this rank is exactly 2. Since the rank is 2 there exist p^{ijkl}, b_{ijkl}, and c_{ijkl}, unique up to a common constant multiple such that:

$$p^{ijkl}\vec{\theta}_2(\xi_{iklj}, \Omega) + b_{ijkl}\vec{\theta}_2(\xi_{ijkl}, \Omega) = c_{ijkl}\vec{\theta}_2(\xi_{iljk}, \Omega). \tag{2.1.7}$$

Now we make an observation which although simple is full of consequences. By proposition 2.1.6, the rank of any distinct pair of the $\vec{\theta}_2(\xi_\bullet, \Omega)$ is two so that each coefficient p^{ijkl}, b_{ijkl}, and c_{ijkl}, must be nonzero. The coefficients are hence uniquely determined if we take $c_{ijkl} = 1$. If we switch j and k and use this uniqueness we see that $b_{ijkl} = p^{ikjl}$; therefore we have deduced (5) of theorem 2.1.1,

$$\begin{aligned} &\exists_1 \; p^{ijkl} \neq 0 \text{ for all distinct } i, j, k, l \in B: \\ &p^{ijkl}\vec{\theta}_2(\xi_{iklj}, \Omega) + p^{ikjl}\vec{\theta}_2(\xi_{ijkl}, \Omega) = \vec{\theta}_2(\xi_{iljk}, \Omega). \end{aligned} \tag{2.1.8}$$

The symmetries (1)–(3) of $p^\bullet$ are consequences of 2.1.8. The four–group symmetries of (1) leave the vectors of second order theta constants unchanged because $\vec{\theta}_2(w, \Omega)$ is an even function of w. By the uniqueness of the $p^\bullet$ we then have (1). Switching i and j in 2.1.8 gives:

$$\begin{aligned} p^{jikl}\vec{\theta}_2(\xi_{jkli}, \Omega) + p^{jkil}\vec{\theta}_2(\xi_{jikl}, \Omega) &= \vec{\theta}_2(\xi_{jlik}, \Omega), \\ \text{or } -\vec{\theta}_2(\xi_{iklj}, \Omega) + p^{jkil}\vec{\theta}_2(\xi_{ijkl}, \Omega) &= -p^{jikl}\vec{\theta}_2(\xi_{iljk}, \Omega). \end{aligned}$$

By the uniqueness of the $p^\bullet$ we have: $p^{ijkl} = (-1)/(-p^{jikl})$, which is (2), and $p^{ikjl} = p^{jkil}/(-p^{jikl})$. This last equation is (3) after applying the symmetries of (1) and (2).

The final symmetry (4) has a more involved derivation which uses the irreducibility hypothesis again, and which when given will complete the proof of theorem 2.1.1. Dotting of each side of equation 2.1.8 with $\vec{\theta}_2(w,\Omega)$ we obtain:

$$\begin{aligned} &p^{ijkl}\theta(w+\xi_{iklj},\Omega)\theta(w-\xi_{iklj},\Omega)+p^{ikjl}\theta(w+\xi_{ijkl},\Omega)\theta(w-\xi_{ijkl},\Omega)\\ =&\theta(w+\xi_{iljk},\Omega)\theta(w-\xi_{iljk},\Omega).\end{aligned}$$

Let $w = \alpha + \xi_{ijkl}$ and restrict α to Θ so that the second term vanishes. We then have

$$\begin{aligned} &p^{ijkl}\theta(\alpha+\xi_{ijkl}+\xi_{iklj},\Omega)\theta(\alpha+\xi_{ijkl}-\xi_{iklj},\Omega)\\ =&\theta(\alpha+\xi_{ijkl}+\xi_{iljk},\Omega)\theta(\alpha+\xi_{ijkl}-\xi_{iljk},\Omega), \text{ or}\\ &p^{ijkl}\theta(\alpha+(\Omega I)(\eta_i-\eta_l),\Omega)\theta(\alpha+(\Omega I)(\eta_j-\eta_k),\Omega)\\ =&\theta(\alpha+(\Omega I)(\eta_i-\eta_k),\Omega)\theta(\alpha+(\Omega I)(\eta_j-\eta_l),\Omega).\end{aligned} \tag{2.1.9}$$

We now show that the product $\theta(w+(\Omega I)(\eta_i-\eta_l),\Omega)\ \theta(w+(\Omega I)(\eta_j-\eta_k),\Omega)$ does not vanish identically for $w \in \Theta$. The irreducibility of Θ would imply that one of these factors vanished identically on Θ, and hence that $\eta_j - \eta_k \in \mathbb{Z}^{2g}$, or $\eta_i - \eta_l \in \mathbb{Z}^{2g}$, by lemma 1.1.4. For distinct i, j, k, l this is impossible so that $\theta(\alpha + (\Omega I)(\eta_i - \eta_l), \Omega)$ $\theta(\alpha + (\Omega I)(\eta_j - \eta_k), \Omega)$ cannot vanish identically for $\alpha \in \Theta$. We conclude that

$$p^{ijkl} = \frac{\theta(\alpha+(\Omega I)(\eta_i-\eta_k),\Omega)}{\theta(\alpha+(\Omega I)(\eta_i-\eta_l),\Omega)}\frac{\theta(\alpha+(\Omega I)(\eta_j-\eta_l),\Omega)}{\theta(\alpha+(\Omega I)(\eta_j-\eta_k),\Omega)}$$

as a meromorphic function on Θ. This is in fact corollary 2.1.2. We also have

$$\begin{aligned} p^{ijkl}p^{ijlm} =&\frac{\theta(\alpha+(\Omega I)(\eta_i-\eta_k),\Omega)}{\theta(\alpha+(\Omega I)(\eta_i-\eta_l),\Omega)}\frac{\theta(\alpha+(\Omega I)(\eta_j-\eta_l),\Omega)}{\theta(\alpha+(\Omega I)(\eta_j-\eta_k),\Omega)}\\ &\frac{\theta(\alpha+(\Omega I)(\eta_i-\eta_l),\Omega)}{\theta(\alpha+(\Omega I)(\eta_i-\eta_m),\Omega)}\frac{\theta(\alpha+(\Omega I)(\eta_j-\eta_m),\Omega)}{\theta(\alpha+(\Omega I)(\eta_j-\eta_l),\Omega)}\\ =&\frac{\theta(\alpha+(\Omega I)(\eta_i-\eta_k),\Omega)}{\theta(\alpha+(\Omega I)(\eta_i-\eta_m),\Omega)}\frac{\theta(\alpha+(\Omega I)(\eta_j-\eta_m),\Omega)}{\theta(\alpha+(\Omega I)(\eta_j-\eta_k),\Omega)} = p^{ijkm}.\end{aligned} \tag{2.1.10}$$

Now equation 2.1.10 is an identity among constant meromorphic functions on Θ and hence an identity among constants. This completes the proof of theorem 2.1.1 and corollary 2.1.2. □

Remark. We could enlarge the domain of p^{ijkl} to certain coincident values, namely $p^{iikl} = 1$ and $p^{ikil} = 0$.

§2.2 Multisecants and the $p^{A\,B\,ab}$.

In this section we prove theorem 2.2.1 which is our version of the multisecant formula. The multisecant formula follows from the trisecant formula of section §2.1 and the irreducibility of Ω. Also, numbers that would be values of Gunning's cross-ratio functions if Ω were hyperelliptic , the $p^{A\,B\,ab}$, are related to the thetanullwerte of Ω in corollary 2.2.5. The proofs in this section are parallel to Fay's delevopment of the multisecant formula in [4].

2.2.1 Theorem. *Let $\Omega \in \mathcal{H}_g$ be irreducible and let $\eta \in \Xi_g$. Assume that $p^{ijkl} \in \mathbb{C}^*$ exist for all distinct $i,j,k,l \in B$, and that these $p^\bullet$ satisfy the cross-ratio symmetries of lemma 1.2.6 as well as*

$$\vec{\theta}_2(\xi_{iljk},\Omega) = p^{ikjl}\,\vec{\theta}_2(\xi_{ijkl},\Omega) + p^{ijkl}\,\vec{\theta}_2(\xi_{iklj},\Omega). \tag{2.2.2 }$$

Then for $N : 2 \le N \le g+1$, and for all distinct $k_1,\dots,k_N,l_1,\dots,l_N \in B$, we have

$$\vec{\theta}_2(t+(\Omega I)\eta_{k_N},\Omega) = \sum_{j=1}^{N} p^{L\setminus l_j\;K\setminus k_N\;k_N l_j}\,\vec{\theta}_2(t+(\Omega I)\eta_{l_j},\Omega), \tag{2.2.3 }$$

where $t = \frac{1}{2}(\Omega I)(\eta_{k_1}+\dots+\eta_{k_{N-1}}-\eta_{l_1}-\dots-\eta_{l_N}-\eta_{k_N})$, and where

$$p^{A\,\bar{B}\,ab} = \prod_{i=1}^{n} p^{a_i b_i ab}, \qquad \text{for } A=\{a_i\}_1^n, \bar{B}=\{b_i\}_1^n. \tag{2.2.4 }$$

2.2.5 Corollary. *Let the assumptions be as in theorem 2.2.1. For any disjoint union $B=\{a\}\amalg A \amalg \{b\} \amalg \bar{B}$, where $A=\{a_i\}_1^g$ and $\bar{B}=\{b_i\}_1^g$, we have*

$$\begin{aligned} p^{A\,\bar{B}\,ab}&\theta((\Omega\,I)(\eta_U+\eta_a-\eta_{\bar{B}}),\Omega)\theta((\Omega\,I)(\eta_U+\eta_b-\eta_A),\Omega)\\ &=\theta((\Omega\,I)(\eta_U+\eta_a-\eta_A),\Omega)\theta((\Omega\,I)(\eta_U+\eta_b-\eta_{\bar{B}}),\Omega). \end{aligned}$$

2.2.6 Lemma. *Let $\Omega \in \mathcal{H}_g$ and $\eta \in \Xi_g$. Let $p^{ijkl} \in \mathbb{C}^*$ be given for all distinct $i,j,k,l \in B$. The following two statements are equivalent.*

(1) *The $p^\bullet$ satisfy the crossratio symmetries of lemma 1.2.6 and*

$$\vec{\theta}_2(\xi_{iljk},\Omega) = p^{ikjl}\,\vec{\theta}_2(\xi_{ijkl},\Omega) + p^{ijkl}\,\vec{\theta}_2(\xi_{iklj},\Omega).$$

(2) *$\forall$ distinct $i,j \in B$, $\exists\, q^{ij} \in \mathbb{C}^* : q^{ij} = -q^{ji}$, $p^{ijkl} = \frac{q^{ik}q^{jl}}{q^{il}q^{jk}}$, and $\forall\ y \in \mathbb{C}^g$,*

$$\begin{aligned} &\frac{q^{il}q^{kj}}{q^{ij}q^{lk}q^{ik}q^{lj}}\theta(y,\Omega)\theta(y+(\Omega I)(\eta_i+\eta_l-\eta_j-\eta_k),\Omega)\\ &\qquad=\begin{vmatrix} \dfrac{\theta(y+(\Omega I)(\eta_i-\eta_j),\Omega)}{q^{ij}} & \dfrac{\theta(y+(\Omega I)(\eta_l-\eta_j),\Omega)}{q^{lj}} \\ \dfrac{\theta(y+(\Omega I)(\eta_i-\eta_k),\Omega)}{q^{ik}} & \dfrac{\theta(y+(\Omega I)(\eta_l-\eta_k),\Omega)}{q^{lk}} \end{vmatrix}. \end{aligned}$$

Proof of lemma 2.2.6. We use proposition 3.4.1, with $(\mathbb{C}^*;-1)$ being the abelian group with distinguished idempotent, to conclude the existence of skew $q^{ij}\in\mathbb{C}^*$ such that $p^{ijkl}=\frac{q^{ik}q^{jl}}{q^{il}q^{jk}}$ if and only if the $p^{ijkl}\in\mathbb{C}^*$ satisfy the crossratio symmetries. The remaining identity 2.2.2 becomes:

$$\vec{\theta}_2(\xi_{iljk},\Omega)=\frac{q^{ij}q^{kl}}{q^{il}q^{kj}}\vec{\theta}_2(\xi_{ijkl},\Omega)+\frac{q^{ik}q^{jl}}{q^{il}q^{jk}}\vec{\theta}_2(\xi_{iklj},\Omega),\ \text{or}$$
$$q^{il}q^{kj}\vec{\theta}_2(\xi_{iljk},\Omega)+q^{ij}q^{lk}\vec{\theta}_2(\xi_{ijkl},\Omega)+q^{ik}q^{jl}\vec{\theta}_2(\xi_{iklj},\Omega)=0.$$

We dot both sides of this last equation with $\vec{\theta}_2(y+\xi_{iljk},\Omega)$, and use the addition formula of proposition 1.1.15 to obtain:

$$\begin{aligned} &q^{ik}q^{jl}\theta(y+(\Omega I)(\eta_i-\eta_j),\Omega)\theta(y+(\Omega I)(\eta_l-\eta_k),\Omega)\\ &\quad+q^{ij}q^{lk}\theta(y+(\Omega I)(\eta_i-\eta_k),\Omega)\theta(y+(\Omega I)(\eta_l-\eta_j),\Omega)\\ &\quad\quad+q^{il}q^{kj}\theta(y+(\Omega I)(\eta_i-\eta_j-\eta_k+\eta_l),\Omega)\theta(y,\Omega)=0. \end{aligned}\tag{2.2.7}$$

Dividing both sides of equation 2.2.7 by $q^{ik}q^{jl}q^{ij}q^{lk}$, we obtain the determinant formula of lemma 2.2.6. This shows that the two formulas are equivalent once we have $p^{ijkl}=\frac{q^{ik}q^{jl}}{q^{il}q^{jk}}$. The reverse implication relies on the fact that $\vec{\theta}_2(z,\Omega)$ is a vector of all members of a basis for the vector space of second order theta functions. □

2.2.8 Lemma. *Let $\Omega\in\mathcal{H}_g$, and $\eta\in\Xi_g$, and $1\le N\le g+1$. Let $p^{ijkl}\in\mathbb{C}^*$ be given for all distinct $i,j,k,l\in B$ such that the $p^\bullet$ satisfy the crossratio symmetries and*

$$\vec{\theta}_2(\xi_{iljk},\Omega)=p^{ikjl}\,\vec{\theta}_2(\xi_{ijkl},\Omega)+p^{ijkl}\,\vec{\theta}_2(\xi_{iklj},\Omega).$$

Then for all distinct $k_1,\dots,k_N,l_1,\dots,l_N\in B$, there exist unique $C^{KL}\in\mathbb{C}$, such that if we write $K=\{k_m\}_1^N$, $L=\{l_m\}_1^N$ we have:

$$\det_{1\le m,n\le N}\{\frac{\theta(y+(\Omega I)(\eta_{k_m}-\eta_{l_n}),\Omega)}{q^{k_ml_n}}\}=C^{KL}\theta(y,\Omega)^{N-1}\theta(y+(\Omega I)(\eta_K-\eta_L),\Omega).$$

Proof of lemma 2.2.8. The case $N=1$ is easily seen to be valid with $C^{\{i\}\{j\}}=1/q^{ij}$. For $N\ge2$ consider the matrix $\{\frac{\theta(y+(\Omega I)(\eta_{k_m}-\eta_{l_n}),\Omega)}{q^{k_ml_n}}\}_{1\le m,n\le N}$. For $y\in\Theta$, any two by two minor of this matrix vanishes by lemma 2.2.6, and so the rank of the matrix is one on the theta locus. The determinant thus has a zero of order $N-1$ on Θ because the coefficients of the matrix are analytic in y. The theta function vanishes simply on Θ; so

$$h(y)=\det_{1\le m,n\le N}\{\frac{\theta(y+(\Omega I)(\eta_{k_m}-\eta_{l_n}),\Omega)}{q^{k_ml_n}}\}\frac{1}{\theta(y,\Omega)^{N-1}}$$

is a holomorphic function on $\mathbb{C}^g$ which transforms by the factor of automorphy $\xi\,\rho_{(\Omega I)(\eta_K-\eta_L)}$. So h is a first order theta function with character $\rho_{(\Omega I)(\eta_K-\eta_L)}$ and thus is equal to $C^{KL}\theta(y+(\Omega I)(\eta_K-\eta_L),\Omega)$ for some unique constants C^{KL}. This is the assertion of lemma 2.2.8. □

The notation C^{KL} appears to imply that C^{KL} is independent of the ordering of K and L, which is not true. A permutation of K or of L multiplies C^{KL} by the sign of the permutation, but we will still use the symbol C^{KL} for notational brevity. We now prove theorem 2.2.1.

Proof of theorem 2.2.1. Lemma 2.2.8 and the hypotheses in theorem 2.2.1 imply, for all distinct $k_1,\dots,k_N,l_1,\dots,l_N \in B$, that if we write $K=\{k_m\}$, $L=\{l_m\}$ then there exist unique $C^{KL}\in\mathbb{C}$ such that

(2.2.9)
$$\det_{1\le m,n\le N}\{\frac{\theta(y+(\Omega I)(\eta_{k_m}-\eta_{l_n}),\Omega)}{q^{k_m l_n}}\} = C^{KL}\theta(y,\Omega)^{N-1}\theta(y+(\Omega I)(\eta_K-\eta_L),\Omega).$$

We expand the determinant in 2.2.9 out by minors of the last row (the row with k_N), and use 2.2.9 again to evaluate these minors. We have

$$\begin{aligned}
&C^{KL}\theta(y,\Omega)^{N-1}\theta(y+(\Omega I)(\eta_K-\eta_L),\Omega)\\
=&\sum_{j=1}^{N}\{(-1)^{N+j}\frac{\theta(y+(\Omega I)(\eta_{k_N}-\eta_{l_j}),\Omega)}{q^{k_N l_j}}\cdot\\
&\det_{1\le m\le N-1,1\le n\le N,n\ne j}\{\frac{\theta(y+(\Omega I)(\eta_{k_m}-\eta_{l_n}),\Omega)}{q^{k_m l_n}}\}\}\\
=&\sum_{j=1}^{N}\{(-1)^{N+j}\frac{\theta(y+(\Omega I)(\eta_{k_N}-\eta_{l_j}),\Omega)}{q^{k_N l_j}}C^{K\backslash k_N L\backslash l_j}\\
&\cdot\theta(y,\Omega)^{N-2}\theta(y+(\Omega I)(\eta_{K\backslash k_N}-\eta_{L\backslash l_j}),\Omega)\}.
\end{aligned}$$

We cancel the factor $\theta(y,\Omega)^{N-2}$ in the above equation to obtain:

$$C^{KL}\theta(y,\Omega)\theta(y+(\Omega I)(\eta_K-\eta_L),\Omega) =$$

(2.2.10)
$$\sum_{j=1}^{N}(-1)^{N+j}\frac{C^{K\backslash k_N L\backslash l_j}}{q^{k_N l_j}}\theta(y+(\Omega I)(\eta_{k_N}-\eta_{l_j}),\Omega)\theta(y+(\Omega I)(\eta_K-\eta_L+\eta_{l_j}-\eta_{k_N}),\Omega).$$

We can now use the addition formula to convert 2.2.10 back into a second order theta function identity; we let $t=\frac{1}{2}(\Omega I)(\eta_K-\eta_L-2\eta_{k_N})$ and obtain:

$$\begin{aligned}
&C^{KL}\vec{\theta}_2(y+\tfrac{1}{2}(\Omega I)(\eta_K-\eta_L),\Omega)\cdot\vec{\theta}_2(t+(\Omega I)\eta_{k_N},\Omega)\\
=&\sum_{j=1}^{N}(-1)^{N+j}\frac{C^{K\backslash k_N L\backslash l_j}}{q^{k_N l_j}}\vec{\theta}_2(y+\tfrac{1}{2}(\Omega I)(\eta_K-\eta_L),\Omega)\cdot\vec{\theta}_2(t+(\Omega I)\eta_{l_j},\Omega).
\end{aligned}$$

A basis for the second order theta functions is given by the entries of $\vec{\theta}_2(y,\Omega)$, and so the above equation may be written as the vector equation,

$$C^{KL}\vec{\theta}_2(t+(\Omega I)\eta_{k_N},\Omega)=\sum_{j=1}^{N}(-1)^{N+j}\frac{C^{K\backslash k_N L\backslash l_j}}{q^{k_N l_j}}\vec{\theta}_2(t+(\Omega I)\eta_{l_j},\Omega). \tag{2.2.11}$$

Equation 2.2.11 is true for some C^{KL} which have been uniquely defined. The conclusion of theorem 2.2.1 will follow from 2.2.11 if we show that:

$$C^{KL}\neq 0 \text{ and } C^{KL}p^{L\backslash l_j\ K\backslash k_N\ k_N l_j}=(-1)^{N+j}\frac{C^{K\backslash k_N L\backslash l_j}}{q^{k_N l_j}}. \tag{2.2.12}$$

Along with $C^{\{i\}\{j\}}=1/q^{ij}$ and $p^{ijkl}=\frac{q^{ik}q^{jl}}{q^{il}q^{jk}}$, the equation 2.2.12 is in fact equivalent to another equation; since this equivalence is "well–known" and combinatorial we omit the verification here and place it in lemma 3.4.9. The equivalent equation is:

$$C^{KL}=\frac{\prod_{1\le m<n\le N}q^{k_m k_n}q^{l_n l_m}}{\prod_{\substack{1\le m\le N\\ 1\le n\le N}}q^{k_m l_n}}. \tag{2.2.13}$$

The reader may also check, or refer to lemma 3.4.9, that either 2.2.12 or 2.2.13 implies the following relation:

$$p^{L\backslash l_1\ K\backslash k_N\ k_N l_1}\frac{C^{K\backslash k_N L\backslash l_j}}{q^{k_N l_j}}=\frac{C^{K\backslash k_N (L+k_N)\backslash(l_1,l_j)}}{q^{l_1 l_j}}. \tag{2.2.14}$$

We will prove 2.2.12 and its equivalent 2.2.13 by induction on N. Equation 2.2.11 is true for all N such that $2\le N\le g+1$. Equation 2.2.13 for $N-1$ implies equation 2.2.14 for N. We will use the irreducibility hypothesis to show that equation 2.2.14 for N implies equation 2.2.12 for N; this then implies equation 2.2.13 for N. To begin the induction we see that the case $N=2$ of equation 2.2.13 is given by lemma 2.2.6 as

$$C^{(i,l)(j,k)}=\frac{q^{il}q^{kj}}{q^{ij}q^{ik}q^{lj}q^{lk}}\neq 0.$$

Consider the equation obtained from 2.2.11 by switching l_1 and k_N; then $t=\frac{1}{2}(\Omega I)(\eta_K-\eta_L-2\eta_{k_N})$ remains unchanged, and we may profitably compare the two equations, which are:

$$C^{KL}\vec{\theta}_2(t+(\Omega I)\eta_{k_N},\Omega)=(-1)^{N+1}\frac{C^{K\backslash k_N L\backslash l_1}}{q^{k_N l_1}}\vec{\theta}_2(t+(\Omega I)\eta_{l_1},\Omega)$$
$$+\sum_{j=2}^{N}(-1)^{N+j}\frac{C^{K\backslash k_N L\backslash l_j}}{q^{k_N l_j}}\vec{\theta}_2(t+(\Omega I)\eta_{l_j},\Omega),\qquad\text{and}$$

$$C^{(K+l_1)\backslash k_N (L+k_N)\backslash l_1}\vec{\theta}_2(t+(\Omega I)\eta_{l_1},\Omega)=(-1)^{N+1}\frac{C^{K\backslash k_N L\backslash l_1}}{q^{l_1 k_N}}\vec{\theta}_2(t+(\Omega I)\eta_{k_N},\Omega)$$
$$+\sum_{j=2}^{N}(-1)^{N+j}\frac{C^{K\backslash k_N (L+k_N)\backslash(l_1,l_j)}}{q^{l_1 l_j}}\vec{\theta}_2(t+(\Omega I)\eta_{l_j},\Omega).$$

We multiply the first equation by $p^{L\backslash l_1\, K\backslash k_N\, k_N l_1}$, and subtract the second equation to obtain:

$$(p^{L\backslash l_1\, K\backslash k_N\, k_N l_1} C^{KL} - (-1)^{N+1}\frac{C^{K\backslash k_N L\backslash l_1}}{q^{k_N l_1}})\vec{\theta}_2(t+(\Omega I)\eta_{k_N},\Omega)$$

$$=((-1)^{N+1}\frac{C^{K\backslash k_N L\backslash l_1}}{q^{k_N l_1}} p^{L\backslash l_1\, K\backslash k_N\, k_N l_1} + \frac{C^{(K+l_1)\backslash k_N (L+k_N)\backslash l_1}}{1})\vec{\theta}_2(t+(\Omega I)\eta_{l_1},\Omega)$$

(2.2.15)

$$+\sum_{j=2}^{N}(-1)^{N+j}(p^{L\backslash l_1\, K\backslash k_N\, k_N l_1}\frac{C^{K\backslash k_N L\backslash l_j}}{q^{k_N l_j}} - \frac{C^{K\backslash k_N (L+k_N)\backslash (l_1,l_j)}}{q^{l_1 l_j}})\vec{\theta}_2(t+(\Omega I)\eta_{l_j},\Omega).$$

As a consequence of the induction hypothesis we may use the equation 2.2.14 for N,

$$p^{L\backslash l_1\, K\backslash k_N\, k_N l_1}\frac{C^{K\backslash k_N L\backslash l_j}}{q^{k_N l_j}} = \frac{C^{K\backslash k_N (L+k_N)\backslash (l_1,l_j)}}{q^{l_1 l_j}},$$

which we apply for $2 \le j \le N$ to equation 2.2.15. Only two terms remain in the summation, and so we obtain:

$$(p^{L\backslash l_1\, K\backslash k_N\, k_N l_1} C^{KL} - (-1)^{N+1}\frac{C^{K\backslash k_N L\backslash l_1}}{q^{k_N l_1}})\vec{\theta}_2(t+(\Omega I)\eta_{k_N},\Omega)$$

(2.2.16)

$$=((-1)^{N+1}\frac{C^{K\backslash k_N L\backslash l_1}}{q^{k_N l_1}} p^{L\backslash l_1\, K\backslash k_N\, k_N l_1} + \frac{C^{(K+l_1)\backslash k_N (L+k_N)\backslash l_1}}{1})\vec{\theta}_2(t+(\Omega I)\eta_{l_1},\Omega).$$

We now use the irreducibility assumption on Ω to apply lemma 2.1.3 from section §2.1, and to conclude that the coefficients in 2.2.16 vanish unless $t+(\Omega I)\eta_{k_N} = \pm(t+(\Omega I)\eta_{l_1}) \mod \mathcal{L}$. That is: $\eta_{k_N} \equiv \eta_{l_1} \mod \mathbb{Z}^{2g}$, or $\eta_{k_1}+\cdots+\eta_{k_{N-1}} \equiv \eta_{l_2}+\cdots+\eta_{l_N} \mod \mathbb{Z}^{2g}$. By choosing the $\{k_m\}$ and $\{l_m\}$ to be disjoint we avoid $\eta_{k_N} \equiv \eta_{l_1}$, and $\eta_{K\backslash k_N} \equiv \eta_{L\backslash l_1}$ is impossible so long as $2(N-1) \le 2g$, or $N \le g+1$. We conclude that the coefficients of 2.2.16 are zero and so we have

$$p^{L\backslash l_1\, K\backslash k_N\, k_N l_1} C^{KL} = (-1)^{N+1}\frac{C^{K\backslash k_N L\backslash l_1}}{q^{k_N l_1}}. \tag{2.2.17}$$

This shows that $C^{KL} \neq 0$ and is equation 2.2.12 with l_1 in place of l_j; since the integer l_1 was arbitrary this shows the general validity of 2.2.12. This completes the evaluation of the C^{KL} by induction for $2 \le N \le g+1$; and, again, equations 2.2.11 and 2.2.12 together imply the conclusion of theorem 2.2.1.

Proof of corollary 2.2.5. Consider equation 2.2.10 for $N = g+1$ in the proof of theorem 2.2.1. Now that the C^{KL} are known we may write equation 2.2.10 as:

$$\theta(y,\Omega)\theta(y+(\Omega I)(\eta_K-\eta_L),\Omega) =$$

(2.2.18)

$$\sum_{j=1}^{g+1} p^{L\backslash l_j\, K\backslash k_{g+1}\, k_{g+1} l_j}\theta(y+(\Omega I)(\eta_{k_{g+1}}-\eta_{l_j}),\Omega)\theta(y+(\Omega I)(\eta_K-\eta_L+\eta_{l_j}-\eta_{k_{g+1}}),\Omega).$$

Let $y = (\Omega I)(\eta_U - \eta_K + \eta_{k_{g+1}} + \eta_{l_{j_0}})$ in 2.2.18, then we have

$$\begin{aligned}&\theta((\Omega I)(\eta_U - \eta_K + \eta_{k_{g+1}} + \eta_{l_{j_0}}),\Omega)\theta((\Omega I)(\eta_U - \eta_L + \eta_{k_{g+1}} + \eta_{l_{j_0}}),\Omega)\\ =&\sum_{j=1}^{g+1}\{p^{L\backslash l_j\; K\backslash k_{g+1}\; k_{g+1} l_j}\theta((\Omega I)(\eta_U - \eta_K + 2\eta_{k_{g+1}} + \eta_{l_{j_0}} - \eta_{l_j}),\Omega)\\ &\cdot\theta((\Omega I)(\eta_U - \eta_L + \eta_{l_j} + \eta_{l_{j_0}}),\Omega)\}.\end{aligned}$$

Consider the factor $\theta((\Omega I)(\eta_U - \eta_L + \eta_{l_j} + \eta_{l_{j_0}}),\Omega)$ for $1 \le j \le g+1$. Since $|L| = g+1$ we know that except in the case $j = j_0$, the theta characteristic $\eta_U - \eta_L + \eta_{l_j} + \eta_{l_{j_0}} \equiv \eta_U - \eta_{L\circ\{l_j,l_0\}}$ is odd and hence $\theta((\Omega I)(\eta_U - \eta_{L\circ\{l_j,l_0\}}),\Omega) = 0$. Using this fact we obtain

$$\begin{aligned}&\theta((\Omega I)(\eta_U - \eta_K + \eta_{k_{g+1}} + \eta_{l_{j_0}}),\Omega)\theta((\Omega I)(\eta_U - \eta_L + \eta_{k_{g+1}} + \eta_{l_{j_0}}),\Omega) =\\ &p^{L\backslash l_{j_0}\; K\backslash k_{g+1}\; k_{g+1} l_{j_0}}\theta((\Omega I)(\eta_U - \eta_K + 2\eta_{k_{g+1}}),\Omega)\theta((\Omega I)(\eta_U - \eta_L + 2\eta_{l_{j_0}}),\Omega), \text{ or}\\ &p^{L\backslash l_{j_0}\; K\backslash k_{g+1}\; k_{g+1} l_{j_0}}\theta((\Omega I)(\eta_U + \eta_{l_{j_0}} - \eta_{L\backslash l_{j_0}}),\Omega)\theta((\Omega I)(\eta_U + \eta_{k_{g+1}} - \eta_{K\backslash k_{g+1}}),\Omega)\\ (2.2.19\;\;)&\\ &= \theta((\Omega I)(\eta_U + \eta_{l_{j_0}} - \eta_{K\backslash k_{g+1}}),\Omega)\theta((\Omega I)(\eta_U + \eta_{k_{g+1}} - \eta_{L\backslash l_{j_0}}),\Omega).\end{aligned}$$

If we set $j_0 = g+1$, $l_{g+1} = b$, $k_{g+1} = a$, and $A = \{l_i\}_{i=1}^{g}$, $\bar{B} = \{k_i\}_{i=1}^{g}$, then equation 2.2.19 becomes the conclusion of corollary 2.2.5,

$$\begin{aligned}&p^{A\,\bar{B}\,ab}\theta((\Omega I)(\eta_U + \eta_b - \eta_A),\Omega)\theta((\Omega I)(\eta_U + \eta_a - \eta_{\bar{B}}),\Omega)\\ =&\theta((\Omega I)(\eta_U + \eta_b - \eta_{\bar{B}}),\Omega)\theta((\Omega I)(\eta_U + \eta_a - \eta_A),\Omega).\end{aligned}$$

§2.3 Nonvanishing.

The purpose of this section is to prove theorem 2.3.1. For irreducible Ω satisfying $F_{g,\eta}$ this theorem gives the nonvanishing of those first order thetanullwerte with hyperelliptic η-order zero. The fact that the p^{ijkl} of section §2.1 are nonzero follows from the nondegeneracy of the second order theta constants, in turn a consequence of the irreducibility of Ω. The corollary 2.2.5 of section §2.2 sufficiently intertwines the p^{ijkl} and the first order thetanullwerte so that the nondegeneracy of the $p^{\bullet}$ implies that of the thetanullwerte. This is how we derive nonvanishing conditions from irreducibility conditions.

2.3.1 Theorem. *Let $\Omega \in \mathcal{H}_g$ be irreducible and $\eta \in \Xi_g$. If $\Omega \in F'_{g,\eta}$ then for all $S \subseteq B$ such that $|S|$ is even and $|U \circ S| = g+1$, we have $\theta[\eta_S](0,\Omega) \neq 0$. In other words, each thetanullwerte of hyperelliptic η-order zero does not vanish.*

2.3.2 Corollary. *Let $\Omega \in \mathcal{H}_g$ be irreducible and $\eta \in \Xi_g$. If $\Omega \in F'_{g,\eta}$ then for all distinct $i,j,k,l \in B$, and $V \subseteq B$ such that $i,j,k,l \notin V$ and $|V| = g-1$, we have*

$$p^{ijkl} = \frac{\theta((\Omega I)(\eta_U - \eta_V + \eta_i - \eta_k),\Omega)}{\theta((\Omega I)(\eta_U - \eta_V + \eta_i - \eta_l),\Omega)}\frac{\theta((\Omega I)(\eta_U - \eta_V + \eta_j - \eta_l),\Omega)}{\theta((\Omega I)(\eta_U - \eta_V + \eta_j - \eta_k),\Omega)}.$$

Recall the discussion of hyperelliptic η-order in section §1.4, and the definition 1.4.16. The hyperelliptic η-order of $\zeta \in \frac{1}{2}\mathbb{Z}^{2g}$ is characterized as the vanishing order of $\theta[\zeta](0,\Omega)$ for a marked *hyperelliptic* Ω whose Abel–Jacobi map corresponds to η. Given an arbitrary $\Omega \in \mathcal{H}_g$ it is not too surprising that at least one of the thetanullwerte of hyperelliptic η-order zero does not vanish, and this is the assertion of the next lemma.

2.3.3 Lemma. *Let $\Omega \in \mathcal{H}_g$ and $\eta \in \Xi_g$. Then there is a $\zeta \in \frac{1}{2}\mathbb{Z}^{2g}$ with hyperelliptic η-order zero such that $\theta[\zeta](0,\Omega) \neq 0$.*

Proof. From lemma 1.4.26 in section §1.4 we see that there is a göpel system all of whose elements have hyperelliptic η-order zero. Then lemma 1.1.18 in section §1.1 assures us that every göpel system has a nonvanishing thetanull.

2.3.4 Lemma. *Let $\Omega \in \mathcal{H}_g$ be irreducible, and $\eta \in \Xi_g$, and $\Omega \in F'_{g,\eta}$. Let $S = \{i, k_1, \dots, k_g\} \subseteq B$ and $S^{\mathbf{c}} = \{j, l_1, \dots, l_g\} \subseteq B$ give the disjoint union $B = S \amalg S^{\mathbf{c}}$. The following is then true,*

$$\theta((\Omega I)(\eta_U + \eta_S),\Omega)^2 = \text{(unit)}\ \theta((\Omega I)(\eta_U + \eta_{S\circ\{i,j\}}),\Omega)^2 \prod_{n=1}^{g} p^{ij\,k_n\,l_n}.$$

Proof. Since Ω is irreducible and $\Omega \in F'_{g,\eta}$ we may use the conclusion of theorem 2.1.1, which implies theorem 2.2.1 and its corollary 2.2.5. Apply corollary 2.2.5 from section §2.2 with $a = i$, $b = j$, $A = \{k_1, \dots, k_g\}$, and $\bar{B} = \{l_1, \dots, l_g\}$.

$$\begin{aligned}
p^{A\,\bar{B}\,ab}&\theta((\Omega\, I)(\eta_U + \eta_a - \eta_{\bar{B}}),\Omega)\theta((\Omega\, I)(\eta_U + \eta_b - \eta_A),\Omega)\\
&=\theta((\Omega\, I)(\eta_U + \eta_a - \eta_A),\Omega)\theta((\Omega\, I)(\eta_U + \eta_b - \eta_{\bar{B}}),\Omega) \quad \text{or}\\
p^{S\setminus i\, S^{\mathbf{c}}\setminus j\, ij}&\theta((\Omega\, I)(\eta_U + \eta_i - \textstyle\sum \eta_{l_n}),\Omega)\theta((\Omega\, I)(\eta_U + \eta_j - \textstyle\sum \eta_{k_n}),\Omega)\\
&=\theta((\Omega\, I)(\eta_U + \eta_i - \textstyle\sum \eta_{k_n}),\Omega)\theta((\Omega\, I)(\eta_U + \eta_j - \textstyle\sum \eta_{l_n}),\Omega)
\end{aligned}$$

From $B = S \amalg S^{\mathbf{c}}$ we obtain $\eta_S + \eta_{S^{\mathbf{c}}} = \eta_B \in \mathbb{Z}^{2g}$, as well as $\eta_{S\circ\{i,j\}} + \eta_{S^{\mathbf{c}}\circ\{i,j\}} = \eta_B \in \mathbb{Z}^{2g}$. These give us the following congruences modulo $\mathbb{Z}^{2g}$:

$$\begin{aligned}
\eta_U + \eta_i - \textstyle\sum \eta_{l_n} &= \eta_U - \eta_{S^{\mathbf{c}}\circ\{i,j\}} + 2\eta_i &&\equiv \eta_U + \eta_{S\circ\{i,j\}},\\
\eta_U + \eta_j - \textstyle\sum \eta_{k_n} &= \eta_U - \eta_{S\circ\{i,j\}} &&\equiv \eta_U + \eta_{S\circ\{i,j\}},\\
\eta_U + \eta_i - \textstyle\sum \eta_{k_n} &= \eta_U - \eta_S + 2\eta_i &&\equiv \eta_U + \eta_S,\\
\eta_U + \eta_j - \textstyle\sum \eta_{l_n} &= \eta_U - \eta_{S^{\mathbf{c}}} + 2\eta_j &&\equiv \eta_U + \eta_S.
\end{aligned}$$

Recall that by proposition 1.1.3, $\theta(w + (\Omega I)\lambda,\Omega) =$ (unit) $\theta(w,\Omega)$ for $\lambda \in \mathbb{Z}^{2g}$. At the expense of introducing units we then have the following:

$$\begin{aligned}
p^{S\setminus i\, S^{\mathbf{c}}\setminus j\, ij}&\theta((\Omega\, I)(\eta_U + \eta_{S\circ\{i,j\}}),\Omega)\theta((\Omega\, I)(\eta_U + \eta_{S\circ\{i,j\}}),\Omega)\\
&= \text{(unit)}\ \theta((\Omega\, I)(\eta_U + \eta_S),\Omega)\theta((\Omega\, I)(\eta_U + \eta_S),\Omega).
\end{aligned}$$

Simply realizing that $p^{S\setminus i\, S^{\mathbf{c}}\setminus j\, ij} = \prod_n p^{k_n\, l_n\, ij}$ gives us the conclusion of lemma 2.3.4. The actual value of the unit is $(-1)^{|U\cap\{i,j\}|+1}\boldsymbol{e}^{-4\pi i(\eta_i-\eta_j)'\Omega(\eta_i-\eta_j+4\eta_S)'}$, but we certainly do not need this.

Proof of theorem 2.3.1. Let $S_0=\{i,k_1,\dots,k_g\}$, and $S_0^{\mathbf{c}}=\{j,l_1,\dots,l_g\}$, and note that we have now labelled every element in B. Apply lemma 2.3.4 to these two sets,

$$(2.3.5\ \ )\quad \theta((\Omega I)(\eta_U+\eta_{S_0}),\Omega)^2 = \text{(unit)}\ \theta((\Omega I)(\eta_U+\eta_{S_0\circ\{i,j\}}),\Omega)^2\prod_{n=1}^{g}p^{ij\,k_n\,l_n}.$$

Since the i,j,k_n,l_n are all distinct we have that the $p^{ijk_nl_n}$ are nonzero by theorem 2.1.1. Notice also that $\theta[\eta_{U\circ S}](0,\Omega)$ and $\theta((\Omega I)(\eta_U+\eta_S),\Omega)$ differ only by an exponential factor. Hence from 2.3.5 we can conclude:

(2.3.6)

$$\forall\ S_0 : |S_0|=g+1, i\in S_0, j\notin S_0,\quad \theta[\eta_{U\circ S_0}](0,\Omega)=0 \iff \theta[\eta_{U\circ S_0\circ\{i,j\}}](0,\Omega)=0.$$

Now by lemma 2.3.3 there is some S_0 such that $|S_0|=g+1$ and $\theta[\eta_{U\circ S_0}](0,\Omega)\neq 0$. Property 2.3.6 allows us to replace any $i\in S_0$ with any $j\notin S_0$ and conclude that $\theta[\eta_{U\circ S_0\circ\{i,j\}}](0,\Omega)$ is also nonzero. In this way we can replace the elements of S_0, one by one, until any $S\subseteq B$ with $|S|=g+1$ is reached. Hence for all $\eta_{U\circ S}$ of η-order zero we have $\theta[\eta_{U\circ S}](0,\Omega)\neq 0$.

Proof of corollary 2.3.2. Now that we have the nonvanishing of the thetanullwerte of hyperelliptic η-order zero, this corollary is a direct consequence of corollary 2.1.2 in section §2.1. The characteristic $\eta_U-\eta_V$ has hyperelliptic η-order $\frac{1}{2}(g+1-|V|)=1$ and so is odd. Therefore $\alpha=(\Omega I)(\eta_U-\eta_V)\in\Theta$ and we may evaluate at this α in corollary 2.1.2 since the denominator here does not vanish.

2.3.7 Remark. There are three equivalent forms of corollary 2.3.2, all of which we will unfortunately use. We give these here and the reader should compare lemma 1.1.20. Notice that this corollary shows that these thetanullwerte quotients are independent of the choice of V.

$$\begin{aligned}
p^{ijkl} &= \frac{\theta((\Omega\, I)(\eta_U-\eta_V+\eta_i-\eta_k),\Omega)}{\theta((\Omega\, I)(\eta_U-\eta_V+\eta_i-\eta_l),\Omega)}\frac{\theta((\Omega\, I)(\eta_U-\eta_V+\eta_j-\eta_l),\Omega)}{\theta((\Omega\, I)(\eta_U-\eta_V+\eta_j-\eta_k),\Omega)}\\
&= \frac{\theta[\eta_U-\eta_V]((\Omega\, I)(\eta_i-\eta_k),\Omega)}{\theta[\eta_U-\eta_V]((\Omega\, I)(\eta_i-\eta_l),\Omega)}\frac{\theta[\eta_U-\eta_V]((\Omega\, I)(\eta_j-\eta_l),\Omega)}{\theta[\eta_U-\eta_V]((\Omega\, I)(\eta_j-\eta_k),\Omega)}\\
&= \boldsymbol{e}^{2\pi i(\eta_i-\eta_j)'\Omega(\eta_k-\eta_l)'}\,\boldsymbol{e}^{2\pi i\{(\eta_i-\eta_j)'\cdot(\eta_k-\eta_l)''+(\eta_k-\eta_l)'\cdot(\eta_i-\eta_j)''\}}\\
&\qquad \frac{\theta[\eta_U-\eta_V+\eta_i-\eta_k](0,\Omega)}{\theta[\eta_U-\eta_V+\eta_i-\eta_l](0,\Omega)}\frac{\theta[\eta_U-\eta_V+\eta_j-\eta_l](0,\Omega)}{\theta[\eta_U-\eta_V+\eta_j-\eta_k](0,\Omega)}
\end{aligned}$$

2.3.8 Remark. Corollary 2.3.2 shows that the $\theta[\eta_U-\eta_V]((\Omega\, I)(\eta_i-\eta_j),\Omega)$ are a concrete realization of the q^{ij} whose existence was ensured by lemma 2.2.6 in section §2.2. They are skew in i and j because $[\eta_U-\eta_V]$ is an odd characteristic.

§2.4 Recovery of the branch points.

In the preceding three sections we have established the nonvanishing of the thetanullwerte of hyperelliptic order zero and a version of the multisecant formula. In the multisecant formula the p^{KLij} are beginning to look very much like the values of Gunning's higher crossratio function at the branch points of a hyperelliptic curve. In theorem 2.4.1 we begin to recover the hyperelliptic curve itself by constructing what will be the projective crossratios of its branch points. Proposition 3.2.1 of section 3.2 plays an important role here.

2.4.1 Theorem. *Let $\Omega \in \mathcal{H}_g$ and $\eta \in \Xi_g$. Assume that $p^{ijkl} \in \mathbb{C}^*$ exist as in the conclusion of theorem 2.1.1 in section §2.1. Assume that $\theta[\zeta](0,\Omega) \neq 0$ for all ζ of hyperelliptic η-order zero as in theorem 2.3.1 in section §2.3. Assume the conclusion of corollary 2.3.2 in §2.3. Then there exist $2g+2$ distinct $a_i \in \mathbb{P}^1$, for $i \in B$, such that for all distinct $i,j,k,l \in B$, we have*

$$\frac{a_i - a_k}{a_i - a_l} \cdot \frac{a_j - a_l}{a_j - a_k} = \boldsymbol{e}^{-4\pi i(\eta_i - \eta_j)'(\Omega I)(\eta_k - \eta_l)}(p^{ijkl})^2$$

$$= \boldsymbol{e}^{4\pi i(\eta_i - \eta_j)' \cdot (\eta_k - \eta_l)''}\left(\frac{\theta[\eta_U - \eta_V + \eta_i - \eta_k](0,\Omega)}{\theta[\eta_U - \eta_V + \eta_i - \eta_l](0,\Omega)}\frac{\theta[\eta_U - \eta_V + \eta_j - \eta_l](0,\Omega)}{\theta[\eta_U - \eta_V + \eta_j - \eta_k](0,\Omega)}\right)^2.$$

2.4.2 Definition. *Let $\Omega \in \mathcal{H}_g$ and $\eta \in \Xi_g$. Assume that $p^{ijkl} \in \mathbb{C}^*$ exist as in the conclusion of theorem 2.1.1 in section §2.1. For all distinct $i,j,k,l \in B$, define $c^{ijkl} \in \mathbb{C}^*$ as follows:*

$$c^{ijkl} = e^{-4\pi i(\eta_i - \eta_j)'(\Omega I)(\eta_k - \eta_l)}(p^{ijkl})^2.$$

2.4.3 Lemma. *Let $\Omega \in \mathcal{H}_g$ and $\eta \in \Xi_g$. Assume that $p^{ijkl} \in \mathbb{C}^*$ exist as in the conclusion of theorem 2.1.1. Assume the conclusions of theorem 2.3.1 and its corollary 2.3.2. Then for all distinct $i,j,k,l \in B$, and $V \subseteq B$ such that $|V| = g-1$ and $i,j,k,l \notin V$ we have:*

$$c^{ijkl} = \boldsymbol{e}^{4\pi i(\eta_i - \eta_j)' \cdot (\eta_k - \eta_l)''}\left(\frac{\theta[\eta_U - \eta_V + \eta_i - \eta_k](0,\Omega)}{\theta[\eta_U - \eta_V + \eta_i - \eta_l](0,\Omega)}\frac{\theta[\eta_U - \eta_V + \eta_j - \eta_l](0,\Omega)}{\theta[\eta_U - \eta_V + \eta_j - \eta_k](0,\Omega)}\right)^2.$$

Proof. We use corollary 2.3.2 to substitute for the p^{ijkl} in the definition $c^{ijkl} = e^{-4\pi i(\eta_i - \eta_j)'(\Omega I)(\eta_k - \eta_l)}\ (p^{ijkl})^2$. To do so we use the last equivalent form in remark 2.3.7, so that we have:

$$c^{ijkl} = e^{-4\pi i(\eta_i - \eta_j)'(\Omega I)(\eta_k - \eta_l)}$$

$$\boldsymbol{e}^{4\pi i(\eta_i - \eta_j)'\Omega(\eta_k - \eta_l)'}\,\boldsymbol{e}^{4\pi i\{(\eta_i - \eta_j)' \cdot (\eta_k - \eta_l)'' + (\eta_k - \eta_l)' \cdot (\eta_i - \eta_j)''\}}$$

$$\left(\frac{\theta[\eta_U - \eta_V + \eta_i - \eta_k](0,\Omega)}{\theta[\eta_U - \eta_V + \eta_i - \eta_l](0,\Omega)}\frac{\theta[\eta_U - \eta_V + \eta_j - \eta_l](0,\Omega)}{\theta[\eta_U - \eta_V + \eta_j - \eta_k](0,\Omega)}\right)^2$$

$$= e^{+4\pi i(\eta_k - \eta_l)' \cdot (\eta_i - \eta_j)''}\left(\frac{\theta[\eta_U - \eta_V + \eta_i - \eta_k](0,\Omega)}{\theta[\eta_U - \eta_V + \eta_i - \eta_l](0,\Omega)}\frac{\theta[\eta_U - \eta_V + \eta_j - \eta_l](0,\Omega)}{\theta[\eta_U - \eta_V + \eta_j - \eta_k](0,\Omega)}\right)^2.$$

The proof of this lemma will be complete when we show $e^{4\pi i(\eta_k - \eta_l)' \cdot (\eta_i - \eta_j)''} = e^{4\pi i(\eta_i - \eta_j)' \cdot (\eta_k - \eta_l)''}$. This says exactly that $e_2(\eta_{\{i,j\}}, \eta_{\{k,l\}}) = (-1)^0 = 1$, which follows from definition 1.4.11 item (3).

2.4.4 Lemma. *The c^{ijkl} satisfy the crossratio identities. Namely, for all distinct $i,j,k,l,m \in B$, we have (1)-(4).*

(1) $c^{ijkl} = c^{jilk} = c^{klij} = c^{lkji}$
(2) $c^{ijkl}c^{jikl} = 1$
(3) $c^{ijkl}c^{iklj}c^{iljk} = -1$
(4) $c^{ijkl}c^{ijlm} = c^{ijkm}$

Proof. This has already been verified in the proof of proposition 1.5.4 in §1.5.

2.4.5 Lemma. *For all distinct $i,j,k,l \in B$, we have $c^{ijkl} + c^{ikjl} = 1$.*

Proof. Consider equation (5) from theorem 2.1.1,

$$p^{ijkl}\vec{\theta}_2(\xi_{ikjl},\Omega) + p^{ikjl}\vec{\theta}_2(\xi_{ijkl},\Omega) = \vec{\theta}_2(\xi_{iljk},\Omega). \tag{2.4.6}$$

We are going to use the addition theorem by dotting both sides of equation 2.4.6 with $\vec{\theta}_2(w,\Omega)$. We take $w = (\Omega I)(\eta_U - \eta_V) + \frac{1}{2}(\Omega I)(\eta_i + \eta_j + \eta_k + \eta_l)$, and let $V \subseteq B$ be any set disjoint from i,j,k,l with $|V| = g-1$. We have the following result:

$$\begin{aligned} &p^{ijkl}\theta((\Omega I)(\eta_U - \eta_V + \eta_i + \eta_k),\Omega)\theta((\Omega I)(\eta_U - \eta_V + \eta_j + \eta_l),\Omega) \\ &+p^{ikjl}\theta((\Omega I)(\eta_U - \eta_V + \eta_i + \eta_j),\Omega)\theta((\Omega I)(\eta_U - \eta_V + \eta_k + \eta_l),\Omega) \\ &=\theta((\Omega I)(\eta_U - \eta_V + \eta_i + \eta_l),\Omega)\theta((\Omega I)(\eta_U - \eta_V + \eta_j + \eta_k),\Omega). \end{aligned} \tag{2.4.7}$$

We now use the assumption that all of the thetanullwerte of hyperelliptic η-order zero do not vanish. This implies that all of the theta constants appearing in 2.4.7 are nonzero. For example, $\theta((\Omega I)(\eta_U - \eta_V + \eta_i + \eta_k),\Omega)$ differs by an exponential factor from $\theta[\eta_{U\circ V\circ\{i,k\}}](0,\Omega)$, and $\eta_{U\circ V\circ\{i,k\}}$ is of hyperelliptic η-order zero. The nonvanishing provided by the assumption of the conclusion of theorem 2.3.1 allows us to write 2.4.7 as:

$$\begin{aligned} &p^{ijkl}\frac{\theta((\Omega I)(\eta_U - \eta_V + \eta_i + \eta_k),\Omega)}{\theta((\Omega I)(\eta_U - \eta_V + \eta_i + \eta_l),\Omega)}\frac{\theta((\Omega I)(\eta_U - \eta_V + \eta_j + \eta_l),\Omega)}{\theta((\Omega I)(\eta_U - \eta_V + \eta_j + \eta_k),\Omega)} \\ &+p^{ikjl}\frac{\theta((\Omega I)(\eta_U - \eta_V + \eta_i + \eta_j),\Omega)}{\theta((\Omega I)(\eta_U - \eta_V + \eta_i + \eta_l),\Omega)}\frac{\theta((\Omega I)(\eta_U - \eta_V + \eta_k + \eta_l),\Omega)}{\theta((\Omega I)(\eta_U - \eta_V + \eta_k + \eta_j),\Omega)} = 1. \end{aligned} \tag{2.4.8}$$

A 25 line calculation using the quasi–periodicity of the theta function reveals:

$$\begin{aligned} &\frac{\theta((\Omega I)(\eta_U - \eta_V + \eta_i + \eta_k),\Omega)}{\theta((\Omega I)(\eta_U - \eta_V + \eta_i + \eta_l),\Omega)}\frac{\theta((\Omega I)(\eta_U - \eta_V + \eta_j + \eta_l),\Omega)}{\theta((\Omega I)(\eta_U - \eta_V + \eta_j + \eta_k),\Omega)} \\ =\ &\{e^{-4\pi i(\eta_k - \eta_l)'(\Omega I)(\eta_i - \eta_j)} \\ &\cdot\frac{\theta((\Omega I)(\eta_U - \eta_V + \eta_i - \eta_k),\Omega)}{\theta((\Omega I)(\eta_U - \eta_V + \eta_i - \eta_l),\Omega)}\frac{\theta((\Omega I)(\eta_U - \eta_V + \eta_j - \eta_l),\Omega)}{\theta((\Omega I)(\eta_U - \eta_V + \eta_j - \eta_k),\Omega)}\}. \end{aligned}$$

By the corollary 2.3.2 to theorem 2.3.1 the second term above is precisely $e^{-4\pi i(\eta_k-\eta_l)'(\Omega I)(\eta_i-\eta_j)}\ p^{ijkl}$, so that equation 2.4.8 becomes:

$$(2.4.9\ )\qquad e^{-4\pi i(\eta_k-\eta_l)'(\Omega I)(\eta_i-\eta_j)}(p^{ijkl})^2+e^{-4\pi i(\eta_j-\eta_l)'(\Omega I)(\eta_i-\eta_k)}(p^{ikjl})^2=1.$$

Recalling that $c^{ijkl}=e^{-4\pi i(\eta_i-\eta_j)'(\Omega I)(\eta_k-\eta_l)}(p^{ijkl})^2$, all that remains to conclude the proof that $c^{ijkl}+c^{ikjl}=1$ is that $e^{4\pi i(\eta_k-\eta_l)'\cdot(\eta_i-\eta_j)''}=e^{4\pi i(\eta_i-\eta_j)'\cdot(\eta_k-\eta_l)''}$. This follows from $e^{4\pi i(\eta_k-\eta_l)'\cdot(\eta_i-\eta_j)''}=e^{4\pi i(\eta_i-\eta_j)'\cdot(\eta_k-\eta_l)''}$, which we have already mentioned in the proof of lemma 2.4.3. □

Proof of theorem 2.4.1. With lemmas 2.4.4 and 2.4.5 at our disposal theorem 2.4.1 follows from proposition 1.5.4, or even from a separate appeal to proposition 3.2.1. The c^{ijkl} are nonzero because the p^{ijkl} are. Lemmas 2.4.4 and 2.4.5 state that the nonzero c^{ijkl} satisfy the characterizing identities for a projective crossratio. The number of indices here is $|B|=2g+2\geq 4$, so by proposition 3.2.1 there exist $2g+2$ distinct $a_i\in\mathbb{P}$ such that $c^{ijkl}=\langle a_i,a_j,a_k,a_l\rangle$. This is the assertion of theorem 2.4.1.

§2.5 Recovery of the hyperelliptic curve.

In theorem 2.5.1 we construct a hyperelliptic curve M with period matrix Ω_1 and projective crossratios of branch points equal to those crossratios in theorem 2.4.1. As an immediate consequence we display for the first time the equality of some thetanullwerte quotients for an Ω satisfying some identities and a *hyperelliptic* Ω_1. These relations between the nullwerte of hyperelliptic order zero for Ω and Ω_1 are given in proposition 2.5.2. The use of propositions 3.3.3 and 3.3.7 in this section should be noted as our third invariant theory calculation.

2.5.1 Theorem. *Let $\eta\in\Xi_g$ and assume $\Omega\in\mathcal{H}_g$ is irreducible and $\Omega\in F'_{g,\eta}$. Let the p^{ijkl} be as in theorem 2.1.1 of section §2.1, and the a_i be as in theorem 2.4.1 of section §2.4. Then we have the following conclusions* (1)-(5)*.*

(1) *There exists a hyperelliptic curve M of genus g which is modeled by $y^2=\prod_{i\in B}(x-a_i)$.*
(2) *There is a marking, m_1, of M with basepoint $\dot{a}_\infty$ over a_∞, and $\pi:\widehat{M}\to M$.*
(3) *There are lifts $\dot{a}_i\in\widehat{M}:\pi\dot{a}_i=a_i$.*
(4) *There is a period matrix $\Omega_1\in\mathcal{H}_g$ computed from (M,m_1) such that the Abel–Jacobi map $w:\widehat{M}\to J_1=\mathbb{C}^g/(\mathbb{Z}^g+\Omega_1\mathbb{Z}^g)$ satisfies $w(\dot{a}_i)=(\Omega_1 I)\eta_i$, for all $i\in B$.*
(5) *Gunning's cross–ratio function $p:\widehat{M}^4\to\mathbb{P}^1$ satisfies: $\forall$ distinct $i,j,k,l\in B$,*

$$e^{-4\pi i(\eta_i-\eta_j)'(\Omega I)(\eta_k-\eta_l)}(p^{ijkl})^2=e^{-4\pi i(\eta_i-\eta_j)'(\Omega_1 I)(\eta_k-\eta_l)}(p(\dot{a}_i,\dot{a}_j,\dot{a}_k,\dot{a}_l))^2.$$

2.5.2 Proposition. *In theorem 2.5.1 we may select a $\sigma \in \Gamma_2$ such that for all V such that $|V| = g-1$, and for all distinct $i,j,k,l \notin V$, we have:*

$$\frac{\theta[\eta_U-\eta_V+\eta_i-\eta_k](0,\sigma\cdot\Omega_1)}{\theta[\eta_U-\eta_V+\eta_i-\eta_l](0,\sigma\cdot\Omega_1)}\frac{\theta[\eta_U-\eta_V+\eta_j-\eta_l](0,\sigma\cdot\Omega_1)}{\theta[\eta_U-\eta_V+\eta_j-\eta_k](0,\sigma\cdot\Omega_1)}$$
$$=\frac{\theta[\eta_U-\eta_V+\eta_i-\eta_k](0,\Omega)}{\theta[\eta_U-\eta_V+\eta_i-\eta_l](0,\Omega)}\frac{\theta[\eta_U-\eta_V+\eta_j-\eta_l](0,\Omega)}{\theta[\eta_U-\eta_V+\eta_j-\eta_k](0,\Omega)};$$

furthermore, for all $K, L \subseteq B$, and $i,j \in B$ such that $i,j \notin K\cup L$ and $|K| = |L| = g$, we have:

$$\frac{\theta[\eta_U+\eta_i-\eta_K](0,\sigma\cdot\Omega_1)}{\theta[\eta_U+\eta_i-\eta_L](0,\sigma\cdot\Omega_1)}\frac{\theta[\eta_U+\eta_j-\eta_L](0,\sigma\cdot\Omega_1)}{\theta[\eta_U+\eta_j-\eta_K](0,\sigma\cdot\Omega_1)}$$
$$=\frac{\theta[\eta_U+\eta_i-\eta_K](0,\Omega)}{\theta[\eta_U+\eta_i-\eta_L](0,\Omega)}\frac{\theta[\eta_U+\eta_j-\eta_L](0,\Omega)}{\theta[\eta_U+\eta_j-\eta_K](0,\Omega)}.$$

Remark. Notice from equation 1.1.5 that the equalities in the conclusion of proposition 2.5.2 will still hold if any of the signs of $\pm\eta_i$, $\pm\eta_j$, $\pm\eta_K$, $\pm\eta_L$ are changed.

Proof of theorem 2.5.1. In the proof of this theorem we will actually only use the conclusions of theorems 2.1.1 and 2.4.1 and not the irreducibility hypothesis on Ω. Use theorem 2.4.1 to produce distinct $a_i \in \mathbb{P}^1$ for $i \in B$, so that we have $\langle a_i, a_j, a_k, a_l\rangle = e^{-4\pi i(\eta_i-\eta_j)'(\Omega I)(\eta_k-\eta_l)}\ (p^{ijkl})^2$. Use an auxiliary linear fractional transformation to ensure that the a_i are all finite. Let M be the hyperelliptic Riemann surface associated to the plane curve $y^2 = \prod_{i\in B}(x-a_i)$.

We may mark M in some way and lift the a_i (really $(a_i, 0)$) from M to $a_i' \in \widehat{M}$ so that the Abel–Jacobi map $w : \widehat{M} \to J'$, with basepoint a_∞', satisfies $w(a_i') = (\Omega' I)\eta_i'$, for some $\eta' \in \Xi_g$ as in proposition 1.4.9 in section §1.4. By lemma 1.4.13 in section §1.4 we see that $\mathrm{Sp}_g(\mathbb{Z})$ is transitive on the classes in Ξ_g. Take an element $\sigma \in \mathrm{Sp}_g(\mathbb{Z})$ which sends the class of η' to the class of η, $\sigma[\eta'] = [\eta]$. Use this σ to change the marking of M to a marking we will call m_1. For the Abel–Jacobi map under m_1 we now have $w(a_i') \equiv (\Omega_1 I)\eta_i \mod \mathbb{Z}^g + \Omega_1\mathbb{Z}^g$. Simply alter the choice of lifts for the a_i until we have equality in $\mathbb{C}^g$ instead of congruence modulo the lattice; call these lifts $\dot{a}_i$ so that we have $w(\dot{a}_i) = (\Omega_1 I)\eta_i$ in $\mathbb{C}^g$. We now have a hyperelliptic curve M given by $y^2 = \prod_{i\in B}(x-a_i)$ with marking m_1, basepoint $\dot{a}_\infty \in \widehat{M}$, and lifts $\dot{a}_i \in \widehat{M}$ such that $\pi\dot{a}_i = a_i$, $w(\dot{a}_i) = (\Omega_1 I)\eta_i$ in $\mathbb{C}^g$, and $\eta \in \Xi_g$. Hence all of the results concerning hyperelliptic curves in sections §1.4 and §1.5 apply to Ω_1. Among them are proposition 1.4.17 and corollary 1.5.3:

$$(2.5.5\ )\qquad \theta[\eta_S](0,\Omega_1) \neq 0 \iff |S| \text{ is even and } |U\circ S| = g+1,$$

$$(2.5.6\ )\qquad \forall\ \text{ distinct } i,j,k,l \in B,\qquad \frac{a_i-a_k}{a_i-a_l}\cdot\frac{a_j-a_l}{a_j-a_k} =$$
$$\boldsymbol{e}^{4\pi i(\eta_i-\eta_j)'\cdot(\eta_k-\eta_l)''}\left(\frac{\theta[\eta_U-\eta_V+\eta_i-\eta_k](0,\Omega_1)}{\theta[\eta_U-\eta_V+\eta_i-\eta_l](0,\Omega_1)}\frac{\theta[\eta_U-\eta_V+\eta_j-\eta_l](0,\Omega_1)}{\theta[\eta_U-\eta_V+\eta_j-\eta_k](0,\Omega_1)}\right)^2$$
$$=e^{-4\pi i(\eta_i-\eta_j)'(\Omega_1 I)(\eta_k-\eta_l)}(p(\dot{a}_i,\dot{a}_j,\dot{a}_k,\dot{a}_l))^2.$$

The equation 2.5.6 implies item (5) of theorem 2.5.1 since both sides of the equation in item (5) are equal to $\langle a_i, a_j, a_k, a_l\rangle$. $\square$

2.5.7 Lemma. *Let $\eta \in \Xi_g$ and $\Omega \in \mathcal{H}_g$. For all $V \subseteq B$ such that $|V| = g-1$, and for all distinct $i,j,k,l \notin V$, assume that the following $\gamma^\bullet$ are well–defined and independent of the choice of V:*

$$\gamma^{ijkl}(\Omega) = \frac{\theta[\eta_U - \eta_V + \eta_i - \eta_k](0,\Omega)}{\theta[\eta_U - \eta_V + \eta_i - \eta_l](0,\Omega)} \frac{\theta[\eta_U - \eta_V + \eta_j - \eta_l](0,\Omega)}{\theta[\eta_U - \eta_V + \eta_j - \eta_k](0,\Omega)}.$$

The action of Γ_2 on the γ^{ijkl}, via $\Omega \mapsto \sigma\cdot\Omega$, is then generated by P_{ab} for distinct $a,b \in B$, where

$$P_{ab}(\gamma^{ijkl}) = \begin{cases} +\gamma^{ijkl}, & \{a,b\} \not\subseteq \{i,j,k,l\} \\ +\gamma^{ijkl}, & \{a,b\} = \{i,j\} \text{ or } \{k,l\} \\ -\gamma^{ijkl}, & \{a,b\} = \{i,k\},\{i,l\},\{j,k\} \text{ or } \{j,l\}. \end{cases}$$

Proof. The reader may want to review lemma 1.1.8 in section §1.1 and the discussion afterwards. The action of Γ_2 on $\gamma^{ijkl}(\Omega)$ is determined by the mapping ϕ_{ijkl} of $M_{2g}^{\mathrm{sym}}(\mathbb{Z}) \to \frac14\mathbb{Z}/\mathbb{Z}$ given by:

$$\begin{aligned}\phi_{ijkl} =& (\eta_U-\eta_V+\eta_i-\eta_k)\otimes(\eta_U-\eta_V+\eta_i-\eta_k)+(\eta_U-\eta_V+\eta_j-\eta_l)\otimes(\eta_U-\eta_V+\eta_j-\eta_l)\\ &-(\eta_U-\eta_V+\eta_i-\eta_l)\otimes(\eta_U-\eta_V+\eta_i-\eta_l)-(\eta_U-\eta_V+\eta_j-\eta_k)\otimes(\eta_U-\eta_V+\eta_j-\eta_k)\\ =&(\eta_i-\eta_k)\otimes(\eta_i-\eta_k)+(\eta_j-\eta_l)\otimes(\eta_j-\eta_l)-(\eta_i-\eta_l)\otimes(\eta_i-\eta_l)-(\eta_j-\eta_k)\otimes(\eta_j-\eta_k)\end{aligned}$$

as follows: if $P \in M_{2g}^{\mathrm{sym}}(\mathbb{Z})$ then $P(\gamma^{ijkl}) = \boldsymbol{e}^{2\pi i \phi_{ijkl}(P)}\gamma^{ijkl}$. Let $\mathbf{lg}$ denote the unique homomorphism from $(\{\pm1\},\cdot)$ to $(\mathbb{Z}/2\mathbb{Z},+)$. Then for $\xi,\zeta \in \frac12\mathbb{Z}$ we have $e_2(\xi,\zeta) = e^{4\pi i\,{}^t\xi J\zeta}$, and ${}^t\xi J\zeta = \frac14\,\mathbf{lg}\, e_2(\xi,\zeta)$ in $\frac14\mathbb{Z}/\frac12\mathbb{Z}$. Since $\eta : B \to \frac12\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ is surjective we know that the $[2\eta_i]$ span $\mathbb{Z}^{2g}/2\mathbb{Z}^{2g}$. The $[2J\eta_i]$ also span $\mathbb{Z}^{2g}/2\mathbb{Z}^{2g}$, and so we have $\mathrm{Span}_{\mathbb{Z}}(2J\eta_i) + 2\mathbb{Z}^{2g} = \mathbb{Z}^{2g}$. Therefore, $\mathrm{Span}_{\mathbb{Z}}(2J\eta_i) + 4\mathbb{Z}^{2g} = \mathbb{Z}^{2g}$. Let

$$\begin{aligned} P_a &= (2J\eta_a)\otimes(2J\eta_a) \quad \text{and}\\ P_{ab} &= (2J\eta_a)\otimes(2J\eta_b) + (2J\eta_b)\otimes(2J\eta_a)\end{aligned}$$

be viewed as elements of $M_{2g}^{\mathrm{sym}}(\mathbb{Z})$. We see that $\mathrm{Span}_{\mathbb{Z}}(P_a,P_{ab}) + 4M_{2g}^{\mathrm{sym}}(\mathbb{Z}) = M_{2g}^{\mathrm{sym}}(\mathbb{Z})$, and hence that the map $\phi_{ijkl} : M_{2g}^{\mathrm{sym}}(\mathbb{Z}) \to \frac14\mathbb{Z}/\mathbb{Z}$ has its image generated by its values on the P_a and P_{ab}. The value of $\phi_{ijkl}(P_a)$ in $\frac14\mathbb{Z}/\mathbb{Z}$ is then

$$\begin{aligned}\phi_{ijkl}(P_a) = &\{[2\,{}^t(\eta_i-\eta_k)J\eta_a]^2 + [2\,{}^t(\eta_j-\eta_l)J\eta_a]^2\\ &-[2\,{}^t(\eta_i-\eta_l)J\eta_a]^2 - [2\,{}^t(\eta_j-\eta_k)J\eta_a]^2\}\\ =& -8[{}^t(\eta_i-\eta_j)J\eta_a][{}^t(\eta_k-\eta_l)J\eta_a].\end{aligned}$$

The above factor of 8 allows us to replace ${}^t(\eta_i - \eta_j)J\eta_a$ and ${}^t(\eta_k - \eta_l)J\eta_a$ by their values in $\frac{1}{4}\mathbb{Z}/\frac{1}{2}\mathbb{Z}$. From ${}^t\xi J\zeta = \frac{1}{4}\,\mathbf{lg}\,e_2(\xi,\zeta)$ in $\frac{1}{4}\mathbb{Z}/\frac{1}{2}\mathbb{Z}$ we obtain:

$$\begin{aligned}\phi_{ijkl}(P_a) &= -\,8[\frac{1}{4}\,\mathbf{lg}\,e_2(\eta_i - \eta_j, \eta_a)][\frac{1}{4}\,\mathbf{lg}\,e_2(\eta_k - \eta_l, \eta_a)] \\ &= -\,\frac{1}{2}[\mathbf{lg}\,e_2(\eta_{(i,j)}, \eta_a)][\mathbf{lg}\,e_2(\eta_{(k,l)}, \eta_a)].\end{aligned}$$

Since $\mathbf{lg}\,e_2(\eta_a, \eta_i)$ is equal to 1 for $a \neq i$, and to 0 for $a = i$, we see that $\phi_{ijkl}(P_a)$ is 0 for $a \notin \{i,j,k,l\}$. If $a = i$ we see that $\phi_{ijkl}(P_a) = -\frac{1}{2}\cdot 1 \cdot 0 = 0$. By the inherent symmetry the cases $a = j,k,l$ are the same, and so $\phi_{ijkl}(P_a)$ is identically zero. We now consider the values of $\phi_{ijkl}(P_{ab})$ in $\frac{1}{4}\mathbb{Z}/\mathbb{Z}$. We have,

$$\begin{aligned}&\phi_{ijkl}(P_{ab}) \\ &= \{2[2^t(\eta_i - \eta_k)J\eta_a][2^t(\eta_i - \eta_k)J\eta_b] + 2[2^t(\eta_j - \eta_l)J\eta_a][2^t(\eta_j - \eta_l)J\eta_b] \\ &\quad -2[2^t(\eta_i - \eta_l)J\eta_a][2^t(\eta_i - \eta_l)J\eta_b] - 2[2^t(\eta_j - \eta_k)J\eta_a][2^t(\eta_j - \eta_k)J\eta_b]\} \\ &= -8[{}^t(\eta_i - \eta_j)J\eta_a][{}^t(\eta_k - \eta_l)J\eta_b] - 8[{}^t(\eta_i - \eta_j)J\eta_b][{}^t(\eta_k - \eta_l)J\eta_a] \\ &= -\frac{1}{2}\,\mathbf{lg}\,e_2(\eta_i - \eta_j, \eta_a)\,\mathbf{lg}\,e_2(\eta_k - \eta_l, \eta_b) - \frac{1}{2}\,\mathbf{lg}\,e_2(\eta_i - \eta_j, \eta_b)\,\mathbf{lg}\,e_2(\eta_k - \eta_l, \eta_a).\end{aligned}$$

Unless both $a, b \in \{i,j,k,l\}$ the value of $\phi_{ijkl}(P_{ab})$ is 0. If $(a,b) = (i,j)$ the value is $-\frac{1}{2}\cdot 1\cdot 0 - \frac{1}{2}\cdot 1 \cdot 0 = 0$. If $(a,b) = (i,k)$ then $\phi_{ijkl}(P_{ab}) = -\frac{1}{2}\cdot 1\cdot 1 - \frac{1}{2}\cdot 0\cdot 0 = -\frac{1}{2}$. The rest are similar by symmetry. Since the induced action of the P_{ab} on the γ^{ijkl} is given by $P_{ab}(\gamma^{ijkl}) = \mathbf{e}^{2\pi i \phi_{ijkl}(P_{ab})}\gamma^{ijkl}$ this completes the proof.

2.5.8 Remark. This is the same action one would obtain if

$$\gamma^{ijkl} = (\text{const})\cdot\sqrt{\frac{a_i - a_k}{a_i - a_l}\cdot\frac{a_j - a_l}{a_j - a_k}},$$

and if the P_{mn} acted by analytically continuing a_m around a_n. The need to consider the P_{mn} arises from the need to adjust some signs in our thetanullwerte quotients $\gamma^{ijkl}(\Omega)$. In an earlier version of this paper, these signs were adjusted by moving the liftings of the branch points around on the universal cover of moduli space for hyperelliptic curves. However, as Mumford points out in [14, 3.131], this is equivalent to letting Γ_2 act directly on Ω; so we have introduced the P_{ab} as generators of the Γ_2 action on the $\gamma^{ijkl}(\Omega)$, thereby avoiding considerable detail concerning the universal cover of hyperelliptic moduli space.

2.5.9 Lemma. *Let $\eta \in \Xi_g$ and $\Omega \in \mathcal{H}_g$. Assume that Ω satisfies the nonvanishing conditions:*

$$S \subseteq B : |S| = g+1 \implies \theta[\eta_{U\circ S}](0,\Omega) \neq 0.$$

Let $K, L \subseteq B$ such that $|K| = |L| = g$. Let $i,j \in B$ such that $i,j \notin K \cup L$. Define $r = |K \cap L|$ so that $K \cap L = \{u_1,\dots,u_r\}$, $K = \{u_1,\dots,u_r,k_1,\dots,k_s\}$, and

$L = \{u_1, \ldots, u_r, l_1, \ldots, l_s\}$, for $s + r = g$. Then for all n such that $1 \le n \le r$, there exist $V_n \subseteq B$ such that $|V_n| = g - 1$, and $i, j, k_n, l_n \notin V_n$ such that we have

$$\frac{\theta[\eta_U + \eta_i - \eta_K](0,\Omega)}{\theta[\eta_U + \eta_i - \eta_L](0,\Omega)} \frac{\theta[\eta_U + \eta_j - \eta_L](0,\Omega)}{\theta[\eta_U + \eta_j - \eta_K](0,\Omega)}$$
$$= \prod_{n=1}^{r} \{ \frac{\theta[\eta_U - \eta_{V_n} + \eta_i - \eta_{k_n}](0,\Omega)}{\theta[\eta_U - \eta_{V_n} + \eta_i - \eta_{l_n}](0,\Omega)} \frac{\theta[\eta_U - \eta_{V_n} + \eta_j - \eta_{l_n}](0,\Omega)}{\theta[\eta_U - \eta_{V_n} + \eta_j - \eta_{k_n}](0,\Omega)} \}.$$

Proof. We inductively define V_n for $1 \le n \le r$ by $V_{n+1} = V_n \circ \{l_n, k_{n+1}\}$ and $V_1 = K \circ \{k_1\}$, and we note it follows that $V_r = L \circ \{l_r\}$. For $1 \le n \le r-1$ we have that $\eta_U - \eta_{V_n} + \eta_i - \eta_{l_n} = \eta_U - \eta_{V_{n+1} \circ \{l_n, k_{n+1}\}} + \eta_i - \eta_{l_n} = \eta_U - \eta_{V_{n+1}} + \eta_i - \eta_{k_{n+1}}$; this shows that $\theta[\eta_U - \eta_{V_n} + \eta_i - \eta_{l_n}](0,\Omega) = \theta[\eta_U - \eta_{V_{n+1}} + \eta_i - \eta_{k_{n+1}}](0,\Omega)$, and we note that the same is true when j replaces i. The product in lemma 2.5.9 is a telescoping product and we are left with only two k_1 factors and two l_r factors:

$$\{ \frac{\theta[\eta_U - \eta_{V_1} + \eta_i - \eta_{k_1}]}{1} \frac{1}{\theta[\eta_U - \eta_{V_1} + \eta_j - \eta_{k_1}]} \} \cdot \{ \frac{1}{\theta[\eta_U - \eta_{V_r} + \eta_i - \eta_{l_r}]} \frac{\theta[\eta_U - \eta_{V_r} + \eta_j - \eta_{l_r}]}{1} \}.$$

We observe that $\eta_U - \eta_{V_1} + \eta_i - \eta_{k_1} = \eta_U + \eta_i - \eta_K$ and $\eta_U - \eta_{V_r} + \eta_i - \eta_{l_r} = \eta_U + \eta_i - \eta_L$ to complete the proof of this lemma.

Proof of proposition 2.5.2. We consider the action of Γ_2 on the $\gamma^{ijkl}(\Omega_1)$, and in view of 2.5.7 call $\mathcal{P}$ the group generated the P_{ab}. By lemma 2.5.7 then there exists a $\sigma \in \Gamma_2$ such that $\gamma^{ijkl}(\sigma \cdot \Omega_1) = \gamma^{ijkl}(\Omega)$ if and only if there is an $S \in \mathcal{P}$ such that $S(\gamma^{ijkl}(\Omega_1)) = \gamma^{ijkl}(\Omega)$. By proposition 3.3.7 in section 3.3 such an $S \in \mathcal{P}$ exists, however, only when $\gamma^{ijkl}(\Omega)$ and $\gamma^{ijkl}(\Omega_1)$ yield the same values upon substitution into the invariant generators of proposition 3.3.3. We are again making strong use of a simple invariant theory calculation and we now list the relevant invariant generators once more.

(1) $(\gamma^{ijkl})^2$
(2) $\gamma^{ijkl}(\gamma^{jilk})^{-1}$, $\gamma^{ijkl}(\gamma^{klij})^{-1}$, $\gamma^{ijkl}(\gamma^{lkji})^{-1}$
(3) $\gamma^{ijkl}\gamma^{jikl}$
(4) $\gamma^{ikjl}\gamma^{lijk}(\gamma^{ijkl})^{-1}$
(5) $\gamma^{ijkl}\gamma^{ijlm}(\gamma^{ijkm})^{-1}$

We now demonstrate that $(\gamma^{ijkl}(\Omega))^2 = (\gamma^{ijkl}(\Omega_1))^2$ by using item (5) of theorem 2.5.1. This shows that the equality of the $(\gamma^{ijkl})^2$ in item (1) above is true. We have

$$\begin{aligned}
&\boldsymbol{e}^{4\pi i(\eta_i - \eta_j)' \cdot (\eta_k - \eta_l)''} (\gamma^{ijkl}(\Omega))^2 \\
=&\boldsymbol{e}^{4\pi i(\eta_i - \eta_j)' \cdot (\eta_k - \eta_l)''} (\frac{\theta[\eta_U - \eta_V + \eta_i - \eta_k](0,\Omega)}{\theta[\eta_U - \eta_V + \eta_i - \eta_l](0,\Omega)} \frac{\theta[\eta_U - \eta_V + \eta_j - \eta_l](0,\Omega)}{\theta[\eta_U - \eta_V + \eta_j - \eta_k](0,\Omega)})^2 \\
=&e^{-4\pi i(\eta_i - \eta_j)'(\Omega I)(\eta_k - \eta_l)} (p^{ijkl})^2 \\
=&e^{-4\pi i(\eta_i - \eta_j)'(\Omega_1 I)(\eta_k - \eta_l)} (p(\dot{a}_i, \dot{a}_j, \dot{a}_k, \dot{a}_l))^2
\end{aligned}$$

$$=\boldsymbol{e}^{4\pi i(\eta_i-\eta_j)'\cdot(\eta_k-\eta_l)''}\Big(\frac{\theta[\eta_U-\eta_V+\eta_i-\eta_k](0,\Omega_1)}{\theta[\eta_U-\eta_V+\eta_i-\eta_l](0,\Omega_1)}\frac{\theta[\eta_U-\eta_V+\eta_j-\eta_l](0,\Omega_1)}{\theta[\eta_U-\eta_V+\eta_j-\eta_k](0,\Omega_1)}\Big)^2$$
$$=\boldsymbol{e}^{4\pi i(\eta_i-\eta_j)'\cdot(\eta_k-\eta_l)''}(\gamma^{ijkl}(\Omega_1))^2.$$

The first and fifth equalities follow from the definition of $\gamma^\bullet$ in lemma 2.5.7. The second equality follows from lemma 2.4.3 and from definition 2.4.2. The third equality is item (5) of theorem 2.5.1. The fourth equality follows from equation 2.5.6, which is the same as corollary 1.5.3. Since $\gamma^{ijkl}(\Omega)$ satisfies the crossratio symmetries up to $\pm$ signs independent of Ω, it follows that items (2), (3), (4), and (5) yield the same universal signs for both $\gamma^{ijkl}(\Omega)$ and $\gamma^{ijkl}(\Omega_1)$. We do not need to know these signs but the reader may verify using 1.1.5 that the $\gamma^{ijkl}(\Omega)$ in fact satisfy the crossratio symmetries. This shows that the invariant generators of $\gamma^{ijkl}(\Omega)$ agree with those of $\gamma^{ijkl}(\Omega_1)$, and hence that $S\in\mathcal{P}$ and $\sigma\in\Gamma_2$ exist so that $\gamma^\bullet(\Omega)=\gamma^\bullet(\sigma\cdot\Omega_1)$.

The second conclusion in proposition 2.5.2 follows from the first part and lemma 2.5.9. We may assume that $i\neq j$, for if $i=j$ then both sides are equal to 1 and the conclusion holds. If $i\neq j$ we pick V_n such that $i,j,k_n,l_n\notin V_n$. By lemma 2.5.9 the equality of the

$$\frac{\theta[\eta_U-\eta_{V_n}+\eta_i-\eta_{k_n}](0,\cdot)}{\theta[\eta_U-\eta_{V_n}+\eta_i-\eta_{l_n}](0,\cdot)}\frac{\theta[\eta_U-\eta_{V_n}+\eta_j-\eta_{l_n}](0,\cdot)}{\theta[\eta_U-\eta_{V_n}+\eta_j-\eta_{k_n}](0,\cdot)}$$

implies the equality of the

$$\frac{\theta[\eta_U+\eta_i-\eta_K](0,\cdot)}{\theta[\eta_U+\eta_i-\eta_L](0,\cdot)}\frac{\theta[\eta_U+\eta_j-\eta_L](0,\cdot)}{\theta[\eta_U+\eta_j-\eta_K](0,\cdot)}.\quad\square$$

§2.6 Irreduciblity and vanishing imply hyperelliptic.

In this section we assume for the first time that the irreducible Ω satisfies the vanishing conditions $V_{g,\eta}$. The main theorem 2.6.1 shows that under these assumptions Ω is Γ_2-equivalent to the hyperelliptic Ω_1 and hence that Ω is hyperelliptic. Starting with the special equalities of quotients of thetanullwerte for Ω and Ω_1 given in proposition 2.5.2, we begin the long march to proposition 2.6.2 where all the Γ_2-invariant quotients of thetanullwerte for Ω and Ω_1 are shown to coincide. The author's inability to find a simple induction proof of proposition 2.6.3 accounts for the length of this section.

2.6.1 Main Theorem. *Let $\eta\in\Xi_g$ and $\Omega\in\mathcal{H}_g$. The following two statements are equivalent.*

(1) *Ω is irreducible and $\Omega\in V'_{g,\eta}$.*
(2) *There is a marked hyperelliptic Riemann surface M of genus g which has Ω as its period matrix and $\mathrm{Jac}(M)=\mathbb{C}^g/(\mathbb{Z}^g+\Omega\mathbb{Z}^g)$. Furthermore, there is a model of M, $y^2=\prod_{i\in B}(x-a_i)$, with a_∞ as the basepoint of the Abel–Jacobi map $w:M\to\mathrm{Jac}(M)$ such that $w(a_i)=[(\Omega I)\eta_i]$ in $\mathrm{Jac}(M)$.*

Proof of the main theorem. Assume statement (2). A Jacobian of a marked hyperelliptic curve has an analytically irreducible theta locus, Θ, by corollary 1.2.13, and hence a symplectically irreducible period matrix Ω. From section §1.4 in general, and proposition 1.4.17 in particular, we see that Ω satisfies the vanishing equations for η. This verifies that (2) implies (1), and so we only need to demonstrate the converse, (1) implies (2). Assume Ω is irreducible and satisfies the vanishing equations $V_{g,\eta}$. We assume proposition 2.6.2 which will be proven later in this section. The equations $V_{g,\eta}$ imply the Frobenius theta formula, $\mathrm{Frob}_{g,\eta}$, and the equations $F_{g,\eta}$. These implications are proposition 1.6.10 and theorem 1.6.16 of section §1.6. Since Ω is irreducible and $\Omega \in F'_{g,\eta}$, we may use theorem 2.5.1 to produce a hyperelliptic Riemann surface M with marking m_1, Abel–Jacobi map corresponding to η, and period matrix Ω_1, such that the conclusion of proposition 2.5.2 holds for some $\sigma \in \Gamma_2$. This proposition asserts that for appropriate indices i, j, K, L, the $\frac{\theta[\eta_U+\eta_i-\eta_K](0,\cdot)}{\theta[\eta_U+\eta_i-\eta_L](0,\cdot)} \frac{\theta[\eta_U+\eta_j-\eta_L](0,\cdot)}{\theta[\eta_U+\eta_j-\eta_K](0,\cdot)}$ give the same values for Ω and $\sigma \cdot \Omega_1$.

If we let Γ_2 act on the ring $\mathbb{C}[\theta[\zeta](0,\Omega)]_\zeta$, where $\zeta \in \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$, then the Γ_2-invariant quotient field will give moduli for Γ_2; the precise lemma we use here is lemma 1.1.9 applied to Ω and $\sigma\cdot\Omega_1$. Condition (1) of lemma 1.1.9 is satisfied because Ω and $\sigma \cdot \Omega_1$ have the same vanishing and nonvanishing. Theorem 2.3.1 shows that the thetanullwerte for Ω of hyperelliptic η–order zero do not vanish; the hypothesis $\Omega \in V'_{g,\eta}$ is that the other thetanullwerte of nonzero hyperelliptic η–order do vanish. Proposition 1.4.17 of section §1.4 shows that the hyperelliptic Ω_1 has the same vanishing and nonvanishing as Ω. Since $\sigma \in \Gamma_2$ does not permute the thetanullwerte, we conclude that: $\theta[\zeta](0,\Omega) = 0 \iff \theta[\zeta](0,\sigma\cdot\Omega_1) = 0$. This verifies condition (1) of lemma 1.1.9 and we now consider condition (2) of lemma 1.1.9. By the proposition 2.5.2 just mentioned and by proposition 2.6.2 below, the Γ_2-invariant quotients of monomials in the thetanullwerte of hyperelliptic η-order zero coincide for Ω and $\sigma \cdot \Omega_1$. Actually, the terms $\pm\,\eta_K$, $\pm\,\eta_L$, in proposition 2.5.2 and proposition 2.6.2 differ, but the sign change which this induces in the thetanullwerte quotients is independent of Ω. Furthermore, both Ω and $\sigma \cdot \Omega_1$ satisfy the vanishing conditions, $V_{g,\eta}$, so that if ζ is not of hyperelliptic η-order zero then both $\theta[\zeta](0,\Omega) = \theta[\zeta](0,\sigma\cdot\Omega_1) = 0$. Any Γ_2-invariant quotient of monomials whose nullwerte are not all of hyperelliptic η–order zero therefore vanishes at both Ω and $\sigma\cdot\Omega_1$ when well–defined. The agreement of all the Γ_2-invariant quotients of monomials of thetanullwerte at Ω and $\sigma\cdot\Omega_1$ which are well–defined there, then follows from the agreement at Ω and $\sigma \cdot \Omega_1$ of the Γ_2-invariant quotients of monomials of thetanullwerte of hyperelliptic η–order zero. The Γ_2-invariant quotients of polynomials in the thetanullwerte, $\theta[\zeta]$, will be generated by Γ_2-invariant quotients of monomials because Γ_2 does not permute the thetanullwerte. This is an invariant theory calculation we suppress. This verifies condition (2) of lemma 1.1.9, and proves that Ω is Γ_2-equivalent to $\sigma \cdot \Omega_1$ and hence to Ω_1. The result that Ω is hyperelliptic is the most important; we continue on to relate η to the Abel–Jacobi map.

Let $\sigma_2 \in \Gamma_2$ be such that $\Omega = \sigma_2 \cdot \Omega_1$. If we use σ_2 to change the marking m_1 on M, then Ω will be the period matrix of M in this new marking m. The old Abel–Jacobi map $w_1 : \widehat{M} \to \mathbb{C}^g/(\mathbb{Z}^g + \Omega_1\mathbb{Z}^g)$ changes with the marking to

$w : \widehat{M} \to \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g)$, but since $\sigma_2 \in \Gamma_2$, we have $\sigma_2\eta_i \equiv \eta_i \mod \mathbb{Z}^{2g}$. This shows that $w(a_i) \equiv (\Omega I)\eta_i \mod \mathbb{Z}^g + \Omega\mathbb{Z}^g$. □

2.6.2 Proposition. *Let $\eta \in \Xi_g$ and $\Omega \in \mathcal{H}_g$. Assume $\theta[\zeta](0,\Omega) \neq 0$ for all ζ of hyperelliptic η-order zero. Then the Γ_2-invariant quotients of monomials,*

$$\prod_{i=1}^{N} \frac{\theta[\zeta_i](0,\Omega)}{\theta[\xi_i](0,\Omega)}, \quad \text{where the } \zeta_i, \xi_i \in \frac{1}{2}\mathbb{Z}^{2g} \text{ have hyperelliptic } \eta\text{-order zero,}$$

are uniquely determined by the values,

$$\frac{\theta[\eta_U + \eta_i + \eta_K](0,\Omega)}{\theta[\eta_U + \eta_i + \eta_L](0,\Omega)} \frac{\theta[\eta_U + \eta_j + \eta_L](0,\Omega)}{\theta[\eta_U + \eta_j + \eta_K](0,\Omega)},$$

where $i, j \in B$ and $K, L \subseteq B$ and $i \neq j$ and $i, j \notin K \cup L$ and $|K| = |L| = g$.

2.6.3 Proposition. *Let $\eta \in \Xi_g$ and $\Omega \in \mathcal{H}_g$. Assume $\theta[\zeta](0,\Omega) \neq 0$ for all ζ of hyperelliptic η-order zero. Let $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \frac{1}{2}\mathbb{Z}^{2g}$ have hyperelliptic η-order zero and assume $\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 \in \mathbb{Z}^{2g}$. Define:*

$$Q = \frac{\theta[\zeta_1](0,\Omega)}{\theta[\zeta_3](0,\Omega)} \frac{\theta[\zeta_2](0,\Omega)}{\theta[\zeta_4](0,\Omega)}.$$

Then Q is (up to a sign independent of Ω) equal to a product of elements of the form:

$$\frac{\theta[\eta_U + \eta_i + \eta_K](0,\Omega)}{\theta[\eta_U + \eta_i + \eta_L](0,\Omega)} \frac{\theta[\eta_U + \eta_j + \eta_L](0,\Omega)}{\theta[\eta_U + \eta_j + \eta_K](0,\Omega)},$$

where $i, j \in B$ and $K, L \subseteq B$ and $i \neq j$ and $i, j \notin K \cup L$ and $|K| = |L| = g$.

Proof. Since the ζ_i are of hyperelliptic η-order zero there exist S, $\bar{S}$, $T \subseteq B$ such that $|S| = |\bar{S}| = |T| = g + 1$, and $\zeta_1 \equiv \eta_U + \eta_S$, $\zeta_2 \equiv \eta_U + \eta_T$, $\zeta_3 \equiv \eta_U + \eta_{\bar{S}} \mod \mathbb{Z}^{2g}$. Furthermore if we define $\bar{T} = S \circ \bar{S} \circ T$, then $\eta_U + \eta_{\bar{T}^{\mathbf{c}}} \equiv \eta_U + \eta_{\bar{T}} = \eta_U + \eta_{S\circ\bar{S}\circ T} \equiv \zeta_1 + \zeta_2 + \zeta_3 \equiv \zeta_4$, so that $|\bar{T}| = g + 1$ since ζ_4 has hyperelliptic η-order zero. If we knew that $\zeta_1 \equiv \eta_U + \eta_S \mod 2\mathbb{Z}^{2g}$ then we could conclude that $\theta[\zeta_1](0,\Omega) = \theta[\eta_U + \eta_S](0,\Omega)$; however we only have $\zeta_1 \equiv \eta_U + \eta_S \mod \mathbb{Z}^{2g}$ so that $\theta[\zeta_1](0,\Omega) = \pm\theta[\eta_U + \eta_S](0,\Omega)$. This sign is independent of Ω though, and so to construct $\frac{\theta[\zeta_1](0,\Omega)}{\theta[\zeta_3](0,\Omega)} \frac{\theta[\zeta_2](0,\Omega)}{\theta[\zeta_4](0,\Omega)}$ up to a sign independent of Ω it suffices to construct:

$$Q' = \frac{\theta[\eta_U + \eta_S](0,\Omega)}{\theta[\eta_U + \eta_{\bar{S}}](0,\Omega)} \frac{\theta[\eta_U + \eta_T](0,\Omega)}{\theta[\eta_U + \eta_{\bar{T}^{\mathbf{c}}}](0,\Omega)}\ .$$

Consider $S, \bar{S}, T, \bar{T} \subseteq B$ such that $|S| = |\bar{S}| = |T| = |\bar{T}| = g+1$ and $S \circ \bar{S} \circ T \circ \bar{T} = \emptyset$, or equivalently $\bar{T} = S \circ \bar{S} \circ T$. We label the eight atoms as follows:

$$S \cap \bar{S} \cap T = A_2, \quad S \cap \bar{S}^{\mathbf{c}} \cap T = B_2, \quad S^{\mathbf{c}} \cap \bar{S} \cap T = C_1, \quad S^{\mathbf{c}} \cap \bar{S}^{\mathbf{c}} \cap T = D_1$$
$$S \cap \bar{S} \cap T^{\mathbf{c}} = A_1,\ S \cap \bar{S}^{\mathbf{c}} \cap T^{\mathbf{c}} = B_1,\ S^{\mathbf{c}} \cap \bar{S} \cap T^{\mathbf{c}} = C_2,\ S^{\mathbf{c}} \cap \bar{S}^{\mathbf{c}} \cap T^{\mathbf{c}} = D_2\,.$$

Then we have the following disjoint unions:

$$S = A_1 \amalg A_2 \amalg B_1 \amalg B_2,\ T = D_1 \amalg A_2 \amalg C_1 \amalg B_2,\ \bar{S}^{\mathbf{c}} = D_1 \amalg D_2 \amalg B_1 \amalg B_2$$
$$\bar{S} = A_1 \amalg A_2 \amalg C_1 \amalg C_2,\ \bar{T} = D_1 \amalg A_2 \amalg B_1 \amalg C_2,\ \bar{T}^{\mathbf{c}} = A_1 \amalg D_2 \amalg C_1 \amalg B_2\,.$$

The unions for $\bar{T}$ and $\bar{T}^{\mathbf{c}}$ follow directly from the definition $\bar{T} = S \circ \bar{S} \circ T$. From the pair of equations $|S| = |\bar{S}|$ and $|T| = |\bar{T}|$, we obtain the pair $|B_1|+|B_2| = |C_1|+|C_2|$ and $|C_1| + |B_2| = |B_1| + |C_2|$, which together imply $|B_1| = |C_1|$ and $|B_2| = |C_2|$. In the same way the pair of equations $|S| = |\bar{S}^{\mathbf{c}}|$ and $|T| = |\bar{T}^{\mathbf{c}}|$ yield the pair $|A_1| + |A_2| = |D_1| + |D_2|$ and $|D_1| + |A_2| = |A_1| + |D_2|$, which together imply $|A_1| = |D_1|$ and $|A_2| = |D_2|$. Denote by r, s and t the non–negative integers given by

$$r = |S \cap \bar{S}| = |A_1| + |A_2| = |D_1| + |A_2| = |T \cap \bar{T}|,$$
$$s = |B_1| = |C_1|, \quad \text{and} \quad t = |B_2| = |C_2|\ .$$

Label the disjoint, possibly empty, sets B_1, C_1, B_2, C_2 as: $B_1 = \{b_i\}_{i=1}^{s}$, $B_2 = \{b_i\}_{i=s+1}^{s+t}$, $C_1 = \{c_i\}_{i=1}^{s}$ and $C_2 = \{c_i\}_{i=s+1}^{s+t}$.

We now inductively define two sequences of sets, $T_i, S_i \subseteq B$, such that $|T_i| = |S_i| = g + 1$. For i such that $0 \le i \le s + t - 1$ define $S_{i+1} = S_i \circ \{b_{i+1}, c_{i+1}\}$, and begin the sequence with $S_0 = S = A_1 \amalg A_2 \amalg B_1 \amalg B_2$. Since S contains all of the $s + t$ elements b_i and none of the $s + t$ elements c_i, we see that S_{i+1} always gains c_{i+1} and loses b_{i+1} so that $|S_{i+1}| = g + 1$. Note for later use that $S_{s+t} = S_0 \circ (B_1 \amalg B_2 \amalg C_1 \amalg C_2) = A_1 \amalg A_2 \amalg C_1 \amalg C_2 = \bar{S}$. We now define the sequence of sets T_i. Let $T_0 = T = D_1 \amalg A_2 \amalg C_1 \amalg B_2$, and $T_{i+1} = T_i \circ \{b_{i+1}, c_{i+1}\}$ for i such that $0 \le i \le s - 1$. Note that $T_s = T \circ (B_1 \amalg C_1) = D_1 \amalg A_2 \amalg B_1 \amalg B_2$. Let $T_{s+1} = {T_s}^{\mathbf{c}} \circ \{b_{s+1}, c_{s+1}\}$, and let $T_{i+1} = T_i \circ \{b_{i+1}, c_{i+1}\}$ for i such that $s+1 \le i \le s+t-1$. By the same reasoning as above $|T_i| = g+1$ for i satisfying $0 \le i \le s$ because T contains C_1 but is disjoint from B_1; for i satisfying $s + 1 \le i \le s + t$ we also have $|T_i| = g + 1$ because ${T_s}^{\mathbf{c}} = A_1 \amalg D_2 \amalg C_1 \amalg C_2$ contains C_2 and is disjoint from B_2. Note for later use that $T_{s+t} = {T_s}^{\mathbf{c}} \circ (B_2 \amalg C_2) = A_1 \amalg D_2 \amalg C_1 \amalg B_2 = \bar{T}^{\mathbf{c}}$.

We use these two sequences to write Q' as a telescoping product:

$$\begin{aligned} Q' =& \frac{\theta[\eta_U + \eta_S](0,\Omega)}{\theta[\eta_U + \eta_{\bar{S}}](0,\Omega)} \frac{\theta[\eta_U + \eta_T](0,\Omega)}{\theta[\eta_U + \eta_{\bar{T}^{\mathbf{c}}}](0,\Omega)} \\ =& \frac{\theta[\eta_U + \eta_{S_0}](0,\Omega)}{\theta[\eta_U + \eta_{S_{s+t}}](0,\Omega)} \frac{\theta[\eta_U + \eta_{T_0}](0,\Omega)}{\theta[\eta_U + \eta_{T_{s+t}}](0,\Omega)} \\ =& \prod_{i=0}^{s+t-1} \frac{\theta[\eta_U + \eta_{S_i}](0,\Omega)}{\theta[\eta_U + \eta_{S_{i+1}}](0,\Omega)} \frac{\theta[\eta_U + \eta_{T_i}](0,\Omega)}{\theta[\eta_U + \eta_{T_{i+1}}](0,\Omega)}. \end{aligned}$$

We now conclude the proof by showing that (up to sign) each of the factors in the above product is an element of the form,

$$\frac{\theta[\eta_U + \eta_a + \eta_K](0,\Omega)}{\theta[\eta_U + \eta_b + \eta_K](0,\Omega)} \frac{\theta[\eta_U + \eta_b + \eta_L](0,\Omega)}{\theta[\eta_U + \eta_a + \eta_L](0,\Omega)},$$

where $a \neq b$ and $K, L \subseteq B$ and $a, b \notin K \cup L$ and $|K| = |L| = g$. For i satisfying $0 \le i \le s-1$, we have

$$\frac{\theta[\eta_U+\eta_{S_i}](0,\Omega)}{\theta[\eta_U+\eta_{S_{i+1}}](0,\Omega)}\frac{\theta[\eta_U+\eta_{T_i}](0,\Omega)}{\theta[\eta_U+\eta_{T_{i+1}}](0,\Omega)}$$
$$=\frac{\theta[\eta_U+\eta_{S_i}](0,\Omega)}{\theta[\eta_U+\eta_{S_i\circ\{b_{i+1},c_{i+1}\}}](0,\Omega)}\frac{\theta[\eta_U+\eta_{T_i}](0,\Omega)}{\theta[\eta_U+\eta_{T_i\circ\{b_{i+1},c_{i+1}\}}](0,\Omega)}$$
$$(2.6.4\quad)$$
$$=\frac{\theta[\eta_U+\eta_{b_{i+1}}+\eta_{S_i\smallsetminus b_{i+1}}](0,\Omega)}{\theta[\eta_U+\eta_{c_{i+1}}+\eta_{S_i\smallsetminus b_{i+1}}](0,\Omega)}\frac{\theta[\eta_U+\eta_{c_{i+1}}+\eta_{T_i\smallsetminus c_{i+1}}](0,\Omega)}{\theta[\eta_U+\eta_{b_{i+1}}+\eta_{T_i\smallsetminus c_{i+1}}](0,\Omega)}.$$

Since $B_1 \cap C_1 = \emptyset$ we have $b_{i+1} \neq c_{i+1}$. From $S_i \cap C_1 = \{c_1, \ldots, c_i\}$ and $T_i \cap B_1 = \{b_1, \ldots, b_i\}$ we deduce $b_{i+1}, c_{i+1} \notin (S_i \smallsetminus b_{i+1}) \amalg (T_i \smallsetminus c_{i+1})$. Since $|T_i| = |S_i| = g+1$ we have $|T_i \smallsetminus c_{i+1}| = |S_i \smallsetminus b_{i+1}| = g$.

For i satisfying $s+1 \le i \le s+t-1$, we also have equation 2.6.4 but the explanation that the factor is of the required form differs. Since $B_2 \cap C_2 = \emptyset$ we have $b_{i+1} \neq c_{i+1}$. From $S_i \cap C_2 = \{c_{s+1}, \ldots, c_i\}$ and $T_i \cap B_2 = \{b_{s+1}, \ldots, b_i\}$ we deduce $b_{i+1}, c_{i+1} \notin (S_i \smallsetminus b_{i+1}) \amalg (T_i \smallsetminus c_{i+1})$. As was previously observed, we have $|T_i \smallsetminus c_{i+1}| = |S_i \smallsetminus b_{i+1}| = g$. The only remaining case is $i = s$, and then we are considering the factor

$$\frac{\theta[\eta_U+\eta_{S_s}](0,\Omega)}{\theta[\eta_U+\eta_{S_s\circ\{b_{s+1},c_{s+1}\}}](0,\Omega)}\frac{\theta[\eta_U+\eta_{T_s}](0,\Omega)}{\theta[\eta_U+\eta_{T_s^{\mathsf{c}}\circ\{b_{s+1},c_{s+1}\}}](0,\Omega)}.$$

We have $\theta[\eta_U+\eta_{T_s}](0,\Omega) = \pm\theta[\eta_U+\eta_{T_s{}^{\mathsf{c}}}](0,\Omega)$, where the sign is independent of Ω. The factor

$$\frac{\theta[\eta_U+\eta_{S_s}](0,\Omega)}{\theta[\eta_U+\eta_{S_s\circ\{b_{s+1},c_{s+1}\}}](0,\Omega)}\frac{\theta[\eta_U+\eta_{T_s{}^{\mathsf{c}}}](0,\Omega)}{\theta[\eta_U+\eta_{T_s^{\mathsf{c}}\circ\{b_{s+1},c_{s+1}\}}](0,\Omega)}$$

is of the required form by the same analysis as in the case $s+1 \le i \le s+t-1$. This concludes the proof except when $s = t = 0$ and the product constructed in this proof is empty. In this case $r = g+1 = |S \cap \bar{S}| = |T \cap \bar{T}|$, so that $S = \bar{S}$, $T = \bar{T}$, and $Q' = \pm 1$. $\square$

The proof of propositon 2.6.3 required extensive labelling. The gist however may be seen in any example. For $g = 4$, consider

$$\frac{\theta[\eta_U+\eta_{\{1,2,3,4,9\}}](0,\Omega)}{\theta[\eta_U+\eta_{\{1,5,7,8,\infty\}}](0,\Omega)}\frac{\theta[\eta_U+\eta_{\{2,5,6,8,9\}}](0,\Omega)}{\theta[\eta_U+\eta_{\{3,4,6,7,\infty\}}](0,\Omega)}.$$

Write $\theta[\eta_U+\eta_{\{1,2,3,4,9\}}](0,\Omega) = (12349)$ in an obvious notation so that the quotient of thetanullwerte we are considering becomes $\frac{1;34;29}{1;58;7\infty}\frac{6;58;29}{6;34;7\infty}$. The telescoping product constructed in the proof of Proposition 2.6.3 is then:

$$(\frac{1;34;29}{1;54;29}\cdot\frac{6;58;29}{6;38;29})(\frac{1;54;29}{1;58;29}\cdot\frac{6;38;29}{6;34;29})(\frac{1;58;29}{1;58;79}\cdot\frac{1;58;7\infty}{1;58;2\infty})(\frac{1;58;79}{1;58;7\infty}\cdot\frac{1;58;2\infty}{1;58;29})$$

This product equals $(\frac{1;34;29}{1;58;7\infty}\cdot\frac{6;58;29}{1;58;29})\frac{1;58;7\infty}{6;34;29}$. Now $1;58;29$ is equivalent to $6;34;7\infty$ by taking complements, and similarly $1;58;7\infty$ is equivalent to $6;34;29$. So up to sign the telescoping product is the desired result, namely: $\frac{1;34;29}{1;58;7\infty}\frac{6;58;29}{6;34;7\infty}$.

2.6.5 Lemma. *Let $\Omega \in \mathcal{H}_g$ and $\Sigma \subseteq \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ be a göpel system such that $\theta[\zeta](0,\Omega) \neq 0$ for all $\zeta \in \Sigma$. For all $\{u_i\}_1^n, \{v_i\}_1^n \subseteq \Sigma$ such that $\sum_{i=1}^n u_i = \sum_{i=1}^n v_i$ write $I = \prod_{i=1}^n \frac{\theta[u_i](0,\Omega)}{\theta[v_i](0,\Omega)}$. Then I is a product of elements of the form,*

$$A^{ab}_{cd} = \frac{\theta[a](0,\Omega)\theta[b](0,\Omega)}{\theta[c](0,\Omega)\theta[d](0,\Omega)},$$

where $a, b, c, d \in \Sigma$, and $a + b = c + d$.

Proof. In the case $n = 1$ the hypothesis is $u_1 = v_1$ so that $I = \prod_{i=1}^n \frac{\theta[u_i](0,\Omega)}{\theta[v_i](0,\Omega)}$ $= 1 = A^{ab}_{ab}$. The case $n = 2$ is precisely the case we are allowing, $A^{u_1u_2}_{v_1v_2}$. Assume that $n \geq 3$; we will show by induction on n that I is a product of factors of the form A^{ab}_{cd}, where $a + b = c + d$. We have

$$\begin{aligned} I =& \prod_{i=1}^n \frac{\theta[u_i](0,\Omega)}{\theta[v_i](0,\Omega)} \\ =& (\frac{\theta[u_1](0,\Omega)}{\theta[v_1](0,\Omega)} \frac{\theta[u_2](0,\Omega)}{\theta[u_1+u_2-v_1](0,\Omega)})(\frac{\theta[u_1+u_2-v_1](0,\Omega)}{\theta[v_2](0,\Omega)} \prod_{i=3}^n \frac{\theta[u_i](0,\Omega)}{\theta[v_i](0,\Omega)}). \end{aligned}$$

The first factor is clearly of the required form since $\Sigma + \Sigma + \Sigma = \Sigma$. The second factor also satisfies the hypothesis of the lemma but has only $n-1$ factors, and so by the induction hypothesis is a product $\prod A^{\gamma_i\delta_i}_{\alpha_i\beta_i}$, where $\alpha_i + \beta_i = \gamma_i + \delta_i$. Therefore $I = A^{u_1u_2}_{v_1\,(u_1+u_2-v_1)} \prod A^{\gamma_i\delta_i}_{\alpha_i\beta_i}$ and the induction is complete.

2.6.6 Lemma. *Let $\Omega \in \mathcal{H}_g$ and $\eta \in \Xi_g$. Assume that $\theta[\zeta](0,\Omega) \neq 0$ for all ζ of hyperelliptic η-order zero. For all $\{u_i\}_1^n, \{v_i\}_1^n \subseteq \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ of hyperelliptic η-order zero such that $\sum_{i=1}^n u_i = \sum_{i=1}^n v_i$, write $I = \prod_{i=1}^n \frac{\theta[u_i](0,\Omega)}{\theta[v_i](0,\Omega)}$. Then I is a product of elements of the form*

$$A^{ab}_{cd} = \frac{\theta[a](0,\Omega)\theta[b](0,\Omega)}{\theta[c](0,\Omega)\theta[d](0,\Omega)},$$

where $a, b, c, d \in \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ are of hyperelliptic η-order zero and $a + b = c + d$.

Proof. There exist göpel sytems, Σ_i, of hyperelliptic η-order zero, such that $\Sigma_1 + \Sigma_2 = \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ and $\Sigma_1 \cap \Sigma_2 = \{\mathit{\Delta}\}$. This is lemma 1.4.26 in section §1.4. Then $(\mathit{\Delta}+\Sigma_1) \oplus (\mathit{\Delta}+\Sigma_2) = \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$, and there exist unique $u_i', v_i' \in \Sigma_1$ and $u_i'', v_i'' \in \Sigma_2$ such that

$$\begin{aligned} \mathit{\Delta} + u_i &= (\mathit{\Delta} + u_i') + (\mathit{\Delta} + u_i'') = u_i' + u_i'', \\ \mathit{\Delta} + v_i &= (\mathit{\Delta} + v_i') + (\mathit{\Delta} + v_i'') = v_i' + v_i''. \end{aligned}$$

Furthermore, the directness of the sum implies that $\sum u_i' = \sum v_i'$ and $\sum u_i'' = \sum v_i''$. We may decompose I in the following way:

$$\begin{aligned} I =& \prod_{i=1}^n \frac{\theta[u_i](0,\Omega)}{\theta[v_i](0,\Omega)} \\ =& \prod_{i=1}^n \frac{\theta[u_i](0,\Omega)}{\theta[u_i'](0,\Omega)} \frac{\theta[\mathit{\Delta}](0,\Omega)}{\theta[u_i''](0,\Omega)} \prod_{i=1}^n \frac{\theta[v_i'](0,\Omega)}{\theta[v_i](0,\Omega)} \frac{\theta[v_i''](0,\Omega)}{\theta[\mathit{\Delta}](0,\Omega)} \prod_{i=1}^n \frac{\theta[u_i'](0,\Omega)}{\theta[v_i'](0,\Omega)} \prod_{i=1}^n \frac{\theta[u_i''](0,\Omega)}{\theta[v_i''](0,\Omega)}. \end{aligned}$$

The first $2n$ factors are of the required form, and the last $2n$ may be put in the required form by lemma 2.6.5.

2.6.7 Lemma. *Let* $\xi_i, \zeta_i \in \frac{1}{2}\mathbb{Z}^{2g}$, *for* $i = 1, \ldots, n$. *If* $\prod_{i=1}^n \frac{\theta[\zeta_i](0,\Omega)}{\theta[\xi_i](0,\Omega)}$ *is* Γ_2*-invariant then* $\sum_{i=1}^n \zeta_i \equiv \sum_{i=1}^n \xi_i \mod \mathbb{Z}^{2g}$.

Proof. From the known action of Γ_2, as given in lemma 1.1.8 in section §1.1, we see that $\phi = \sum_i \zeta_i \otimes \zeta_i - \xi_i \otimes \xi_i$ must send all symmetric $S \in M_{2g}(\mathbb{Z})$ into $\mathbb{Z}$. Use the notation $\zeta_i = [\zeta'_{ij}|\zeta''_{ij}]$ and $\xi_i = [\xi'_{ij}|\xi''_{ij}]$ for $j = 1, \ldots, g$. If we apply all diagonal matrices in $M_{2g}(\mathbb{Z})$ to ϕ we obtain the following equations for each j:

$$\text{(2.6.8)} \qquad \sum_i (\zeta'_{ij})^2 - (\xi'_{ij})^2 \in \mathbb{Z}, \text{ and } \sum_i (\zeta''_{ij})^2 - (\xi''_{ij})^2 \in \mathbb{Z}.$$

From 2.6.8 we have $\sum_i (2\zeta'_{ij})^2 - (2\xi'_{ij})^2 \in 4\mathbb{Z}$ which implies $\sum_i (2\zeta'_{ij})^2 - (2\xi'_{ij})^2 \in 2\mathbb{Z}$. Since $x^2 \equiv x \mod 2\mathbb{Z}$, we conclude that $\sum_i (2\zeta'_{ij}) - (2\xi'_{ij}) \in 2\mathbb{Z}$, or $\sum_i \zeta'_{ij} - \xi'_{ij} \in \mathbb{Z}$. This is true for $j = 1, \ldots, g$ so that $\sum_i \zeta'_i - \xi'_i \in \mathbb{Z}^g$. The demonstration of $\sum_i \zeta''_i - \xi''_i \in \mathbb{Z}^g$ is the same.

Proof of proposition 2.6.2. By proposition 2.6.3 the values,

$$\frac{\theta[\eta_U + \eta_i + \eta_K](0,\Omega)}{\theta[\eta_U + \eta_i + \eta_L](0,\Omega)} \frac{\theta[\eta_U + \eta_j + \eta_L](0,\Omega)}{\theta[\eta_U + \eta_j + \eta_K](0,\Omega)},$$

suffice to determine all expressions,

$$\frac{\theta[\zeta_1](0,\Omega)}{\theta[\zeta_3](0,\Omega)} \frac{\theta[\zeta_2](0,\Omega)}{\theta[\zeta_4](0,\Omega)},$$

when $\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 \in \mathbb{Z}^{2g}$, and the $\zeta_\bullet$ have hyperelliptic η–order zero. By lemma 2.6.6 these elements in turn multiplicatively generate all $\prod_{i=1}^n \frac{\theta[u_i](0,\Omega)}{\theta[v_i](0,\Omega)}$ when $\sum u_i = \sum v_i \mod \mathbb{Z}^{2g}$, and the u_i, v_i, have hyperelliptic η–order zero. By lemma 2.6.7 all the Γ_2-invariant quotients of monomials of hyperelliptic η–order zero are of this form. □

§2.7 Mumford's Theorem.

Mumford's characterization of the hyperelliptic locus in terms of vanishing and nonvanishing is apparently not a direct corollary of the characterization in terms of vanishing and irreducibility given in theorem 2.6.1. Neither can theorem 2.6.1 apparently be deduced directly from Mumford's theorem; this requires showing that irreducibility and vanishing imply nonvanishing, which was accomplished at length in sections §2.1, §2.2 and §2.3. However, Mumford's theorem can be given another proof by imitating the proof of theorem 2.6.1 given here. The two methods of proof appear to be quite different since Mumford proceeds via Neumann's dynamical system. If we assume the nonvanishing conditions as well as the vanishing conditions then the proof that Ω is hyperelliptic given here becomes much more direct. We only sketch the relevant steps.

2.7.1 Theorem (Mumford). [14, 3.137] *Let $\Omega \in \mathcal{H}_g$. There is a hyperelliptic curve of genus g which has Ω as a period matrix if and only if there exists an $\eta \in \Xi_g$ such that*

$$\theta[\zeta](0,\Omega) \neq 0 \iff \zeta \text{ is of hyperelliptic } \eta\text{-order zero.}$$

Sketch of proof. To establish the analogue of theorem 2.1.1 we may use the nonvanishing conditions to directly *define* the p^{ijkl} by

$$(2.7.2\ )\quad p^{ijkl} = \frac{\theta((\Omega\, I)(\eta_U - \eta_V + \eta_i - \eta_k), \Omega)}{\theta((\Omega\, I)(\eta_U - \eta_V + \eta_i - \eta_l), \Omega)} \frac{\theta((\Omega\, I)(\eta_U - \eta_V + \eta_j - \eta_l), \Omega)}{\theta((\Omega\, I)(\eta_U - \eta_V + \eta_j - \eta_k), \Omega)}.$$

It is not clear that this definition is independent of the choice of V though, so we are forced to make some definite choice of V for each 4-tuple (i,j,k,l). From the vanishing $V_{g,\eta}$ we derive the $\mathrm{Frob}_{g,\eta}$ which implies lemma 1.6.12 of section §1.6. Given equation 2.7.2, lemma 1.6.12 is essentially (5) of theorem 2.1.1. The *proof* of corollary 2.1.2 to theorem 2.1.1, especially equation 2.1.9, then shows that the definition of the p^{ijkl} in 2.7.2 is independent of the choice of V. Once this is known the crossratio symmetries of the p^{ijkl} follow directly from the definition 2.7.2. Thus we recover the conclusions of theorem 2.1.1, corollary 2.1.2, and corollary 2.3.2 completely, except for the uniqueness of the p^{ijkl}. Theorem 2.2.1 was used only to imply the nonvanishing in theorem 2.3.1 so we do not need to prove an analogue of this if we assume the nonvanishing property. After the nonvanishing was proven in theorem 2.3.1 neither the irreducibility of Ω nor the uniqueness of the $p^\bullet$ was used again.

3. Chapter Three

§3.1 Crossratio Symmetries.

In these last four sections we perform the invariant theory calculations that were needed at various junctures in sections §2.1–§2.6. Despite their elementary nature the ability to perform these calculations is a crucial ingredient in the proof of the main theorem 2.6.1. It will be convenient to use the following notation: given a field F with units F^* and a multiplicative subset $S \subseteq F^*$, we write $a \equiv b \mod S$ when there is an element $s \in S$ such that $a = bs$. This relation is always transitive, it is reflexive if $1 \in S$ and it is symmetric if $S^{-1} = S$.

3.1.1 Lemma. *Let $n \geq 4$ and let any $c^{ijkl} \in F^*$ be given for distinct $i,j,k,l \in \{1,2,\ldots,n\}$. If the following two types of elements (1) and (2) are members of a multiplicative set $S \subseteq F^*$ then each c^{ijkl} is equal to a quotient of monomials in $\{c^{1rst} : r,s,t \in \{2,3,\ldots,n\}\}$ times an element in S. This element in S depends only upon i,j,k,l and the values of the elements in (1) and (2).*

(1) $c^{ijkl}(c^{jilk})^{-1}$, $c^{ijkl}(c^{klij})^{-1}$, $c^{ijkl}(c^{lkji})^{-1}$
(2) $c^{ijkl}c^{ijlm}(c^{ijkm})^{-1}$

Proof. By condition (2), $c^{ijkl}c^{ijl1}(c^{ijk1})^{-1} \in S$ so $c^{ijkl} \equiv \frac{c^{ijk1}}{c^{ijl1}} \mod S$. By condition (1) $c^{ijk1}/c^{ijl1} \equiv c^{1kji}/c^{1lji} \mod S$ is a quotient of the stated type of monomials when $i, j, k, l \neq 1$. When $i = 1$ there is nothing to be proven. When j, k or $l = 1$ we use condition (1) to respectively obtain:

$$c^{i1kl} \equiv c^{1ilk} \mod S, \qquad c^{ij1l} \equiv c^{1lij} \mod S, \qquad c^{ijk1} \equiv c^{1kji} \mod S.$$

Clearly the elements of S have been drawn only from (1) and (2). This completes the proof of lemma 3.1.1.

3.1.2 Lemma. *Let $n \geq 4$ and let any $c^{ijkl} \in F^*$ be given for distinct $i, j, k, l \in \{1, 2, \ldots, n\}$. If the following four types of elements (1)–(4) are members of a multiplicative set $S \subseteq F^*$ then each c^{ijkl} is equal to a quotient of monomials in $\{c^{1rs2} : r, s \in \{3, \ldots, n\}\}$ times an element in S. This element in S depends only upon i, j, k, l and the values of the elements in (1)–(4).*

(1) $c^{ijkl}(c^{jilk})^{-1}$, $c^{ijkl}(c^{klij})^{-1}$, $c^{ijkl}(c^{lkji})^{-1}$
(2) $c^{ijkl}c^{jikl}$
(3) $c^{ijkl}c^{iljk}c^{iklj}$
(4) $c^{ijkl}c^{ijlm}(c^{ijkm})^{-1}$

Proof. By condition (4), $c^{1jkl}c^{1jl2}(c^{1jk2})^{-1} \in S$ so $c^{1jkl} \equiv c^{1jk2}/c^{1jl2} \mod S$ is a quotient of the required type when $j, k, l \neq 2$. When $l = 2$ there is nothing to be proven. When $j = 2$ we use condition (3), $c^{ijkl}c^{iljk}c^{iklj} \in S$ to obtain: $c^{12kl} \equiv 1/(c^{1l2k}c^{1kl2}) \mod S$. By an application of conditions (1) and (2) we obtain $c^{12kl} \equiv c^{1lk2}/c^{1kl2} \mod S$ as desired. When $k = 2$ we have $c^{1j2l} \equiv 1/c^{1jl2} \mod S$ by applications of (1) and (2) also as desired. Reference to lemma 3.1.1 concludes the proof of lemma 3.1.2.

§3.2 Cycles on $\mathbb{P}^1$.

Proposition 3.2.1 in this section characterizes sets of complex numbers which are the crossratios of n distinct points from $\mathbb{P}^1$.

3.2.1 Proposition. *Let $n \geq 4$ and let $c^{ijkl} \in \mathbb{C}^*$ be given for all distinct $i, j, k, l \in \{1, \ldots, n\}$. There exist n distinct $a_i \in \mathbb{P}^1$ such that $c^{ijkl} = \langle a_i, a_j, a_k, a_l \rangle$ if and only if for all distinct $i, j, k, l, m \in \{1, \ldots, n\}$, we have (1)–(5).*

(1) $c^{ijkl} = c^{jilk} = c^{klij} = c^{lkji}$
(2) $c^{ijkl}c^{jikl} = 1$
(3) $c^{ijkl}c^{iljk}c^{iklj} = -1$
(4) $c^{ijkl}c^{ijlm} = c^{ijkm}$
(5) $c^{ijkl} + c^{ikjl} = 1$

3.2.2 Lemma. *Let F be a field, and let $n \geq 4$, and let any $c^{ijkl} \in F^* \backslash \{1\}$ be given for distinct $i, j, k, l \in \{1, \ldots, n\}$, which satisfy the five equations in proposition 3.2.1. Then each c^{ijkl} is equal to a quotient of monomials in $\{c^{21ij} - 1 : i, j \in \{3, \ldots, n\}\}$ times a sign depending only upon the i, j, k, l and not upon the values $c^\bullet$.*

Proof of lemma. We can apply lemma 3.1.2 to the equations in proposition 3.2.1 with the multiplicative set $S = \{\pm 1\}$ being the same for any $c^{ijkl} \in F^* \backslash \{1\}$. We conclude from this that each c^{ijkl} is equal to a quotient of monomials in the c^{1rs2} up to sign. The proof will therefore be concluded if we show that:

$$c^{1rs2} = \frac{1}{1 - c^{21sr}}. \tag{3.2.3 }$$

From (3) we have $c^{1rs2}c^{12rs}c^{1s2r} = -1$, or using (1) and (2), $c^{1sr2} = -c^{1rs2}c^{21sr}$. This allows us to rewrite (5), $c^{1rs2} + c^{1sr2} = 1$, as $c^{1rs2} - c^{1rs2}c^{21sr} = 1$, which is equivalent to 3.2.3.

3.2.4 Definition. *For $n \geq 2$, let $\mathrm{Coin}_n \subseteq \prod_{i=1}^{n} \mathbb{P}^1$ be defined as*

$$\mathrm{Coin}_n = \{(a_i) \in \prod_{i=1}^{n} \mathbb{P}^1 : \exists\, i \neq j : a_i = a_j\}.$$

Consider the map ϕ from $(\prod_{i=1}^{n} \mathbb{P}^1) \setminus \mathrm{Coin}_n$ to $\prod_{i=1}^{N}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$, where $N = n(n-1)(n-2)(n-3)$, given by $\phi(\{a_i\}) = \langle a_i, a_j, a_k, a_l\rangle$ for distinct $i, j, k, l \in (1, \ldots, n)$. Since a crossratio of distinct points cannot be 0, 1, or ∞, we know that ϕ really maps into $\prod_{i=1}^{N}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$.

3.2.5 Definition. *For $n \geq 4$, let $V_n \subseteq \prod_{i=1}^{N}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$ be defined as*

$$V_n = \{c^{ijkl} \in \prod_{i=1}^{N}(\mathbb{P}^1 \setminus \{0, 1, \infty\}) : c^{\bullet} \textit{satisfy equations } (1)\text{–}(5)$$
$$\textit{in proposition 3.2.1, and } \forall i, j, k, l, m \ \ \textit{we have } c^{ijkl} \neq c^{ijkm}\}.$$

3.2.6 Lemma. *We have* $\mathrm{Im}\,\phi \subseteq V_n$.

Proof. The reader may enjoy the verification that crossratios satisfy the equations of proposition 3.2.1. We verify (2) to ensure that the notation has been understood.

$$c^{ijkl}c^{jikl} = \frac{a_i - a_k}{a_i - a_l}\frac{a_j - a_l}{a_j - a_k}\frac{a_j - a_k}{a_j - a_l}\frac{a_i - a_l}{a_i - a_k} = 1 \tag{(2)}$$

The image of ϕ satisfies $c^{ijkl} \neq 1, 0, \infty$, and condition (5), $c^{ijkl}c^{ijlm} = c^{ijkm}$, so that if $c^{ijkl} = c^{ijkm}$, then $c^{ijlm} = 1$ which is impossible. This verifies lemma 3.2.6.

3.2.7 Definition. *For $n \geq 4$, let $\pi(\{c^{ijkl}\}) = \{c^{123r}\}_{r=4}^{n}$ define a map π by*

$$\pi : (\prod_{i=1}^{N}(\mathbb{P}^1 \setminus \{0, 1, \infty\})) \setminus \{c^{\bullet} : \exists m \neq l : c^{ijkl} = c^{ijkm}\}$$
$$\rightarrow (\prod_{i=1}^{n-3}(\mathbb{P}^1 \setminus \{0, 1, \infty\})) \setminus \mathrm{Coin}_{n-3}\,.$$

3.2.8 Definition. *For $n \geq 4$, let $i(\{x^r\}_{r=4}^n) = (\infty, 0, 1; x^r)$ define a map i by*

$$i : (\prod_{i=1}^{n-3} (\mathbb{P}^1 \setminus \{0, 1, \infty\})) \setminus \mathrm{Coin}_{n-3} \to (\prod_{i=1}^{n} \mathbb{P}^1) \setminus \mathrm{Coin}_n\, .$$

Consider the following diagram which is well–defined by lemma 3.2.6.

$$\begin{array}{ccc} (\prod_{i=1}^n \mathbb{P}^1) \setminus \mathrm{Coin}_n & \xrightarrow{\phi} & (\prod_{i=1}^N (\mathbb{P}^1 \setminus \{0,1,\infty\})) \setminus \{c^\bullet : \exists m \neq l : c^{ijkl} = c^{ijkm}\} \\ \mathrm{id} \downarrow & & \downarrow \pi \\ (\prod_{i=1}^n \mathbb{P}^1) \setminus \mathrm{Coin}_n & \xleftarrow{i} & (\prod_{i=1}^{n-3} (\mathbb{P}^1 \setminus \{0,1,\infty\})) \setminus \mathrm{Coin}_{n-3} \end{array}$$

3.2.9 Lemma. *We have $\pi \circ \phi \circ i = \mathrm{id}$ on $(\prod_{i=1}^{n-3} (\mathbb{P}^1 \setminus \{0,1,\infty\})) \setminus \mathrm{Coin}_{n-3}$.*

Proof. Take $\{x^r\}_{r=4}^n \in (\prod_{i=1}^{n-3} (\mathbb{P}^1 \setminus \{0,1,\infty\})) \setminus \mathrm{Coin}_{n-3}$ so that the x^r are distinct and not equal to 0, 1 or ∞. By the definition of i we have $i(\{x^r\}) = (\infty, 0, 1; x^r) \in (\prod_{i=1}^n \mathbb{P}^1) \setminus \mathrm{Coin}_n$. If we define $x^1 = \infty$, $x^2 = 0$ and $x^3 = 1$, then by the definition of ϕ we have

$$\phi \circ i(\{x^r\}) = \phi(\infty, 0, 1; x^r) = \langle x^i, x^j, x^k, x^l \rangle.$$

Applying π to $\phi \circ i(\{x^r\})$ we obtain

$$\pi \circ \phi \circ i(\{x^r\}) = \pi(\langle x^i, x^j, x^k, x^l \rangle) = \{\langle x^1, x^2, x^3, x^r \rangle\} = \{\langle \infty, 0, 1, x^r \rangle\} = \{x^r\}. \quad \square$$

3.2.10 Lemma. *The map π is injective on V_n.*

Proof. Let $\pi(\{c^{ijkl}\}) = \{c^{123r}\}_{r=4}^n$ where the $c^{ijkl} \in V_n$, and so satisfy the five equations of proposition 3.2.1 as well as being not equal to 0, 1 or ∞. For $i, j \in (4, \ldots, n)$ we have equation (5), $c^{21ij} c^{21j3} = c^{21i3}$ so that the $c^{21ij} = c^{123i}/c^{123j}$ are completely determined by the image of π. If $i = 3$ we have $c^{213j} = c^{12j3} = 1/c^{123j}$ by items (1) and (2). If $j = 3$ we have $c^{21i3} = c^{123i}$ by item (1). By lemma 3.2.2, each c^{ijkl} is a quotient of monomials in the $c^{21ij} - 1$ times a sign depending only on i, j, k, l, so that the c^{ijkl} are completely determined by the c^{21ij} and hence by the image of π. $\square$

3.2.11 Lemma. *The map ϕ is onto V_n.*

Proof. By lemma 3.2.9 we have $\pi \circ \phi \circ i = \mathrm{id}$ on $(\prod_{i=1}^{n-3} (\mathbb{P}^1 \setminus \{0,1,\infty\})) \setminus \mathrm{Coin}_{n-3}$. Since by lemmas 3.2.6 and 3.2.10, $\mathrm{Im}\, \phi \subseteq V_n$ and π is injective on V_n, we have $\phi \circ i \circ \pi = \mathrm{id}$ on V_n; therefore ϕ is onto V_n. $\square$

Proof of proposition 3.2.1. We have already mentioned in lemma 3.2.6 that cross-ratios satisfy the equations of proposition 3.2.1. On the other hand, given any $c^{ijkl} \in \mathbb{C}^*$ satisfying these equations we cannot have $c^{ijkl} = 1$ because this would imply $c^{ikjl} = 0$ via equation (4), $c^{ijkl} + c^{ikjl} = 1$. Furthermore we cannot have

$c^{ijkl} = c^{ijkm}$ for $l \neq m$, else $c^{ijlm} = 1$ via equation (5), $c^{ijkl}c^{ijlm} = c^{ijkm}$. Hence $\{c^{ijkl}\} \in V_n$, and the assertion of proposition 3.2.1 is then that ϕ is onto V_n, which is lemma 3.2.11. $\square$

§3.3 Invariant calculations.

Consider the field $E = \mathbb{Q}(\gamma^\bullet)$ generated by the $N = n(n-1)(n-2)(n-3)$ algebraically independent variables γ^{ijkl} for distinct $i, j, k, l \in \{1, 2, \ldots, n\}$. Define automorphisms P_{ab} of $\mathbb{Q}(\gamma^\bullet)$ as follows.

3.3.1 Definition. *For distinct $a, b \in \{1, \ldots, n\}$, let $P_{ab} \in \mathrm{Aut}(\mathbb{Q}(\gamma^\bullet))$ be the automorphism given by:*

$$P_{ab}\gamma^{ijkl} = \begin{cases} +\gamma^{ijkl}, & \text{if } \{a,b\} \not\subseteq \{i,j,k,l\} \\ +\gamma^{ijkl}, & \text{if } \{a,b\} = \{i,j\} \text{ or } \{k,l\} \\ -\gamma^{ijkl}, & \text{if } \{a,b\} = \{i,k\}, \{i,l\}, \{j,k\} \text{ or } \{j,l\}. \end{cases}$$

3.3.2 Definition. *Let $\mathcal{P} \subseteq \mathrm{Aut}(\mathbb{Q}(\gamma^\bullet))$ be the group generated by the P_{ab} for distinct pairs $a, b \in \{1, \ldots, n\}$, $n \geq 4$.*

$\mathcal{P}$ is an abelian group because ± 1 are fixed by each P_{ab}. Since each P_{ab} has order two we see that the abelian $\mathcal{P}$ is a subgroup of $(\mathbb{Z}/2\mathbb{Z})^M$, where $M = \binom{n}{2}$.

3.3.3 Proposition. *The fixed subfield of $\mathbb{Q}(\gamma^\bullet)$ corresponding to the group $\mathcal{P} \subseteq \mathrm{Aut}(\mathbb{Q}(\gamma^\bullet))$ is generated over $\mathbb{Q}$ by the following elements* (1)–(5).

(1) $(\gamma^{ijkl})^2$
(2) $\gamma^{ijkl}(\gamma^{jilk})^{-1}, \gamma^{ijkl}(\gamma^{klij})^{-1}, \gamma^{ijkl}(\gamma^{lkji})^{-1}$
(3) $\gamma^{ijkl}\gamma^{jikl}$
(4) $\gamma^{ikjl}\gamma^{lijk}(\gamma^{ijkl})^{-1}$
(5) $\gamma^{ijkl}\gamma^{ijlm}(\gamma^{ijkm})^{-1}$

One can certainly check that the elements of $\mathbb{Q}(\gamma^\bullet)$ listed in (1) through (5) are left invariant by each P_{ab} and hence by the group $\mathcal{P}$. We leave this to the reader. We check the item (4), $\gamma^{ikjl}\gamma^{lijk}(\gamma^{ijkl})^{-1}$, to ensure that the notation has been understood. Only if $\{a, b\} \subseteq \{i, j, k, l\}$ will any of these $\gamma^\bullet$ alter. P_{ik} negates the second and third factors. P_{ij} negates the first and second factors. P_{il} negates the first and third factors. The rest are symmetrical. On the other hand we will now show in a series of lemmas that the elements listed in proposition 3.3.3 in fact generate the fixed field of the group $\mathcal{P}$. Denote by K the subfield of $E = \mathbb{Q}(\gamma^\bullet)$ generated over $\mathbb{Q}$ by the elements listed in propositon 3.3.3. Item (2) tells us that

$$\gamma^{ijkl} \in K(\gamma^{jilk}), \gamma^{ijkl} \in K(\gamma^{klij}), \text{ and } \gamma^{ijkl} \in K(\gamma^{lkji}),$$

so that we may apply the fourgroup to the indices of γ^{ijkl} when generating a field over K. Similarly item (3) says that we may apply the permutation (12) via $\gamma^{ijkl} \in K(\gamma^{jikl})$. In other words, if $\gamma^{abcd} \in K$ then all the other $\gamma^\bullet$ obtained by fourgroup or (12) permutations of $abcd$ are also in K. These two facts streamline the proof of proposition 3.3.3 and we will use them without further comment.

3.3.4 Lemma. *We have* $\mathbb{Q}(\gamma^\bullet) = K(\gamma^{1ij2} : i, j \in \{3, 4 \ldots, n\})$.

Proof. This follows from lemma 3.1.2 with K^* as the multiplicative subset. Actually (4) of proposition 3.3.3 is not identical to item (3) of lemma 3.1.2, but rather equivalent using items (2) and (3) of proposition 3.3.3 and more convenient for us here.

3.3.5 Lemma. *For all* $L : 3 \le L \le n$*, the fixed field of* $\{P_{ab} : a, b \in (L, \ldots, n)\}$ *is* $K(\gamma^{1ij2}, \gamma^{21kl} : \{i, j\} \not\subseteq \{L, \ldots, n\}, \{k, l\} \subseteq \{L, \ldots, n\})$*. In particular for* $L = 3$*,*

$$\mathrm{Fix}\{P_{ab} : a, b \in (3, \ldots, n)\} = K(\gamma^{21kl} : \{k, l\} \subseteq \{3 \ldots, n\}).$$

Proof. We will prove this by downward induction on L noting that the conclusion of lemma 3.3.4 is the case $L = n$. We assume the truth of lemma 3.3.5 for some $L + 1$ such that $4 \le L + 1 \le n$, and demonstrate it for L. We have easily:

$$\{P_{ab} : a, b \in (L, \ldots, n)\} = \{P_{ab} : a, b \in (L+1, \ldots, n)\} \cup \{P_{Lm} : m \in (L+1, \ldots, n)\}.$$

To obtain the fixed field of $\{P_{ab} : a, b \in (L, \ldots, n)\}$ we consider the action of each P_{Lm} on the fixed field of $\{P_{ab} : a, b \in (L + 1, \ldots, n)\}$, which by the induction hypothesis is $K(\gamma^{1ij2}, \gamma^{21kl} : \{i, j\} \not\subseteq \{L + 1 \ldots, n\}, \{k, l\} \subseteq \{L + 1 \ldots, n\})$. Among the generators of this field the automorphism P_{Lm} alters the sign of only γ^{1Lm2} and γ^{1mL2}. The elements invariant under P_{Lm} will hence be generated by the remaining unaltered generators and by products with an even number of factors of γ^{1Lm2} and γ^{1mL2}. This is the elementary fact which we use repeatedly in this proof. All combinations of γ^{1Lm2} and γ^{1mL2} which occur in a P_{Lm}-invariant expression are then combinations of $(\gamma^{1Lm2})^2$, $\gamma^{1Lm2}\gamma^{1mL2}$ and $(\gamma^{1mL2})^2$. K already contains these squares and by item (4), $\gamma^{1Lm2}\gamma^{21mL}(\gamma^{1mL2})^{-1} \in K$ so that we have $\gamma^{1Lm2}\gamma^{1mL2} \in K(\gamma^{21mL})$. The exponents of ± 1 on the $\gamma^\bullet$ are only to produce convenient indices and are really irrelevant since the squares of the $\gamma^\bullet$ are in K^*. Using this result for each $m \in \{L + 1, \ldots, n\}$ we see that the fixed field of $\{P_{ab} : a, b \in (L + 1, \ldots, n)\} \cup \{P_{Lm} : m \in (L + 1, \ldots, n)\}$ is generated by

$$K(\gamma^{1ij2}, \gamma^{21kl} : \{i, j\} \not\subseteq \{L, \ldots, n\}, \{k, l\} \subseteq \{L + 1, \ldots, n\}) \\ \cup K(\gamma^{21mL} : m \in \{L + 1, \ldots, n\}).$$

Written more succinctly this is the first assertion of lemma 3.3.5 for L. This concludes the proof by induction of the first assertion of lemma 3.3.5 and the second assertion is simply the case $L = 3$.

3.3.6 Lemma. *For all* $M : 2 \le M < n$*, the fixed field of* $\{P_{ab} : a, b \in (3, \ldots, n)$*, or* $a, b \in (2, \ldots, M)\}$ *is* $K(\gamma^{21kl} : k < l, \{k, l\} \subseteq \{M + 1 \ldots, n\})$.

Proof. Lemma 3.3.6 will be proven by forward induction on M. The case $M = 2$ is the final assertion of lemma 3.3.5 except for the proviso that $k < l$ which is permissible since $\gamma^{21kl} \in K(\gamma^{21lk})$. We assume lemma 3.3.6 for $M-1$ such that $1 \le M-1 <$

n, and demonstrate lemma 3.3.6 for M. The set $\{P_{ab} : a, b \in (3, \ldots, n),\ \text{or}\ a, b \in (2, \ldots, M)\}$ is equal to $\{P_{ab} : a, b \in (3, \ldots, n),\ \text{or}\ a, b \in (2, \ldots, M-1)\} \cup \{P_{2M}\}$.

Consider the action of P_{2M} on the fixed field of $\{P_{ab} : a, b \in (3, \ldots, n),$ or $a, b \in (2, \ldots, M-1)\}$ which is by induction hypothesis equal to $K(\gamma^{21kl} : k < l,\ \{k, l\} \subseteq \{M \ldots, n\})$. The only generators of this fixed field altered by P_{2M} are γ^{21Ml} for $l \in \{M+1, \ldots, n\}$. Since P_{2M} alters the sign of each of these generators they occur in the P_{2M}-invariant expressions in products of two as $\gamma^{21Ml}\gamma^{21Mk}$ for $l, k \in \{M+1, \ldots, n\}$. Some of these products are squares and are thus already in K. For $k \neq l$ we use item (5), $\gamma^{21kl}\gamma^{21lM}(\gamma^{21kM})^{-1} \in K$ so that $\gamma^{21Ml}\gamma^{21Mk} \in K(\gamma^{21kl})$. This shows that the fixed field of $\{P_{ab} : a, b \in (3, \ldots, n),\ \text{or}\ a, b \in (2, \ldots, M-1)\} \cup \{P_{2M}\}$ is generated by $K(\gamma^{21kl} : k < l,\ \{k, l\} \subseteq \{M+1 \ldots, n\})$ $\cup\ K(\gamma^{21kl} : \{k, l\} \subseteq \{M+1 \ldots, n\})$. This is equivalent to the assertion of lemma 3.3.6 for M and that completes the induction proof of lemma 3.3.6.

Proof of proposition 3.3.3. The hard direction of proposition 3.3.3 follows from lemma 3.3.6. When $M = n-1$ in lemma 3.3.6 we see that $K = K(\gamma^{21kl} : k < l,\ \{k, l\} \subseteq \{n\})$ is the fixed field of $\{P_{ab} : a, b \in (3, \ldots, n),\ \text{or}\ a, b \in (2, \ldots, n-1)\}$. Since $K \subseteq \mathrm{Fix}(\mathcal{P})$, this shows that K is the fixed field of $\mathcal{P}$ and completes the proof of proposition 3.3.3. This also incidentally shows that the above type of P_{ab} generate $\mathcal{P}$. □

3.3.7 Proposition (Specialization of proposition 3.3.3). *For $n \geq 4$, let a^{ijkl}, $b^{ijkl} \in \mathbb{C}^*$ be given for all distinct $i, j, k, l \in \{1, \ldots, n\}$. There exists an $S \in \mathcal{P}$ such that $S\gamma^{ijkl}|_{\gamma^{ijkl} \doteq a^{ijkl}} = b^{ijkl}$ if and only if the respective substitutions of the a^{ijkl}, b^{ijkl} into the γ^{ijkl} of the invariant generators of proposition 3.3.3 are equal.*

Proof. If such an S exists and $f(\gamma^\bullet)$ is any invariant of $\mathcal{P}$ then $f(S\gamma^\bullet) = f(\gamma^\bullet)$, and we immediately have our result:

$$f(b^\bullet) = f(S\gamma^\bullet)|_{\gamma^\bullet \doteq a^\bullet} = f(\gamma^\bullet)|_{\gamma^\bullet \doteq a^\bullet} = f(a^\bullet).$$

To pursue the other direction of the equivalence let H be the abelian group of automorphisms of $\mathbb{Q}(\gamma^\bullet)$ given by arbitrarily sending any subset of the $M = n(n-1)(n-2)(n-3)$ generators γ^{ijkl} to their negatives. Clearly we have $H \cong (\mathbb{Z}/2\mathbb{Z})^M$. We recall the elementary fact that the fixed field of H is $\mathbb{Q}((\gamma^\bullet)^2) = \mathbb{Q}((\gamma^\bullet)^2 : i, j, k, l \in \{1, \ldots, n\})$. To see this note that $H \subseteq \mathrm{Aut}(\mathbb{Q}(\gamma^\bullet))$ is a group of order 2^M and so by [1, 42], $[\mathbb{Q}(\gamma^\bullet) : \mathrm{Fix}(H)] = 2^M$. By inspection we see that $\mathbb{Q}((\gamma^\bullet)^2) \subseteq \mathrm{Fix}(H)$, and if we can show $[\mathbb{Q}(\gamma^\bullet) : \mathbb{Q}((\gamma^\bullet)^2)] \leq 2^M$ we will have $\mathbb{Q}((\gamma^\bullet)^2) = \mathrm{Fix}(H)$ as desired. We obtain $\mathbb{Q}(\gamma^\bullet)$ from $\mathbb{Q}((\gamma^\bullet)^2)$ by adjoining any solutions of the M quadratic equations over $\mathbb{Q}((\gamma^\bullet)^2)$, $x^2 - (\gamma^{ijkl})^2 = 0$; this shows that $\mathbb{Q}(\gamma^\bullet)$ is splitting over $\mathbb{Q}((\gamma^\bullet)^2)$ and that $[\mathbb{Q}(\gamma^\bullet) : \mathbb{Q}((\gamma^\bullet)^2)] \leq 2^M$. Hence we have $\mathbb{Q}((\gamma^\bullet)^2) = \mathrm{Fix}(H)$ as promised, and H is the galois group of the normal extension $\mathbb{Q}(\gamma^\bullet)/\mathbb{Q}((\gamma^\bullet)^2)$.

Suppose we are given a^{ijkl}, $b^{ijkl} \in \mathbb{C}^*$ whose substitutions into the generators of proposition 3.3.3 are equal. By item (1) in proposition 3.3.3, $(a^{ijkl})^2 = (b^{ijkl})^2$, and since the a^{ijkl}, b^{ijkl} are nonzero we may uniquely define $e_{ijkl} = \pm 1$ via $e_{ijkl}a^{ijkl} =$

b^{ijkl}. This collection of signs, $e_\bullet$, allows us to define a $\sigma \in H$ by $\sigma(\gamma^{ijkl}) = e_{ijkl}\gamma^{ijkl}$. $\mathcal{P}$ is a subgroup of H and the assertion of this direction of proposition 3.3.7 is that $\sigma \in \mathcal{P}$. By the fundamental galois correspondence [1, 46] it suffices to show that σ fixes the invariant field of $\mathcal{P}$. The invariant field of $\mathcal{P}$ has, by proposition 3.3.3, generators over $\mathbb{Q}$ as listed in items (1)–(5) of proposition 3.3.3, so it suffices to show σ fixes these generators. This is obvious. For example consider the generator of item (3), $\gamma^{ijkl}\gamma^{jikl}$; the hypothesis on a^{ijkl} and b^{ijkl} is that $a^{ijkl}a^{jikl} = b^{ijkl}b^{jikl}$, or that $e_{ijkl}e_{jikl} = 1$. What this means for σ is that

$$e_{ijkl}\gamma^{ijkl}e_{jikl}\gamma^{jikl} = \gamma^{ijkl}\gamma^{jikl}, \text{ or that } \sigma(\gamma^{ijkl}\gamma^{jikl}) = \gamma^{ijkl}\gamma^{jikl},$$

so that σ fixes the generator in item (3). The other generators are similar. □

§3.4 Crossratio symmetries and skew factorizations.

3.4.1 Proposition. *Let (G, c) be an abelian group with a distinguished element, c, of order two. Let $n \geq 4$ be an integer. For distinct $i, j, k, l \in \{1, 2, \ldots, n\}$ let $p^{ijkl} \in G$. Then we have:*

$$\forall \text{ distinct } i, j \in \{1, \ldots, n\},\ \exists\, q^{ij} \in G : q^{ij} = cq^{ji}, \text{ and } \frac{q^{ik}}{q^{il}}\frac{q^{jl}}{q^{jk}} = p^{ijkl},$$

if and only if

(1) $p^{ijkl} = p^{jilk} = p^{klij} = p^{lkji}$
(2) $p^{ijkl}p^{jikl} = 1$
(3) $p^{ijkl}p^{iljk}p^{iklj} = c$
(4) $p^{ijkl}p^{ijlm} = p^{ijkm}$.

Proof. It may amuse the reader to verify that the existence of such q^{ij} does imply the familiar crossratio symmetries (1)–(4) among the p^{ijkl}. We verify item (3) to ensure that the notation has been understood.

$$p^{ijkl}p^{iljk}p^{iklj} = \frac{q^{ik}}{q^{il}}\frac{q^{jl}}{q^{jk}}\frac{q^{ij}}{q^{ik}}\frac{q^{lk}}{q^{lj}}\frac{q^{il}}{q^{ij}}\frac{q^{kj}}{q^{kl}} = \frac{q^{ik}}{q^{ik}}\frac{q^{jl}}{q^{lj}}\frac{q^{ij}}{q^{ij}}\frac{q^{lk}}{q^{kl}}\frac{q^{il}}{q^{il}}\frac{q^{kj}}{q^{jk}} = 1{\cdot}c{\cdot}1{\cdot}c{\cdot}1{\cdot}c = c^3 = c$$

This completes the proof of the "only if" implication and we will use it in proving the converse: that such q^{ij} can be constructed from the p^{ijkl}. Suppose $p^\bullet \in G$ are given which satisfy the crossratio symmetries (1)–(4) . Pick any n elements $\alpha_i \in G$. We first define all and only those q^{ij} with 1 or 2 as an index, and then those with $i, j \geq 3$. For these $p^\bullet \in G$ and $\alpha_1, \ldots, \alpha_n \in G$ define:

$$q^{23} = \alpha_1, \quad q^{32} = c\alpha_1, \quad q^{1i} = \alpha_i, \quad q^{i1} = c\alpha_i, \quad \text{for } i \geq 2$$
$$q^{2i} = \alpha_i\alpha_1(\alpha_3)^{-1}p^{123i}, \quad q^{i2} = c\alpha_i\alpha_1(\alpha_3)^{-1}p^{123i}, \quad \text{for } i \geq 4$$

(3.4.2)

$$q^{ij} = q^{1j}q^{i2}(q^{12}p^{1ij2})^{-1}, \quad \text{for } i, j \geq 3.$$

We first show our result for indices of the type $(123i)$ and $(12i3)$,

(3.4.3) $$\text{for } i \geq 4 \text{ we have} \quad p^{123i} = \frac{q^{13}}{q^{1i}}\frac{q^{2i}}{q^{23}}, \quad p^{12i3} = \frac{q^{1i}}{q^{13}}\frac{q^{23}}{q^{2i}}\,.$$

We have $\frac{q^{13}}{q^{1i}}\frac{q^{2i}}{q^{23}} = \frac{\alpha_3}{\alpha_i}\frac{\alpha_i\alpha_1(\alpha_3)^{-1}p^{123i}}{\alpha_1} = p^{123i}$ by definition of the $q^\bullet$. Since $p^{12i3} = (p^{123i})^{-1}$ by (1) and (2) the second equation in 3.4.3 follows from the first. We now show our result for indices of the type $(12ij)$,

(3.4.4) $$\text{for } i, j \geq 3 \text{ we have} \quad p^{12ij} = \frac{q^{1i}}{q^{1j}}\frac{q^{2j}}{q^{2i}}\,.$$

For i or $j = 3$ this is 3.4.3. For $i, j \geq 4$, definition 3.4.2 gives us the following:

$$\frac{q^{1i}}{q^{1j}}\frac{q^{2j}}{q^{2i}} = \frac{\alpha_i}{\alpha_j}\frac{\alpha_j\alpha_1(\alpha_3)^{-1}p^{123j}}{\alpha_i\alpha_1(\alpha_3)^{-1}p^{123i}} = \frac{p^{123j}}{p^{123i}} = p^{12i3}p^{123j} = p^{12ij}.$$

The third equality above follows from (1) and (2); the fourth equality follows from (4). Proceeding to indices of the type $(1ij2)$ we see that our result in this case,

(3.4.5) $$p^{1ij2} = \frac{q^{1j}}{q^{12}}\frac{q^{i2}}{q^{ij}}, \quad \forall i, j \geq 3\,,$$

is immediate from the definition of q^{ij}. What is not immediate from this definition, however, is that the $q^\bullet$ are skew: $q^{ij} = cq^{ji}$, for all i, j. This skewness is clear from the definition 3.4.2 unless $i, j \geq 3$ where it amounts to the following calculation.

$$\frac{q^{ij}}{q^{ji}} = \frac{q^{1j}q^{i2}(q^{12}p^{1ij2})^{-1}}{q^{1i}q^{j2}(q^{12}p^{1ji2})^{-1}} = \frac{q^{1j}}{q^{1i}}\frac{q^{2i}}{q^{2j}}\frac{p^{1ji2}}{p^{1ij2}} = p^{12ji}p^{1ji2}p^{1i2j} = c$$

The third equality above uses 3.4.4 and (1) and (2); the fourth equality is (3). We will now show that for all indices $(ijkl)$ we have our result.

(3.4.6) $$\text{For all } i, j, k, l \text{ we have} \quad p^{ijkl} = \frac{q^{ik}}{q^{il}}\frac{q^{jl}}{q^{jk}}\,.$$

Define $P^{ijkl} = \frac{q^{ik}}{q^{il}}\frac{q^{jl}}{q^{jk}}$, and note that $P^\bullet$ satisfies the crossratio symmetries by the "only if" part of proposition 3.4.1. Both $p^\bullet$ and $P^\bullet$ satisfy the crossratio symmetries (1)–(4) and so, by lemma 3.1.2 of §3.1 with multiplicative set $S = \{1, c\}$, the p^{ijkl} and P^{ijkl} are the same power of c times the same quotients of products of the p^{1ij2} and P^{1ij2} respectively. However by 3.4.5, $p^{1ij2} = P^{1ij2}$ and so $p^{ijkl} = P^{ijkl}$. Since the $q^\bullet$ are skew, 3.4.6 proves the "if" part of proposition 3.4.1. □

Consider again an abelian group, G, with $c \in G$ an element of order two. Assume that for all distinct $i, j \in \{1, \ldots, N\}$ we are given $q(i, j) \in G$ which are *skew* in the sense that $q(j,i) = c\,q(i,j)$. We need some combinatorial consequences of this situation. First define

$$p(i,j,k,l) = \frac{q(i,k)}{q(i,l)}\frac{q(j,l)}{q(j,k)} \tag{3.4.7}$$

and note that p satisfies the crossratio symmetries (1)–(4) of the previous proposition 3.4.1. Furthermore define

$$P(K;L;k,l) = \prod_{s=1}^{n} p(k_s,l_s,k,l) \tag{3.4.8}$$

for n-tuples $K = (k_1,\dots,k_n)$, $L = (l_1,\dots,l_n)$, where for each s, the k_s, l_s, k, l are distinct. With the obvious difference in notation this P satisfies only the symmetries (2) and (4) completely; P satisfies the first equality in (1), and we could insist that P satisfy the second two equalities in (1) by the fiat of extending the definition. P does not satisfy the symmetry (3). P is actually a function of unordered n-tuples as may be seen by expressing it in terms of q,

$$P(K;L,k,l) = (\prod_{i\in K}\frac{q(i,k)}{q(i,l)})(\prod_{j\in L}\frac{q(j,l)}{q(j,k)}).$$

We will be using some alternating functions, $C(K;L)$, which are functions of oriented n-tuples in the sense that an odd permutation of K or of L multiplies $C(K;L)$ by c. Therefore we introduce some (ad hoc) ordering conventions for the deletion, addition, and replacement of members of n-tuples. $L\setminus l_i = (l_1,\dots,\hat{l}_i,\dots,l_n)$ means the $(n-1)$-tuple obtained by deleting l_i from its coordinate and similarly for $L\setminus(l_i,l_j)$; these deletions are well–defined because the l_i, l_j are always distinct. For the m-tuple $B = (b_1,\dots,b_m)$ we let $B\overset{e}{\hookleftarrow}x$ denote the $(m+1)$-tuple $(b_1,\dots,b_{e-1},x,b_e,\dots,b_m)$ which is the insertion of x in the e^{th} coordinate. $(L+x)\setminus(l_i,l_j)$ will denote the deletion of l_j followed by the replacement of l_i by x.

3.4.9 Lemma. *Let G be an abelian group and $c\in G$ an element of order two. For all distinct $i,j\in\{1,\dots,N\}$, let $q(i,j)\in G$ be given such that $q(j,i) = cq(i,j)$. For all n-tuples $K = (k_1,\dots,k_n)$, $L = (l_1,\dots,l_n)$, where $k_i, l_j \in \{1,\dots,N\}$ are distinct, let $C(K;L)\in G$ be given. Let $M \le \frac{1}{2}N$ be an integer. Then* (1) *holds if and only if* (2) *holds, and either implies* (3).

(1) *For all n such that $1\le n\le M$,*

$$C(K;L) = \frac{\prod_{1\le s<t\le n} q(k_s,k_t)q(l_t,l_s)}{\prod_{1\le s,t\le n} q(k_s,l_t)}.$$

(2) $C((k_1);(l_1)) = \frac{1}{q(k_1,l_1)}$, *and for all n such that $2\le n\le M$,*

$$C(K;L)P(L\setminus l_t;K\setminus k_s;k_s,l_t) = c^{s+t}C(K\setminus k_s;L\setminus l_t)q(k_s,l_t)^{-1}.$$

(3) *For all n such that $2\le n\le M+1$,*

$$C(K\setminus k_s;(L+k_s)\setminus(l_u,l_t))q(l_u,l_t)^{-1} = P(L\setminus l_u;K\setminus k_s;k_s,l_u)C(K\setminus k_s;L\setminus l_t)q(k_s,l_t)^{-1}$$

Proof. We begin by demonstrating that (1) for $(n-1)$-tuples and n-tuples implies (2) for n-tuples. When $n = 1$ in (1), we interpret the empty products as 1 and obtain $C((k_1);(l_1)) = \frac{1\cdot 1}{q(k_1,l_1)}$ as needed. For $1 < n$ we calculate the factors in $C(K;L)$ which are absent from $C(K\setminus k_a; L\setminus l_b)$. We have

$$\begin{aligned}
&\frac{C(K;L)}{C(K\setminus k_a;L\setminus l_b)}\\
=&\frac{\prod_{1\le s<t\le n} q(k_s,k_t)\prod_{1\le s<t\le n} q(l_t,l_s)}{\prod_{1\le s,t\le n} q(k_s,l_t)}\frac{\prod_{\substack{1\le s,t\le n\\ s\ne a,t\ne b}} q(k_s,l_t)}{\prod_{\substack{1\le s<t\le n\\ s,t\ne a}} q(k_s,k_t)\prod_{\substack{1\le s<t\le n\\ s,t\ne b}} q(l_t,l_s)}\\
=&\frac{\prod_{t=a+1}^{n} q(k_a,k_t)\prod_{s=1}^{a-1} q(k_s,k_a)\prod_{s=1}^{b-1} q(l_b,l_s)\prod_{t=b+1}^{n} q(l_t,l_b)}{q(k_a,l_b)^{-1}\prod_{t=1}^{n} q(k_a,l_t)\prod_{s=1}^{n} q(k_s,l_b)}\\
=&\frac{c^{n-a}\prod_{s\ne a} q(k_s,k_a)\cdot c^{b-1}\prod_{t\ne b} q(l_t,l_b)}{q(k_a,l_b)\prod_{s\ne a} q(k_s,l_b)\cdot c^{n-1}\prod_{t\ne b} q(l_t,k_a)}\\
=&c^{n-a+b-1-n+1}P(K\setminus k_a;L\setminus l_b;k_a,l_b)\frac{1}{q(k_a,l_b)}.
\end{aligned}$$

We note that $P(K\setminus k_a;L\setminus l_b;k_a,l_b)^{-1} = P(L\setminus l_b;K\setminus k_a;k_a,l_b)$ and obtain (2):

$$c^{a+b}C(K\setminus k_a;L\setminus l_b)q(k_a,l_b)^{-1} = C(K;L)p(L\setminus l_b;K\setminus k_a;k_a,l_b).$$

Conversely this calculation shows that if $C(K\setminus k_a;L\setminus l_b)$ is given by (1) for $(n-1)$-tuples, and if

$$P(K\setminus k_a;L\setminus l_b;k_a,l_b) = \frac{\prod_{s\ne a} q(k_s,k_a)\prod_{t\ne b} q(l_t,l_b)}{\prod_{s\ne a} q(k_s,l_b)\prod_{t\ne b} q(l_t,k_a)}$$

then $C(K;L)$ is given by (1) for n-tuples. Thus (2) inductively defines (1) as long as we provide the case $n = 1$ which we have done. This shows that (1) is equivalent to (2).

Item (3) is stated in the form in which it is used in §2.2; however, to derive (3) it is helpful to make the following change of notation:

$$A = (a_1,\dots,a_{n-1}) = K\setminus k_s,\ B = (b_1,\dots,b_{n-2}) = L\setminus(l_u,l_t),\ x = k_s, y = l_u, z = l_t.$$

Let e be the index at which l_u would have to be inserted into $L\setminus(l_u,l_t)$ to regain $L\setminus l_t$, so that $L\setminus l_t = B\overset{e}{\hookleftarrow}y$, $(L+k_s)\setminus(l_u,l_t) = B\overset{e}{\hookleftarrow}x$, and $B\overset{e}{\hookleftarrow}z$ is some reordering of $L\setminus l_u$. Then (3) may be rewritten as:

$$C(A;B\overset{e}{\hookleftarrow}x)q(y,z)^{-1} = P(B\overset{e}{\hookleftarrow}z;A;x,y)C(A;B\overset{e}{\hookleftarrow}y)q(x,z)^{-1}. \tag{3.4.10}$$

The fact that $B\overset{e}{\hookleftarrow}z$ is a reordering of $L\setminus l_u$ can be ignored because $B\overset{e}{\hookleftarrow}z$ occurs in P which is a function of unordered n-tuples. To verify equation 3.4.10 we compute

$C(A;B\overset{e}{\hookleftarrow}x)/C(A;B\overset{e}{\hookleftarrow}y)$. This quotient is actually independent of the index e because the function C is alternating; hence we may permute x and y to the first coordinate and compute with $e=1$. We have

$$\frac{C(A;B\overset{1}{\hookleftarrow}x)}{C(A;B\overset{1}{\hookleftarrow}y)} =$$
$$\frac{\prod\limits_{i<j} q(a_i,a_j)\prod_{i=1}^{n-2} q(x,b_i)\prod\limits_{i<j} q(b_i,b_j)}{\prod_{i=1}^{n-1} q(a_i,x)\prod\limits_{i,j} q(a_i,b_j)}\,\frac{\prod_{i=1}^{n-1} q(a_i,y)\prod\limits_{i,j} q(a_i,b_j)}{\prod\limits_{i<j} q(a_i,a_j)\prod_{i=1}^{n-2} q(y,b_i)\prod\limits_{i<j} q(b_i,b_j)}$$
$$=\prod_{i=1}^{n-2}\frac{q(x,b_i)}{q(y,b_i)}\prod_{i=1}^{n-1}\frac{q(a_i,y)}{q(a_i,x)} = (\prod_{i=1}^{n-2}\frac{q(b_i,x)}{q(b_i,y)})\cdot\frac{q(z,x)}{q(z,y)}(\prod_{i=1}^{n-1}\frac{q(a_i,y)}{q(a_i,x)})\cdot\frac{q(y,z)}{q(x,z)}$$
$$=P(B\overset{1}{\hookleftarrow}z;A;x,y)\frac{q(y,z)}{q(x,z)}.$$

This gives 3.4.10 and completes the proof of lemma 3.4.9. □

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Math Department, Fordham University, Bronx NY 10458

poor@murray.fordham.edu