ESTIMATES FOR DIMENSIONS OF SPACES OF SIEGEL MODULAR CUSP FORMS

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Abstract. Every Siegel modular form has a Fourier-Jacobi expansion. This paper provides various sets of Fourier coefficients whose vanishing implies that the associated cusp form is identically zero. We call such sets *estimates* because in the Fourier series case, an upper bound for the dimension of the vector space of cusp forms is provided by the cardinality of the set. Our general estimates have, among others, those estimates of Siegel and Eichler as corollaries. In particular, one new corollary of our general estimates appears to be superior for computational purposes to all other known estimates. To illustrate the use of this corollary, we prove the known result that the theta series of the lattices D_{16}^{+} and $E_8 \oplus E_8$ are the same in degree $n = 3$ by computing just one Fourier coefficient.

§1. Introduction and Notation.

Siegel modular forms are holomorphic functions on the Siegel upper half space $\mathcal{H}_n$ automorphic with respect to the action of the Siegel modular group $\Gamma_n = \mathrm{Sp}_n(\mathbb{Z})$ acting on $\mathcal{H}_n$. We let M_n^k denote the vector space of Siegel modular forms of weight k and let S_n^k denote the cusp forms of weight k, the kernel of Siegel's $\mathbb{C}$-linear Φ_n map, $\Phi_n : M_n^k \to M_{n-1}^k$. These notations are standard; for example see [9, pp. 43,47,54,56]. The vector spaces M_n^k and S_n^k are finite dimensional, and an outstanding problem on this topic is to understand the structure of the graded rings $M_n = \bigoplus_k M_n^k$ and $S_n = \bigoplus_k S_n^k$ by giving their generating functions. This problem is answered only for $n \leq 3$, and reliance has been placed instead on methods which provide upper bounds on $\dim S_n^k$ and allow computation in individual cases. An important result is that every Siegel modular form has a Fourier series, or more generally a Fourier–Jacobi series, and that if "enough" of the Fourier or Fourier–Jacobi coefficients are zero then the modular form is zero itself. In this paper we discuss how many Fourier–Jacobi coefficients are "enough," and provide new methods for bounding $\dim S_n^k$ from above.

In the remainder of this section, we will give an overview of the entire paper, and we also give the definition and notation of various standard objects. In section §2, we prove a general estimate theorem (Theorem 2.4), from which we recover estimates of Siegel and Eichler. In section §3, we prove our main theorem (Theorem 3.4), which is a generalization of the theorem in the previous section; we then obtain as a corollary a new practical estimate theorem. In section §4, we apply this new estimate theorem to show that the theta series for the lattices $E_8 \oplus E_8$ and D_{16}^{+} are the same in degree $n = 3$ by computing just one Fourier coefficient.

1991 *Mathematics Subject Classification.* 11F46 (11F30, 11F27).

Key words and phrases. Siegel modular forms, cusp forms, Fourier-Jacobi expansions.

Theorems of the type we will discuss include two due to Siegel and Eichler. In order to present Siegel's Theorem, recall the following notation. Let $\mathcal{F}_n$ be Siegel's fundamental domain [9, p. 29] for the action of Γ_n on $\mathcal{H}_n$. Let $\kappa_n = \sup_{\Omega \in \mathcal{F}_n} \operatorname{tr}(Y^{-1})$, where we always write $\Omega = X + iY$ for $\Omega \in \mathcal{F}_n$ throughout this paper. Every $f \in S_n^k$ has a Fourier expansion of the form $f(\Omega) = \sum_{s>0} a_s e^{2\pi i \operatorname{tr}(s\Omega)}$ where s runs over all symmetric, positive definite, $n \times n$ matrices that are integer valued on $\mathbb{Z}^n$; we denote this set by $\mathcal{X}_n$.

Theorem (Siegel). *Let $f \in S_n^k$ have Fourier expansion $f(\Omega) = \sum_{s>0} a_s e^{2\pi i \operatorname{tr}(s\Omega)}$. The following two conditions are equivalent.*

(1) $f = 0$.
(2) *For all $s \in \mathcal{X}_n$ such that $\operatorname{tr}(s) \le \kappa_n \frac{k}{4\pi}$, we have $a_s = 0$.*

Actually, the above result holds for all $f \in M_n^k$; the theorem for M_n^k follows easily from the theorem for S_n^k. To compute an upper bound for $\dim S_n^k$ or to show that a particular cusp form is zero using Siegel's Theorem, one lists the *classes* $[s]$ such that a representative $s \in \mathcal{X}_n$ exists satisfying $\operatorname{tr}(s) \le \kappa_n \frac{k}{4\pi}$. The number of such classes $[s]$ is then an upper bound for $\dim S_n^k$ and is the number of Fourier coefficients a_s of f that must be computed. The *class* of s is the set $[s] = \{ {}^t vsv \in \mathcal{X}_n : v \in \mathrm{GL}_n(\mathbb{Z})\}$. It suffices to count classes because the Fourier coefficients a_s satisfy $a_{{}^t vsv} = \det(v)^k a_s$ [9, p.45]; so the vanishing of a_s is a class function on $\mathcal{X}_n$. Notice that the fact that the trace, $\operatorname{tr} : \mathcal{X}_n \to \mathbb{R}^+$, is not a class function can be a nuisance. To discuss Eichler's Theorem we need to introduce Hermite's constant μ_n. Let $\mathcal{P}_n(D)$ be the cone of positive definite, symmetric matrices with entries from D, a subring of $\mathbb{R}$. For $A \in \mathcal{P}_n(\mathbb{R})$ let $m(A) = \min_{c \in \mathbb{Z}^n \setminus \{0\}} {}^t cAc$ [9, p.14]. There is a minimal number $\mu_n \in \mathbb{R}^+$, called Hermite's constant, such that for all $A \in \mathcal{P}_n(\mathbb{R})$ we have $m(A) \le \mu_n \det(A)^{1/n}$. For $1 \le n \le 8$ the value of μ_n is known [11, p.28], as exhibited in the following table.

n	1	2	3	4	5	6	7	8
$(\mu_n)^n$	1	$\frac{4}{3}$	2	4	8	$\frac{64}{3}$	64	256

For all n we have Minkowski's bound, $\mu_n \le \frac{4}{(V_n)^{2/n}}$, where $V_n = \frac{\pi^{n/2}}{(n/2)!}$ is the volume of a sphere of radius one in $\mathbb{R}^n$. See Definition 2.1 for the definition of the Fourier–Jacobi expansion. Eichler's Theorem can also be used to provide upper bounds for $\dim M_n^k$ but it is particularly suited to show $S_n^k = 0$ for small k, namely whenever $\frac{2}{\sqrt{3}} \mu_n^2 \frac{k}{4\pi} < 1$.

Theorem (Eichler). [3, p. 286] *Let $f \in S_n^k$ have the type $(n-1,1)$ Fourier-Jacobi expansion $f(\Omega) = \sum_{s=1}^{\infty} a_s(\pi_1(\Omega), \pi_{12}(\Omega)) e^{2\pi i s\, \pi_2(\Omega)}$. The following conditions are equivalent.*

(1) $f = 0$.
(2) *For all $s \in \mathbb{Z}^+ : s \le \frac{2}{\sqrt{3}} \mu_n^2 \frac{k}{4\pi}$, we have $a_s = 0$.*

Our Theorem 2.4 has both Siegel's and Eichler's Theorems as corollaries, as well as the more general estimates 2.6 and 2.8. The proof of Theorem 2.4 can be thought of as an interpolation between the proof of Siegel's Theroem in [6, p.206] and Eichler's Theorem in [3, p.286]. The approach is also similar to calculations in Freitag [5, pp.48-51]. We leave in "free parameters" that may be chosen according to the application in mind. These "free parameters" take the form of certain maps

$$C : \mathcal{F}_n \to \mathcal{P}^{\mathrm{semi}}(\mathbb{Q}) \setminus \{0\} \quad \text{and} \quad U : \mathcal{F}_n \to \mathrm{GL}_n(\mathbb{Z}),$$

for which we define a transform $\mathcal{T}$ (see Definition 2.3) such that $\mathcal{T}[C,U] : \mathcal{P}^{\text{semi}}_{n_2} \to \mathbb{R}_{\geq 0}$. Our unification of Siegel's and Eichler's results then has the following form.

Theorem (2.4). *Let $f \in S^k_n$ have a type (n_1, n_2) Fourier-Jacobi expansion*

$$f(\Omega) = \sum_{s>0} a_s(\pi_1(\Omega), \pi_{12}(\Omega)) e^{2\pi i \operatorname{tr}[s\,\pi_2(\Omega)]}.$$

Let

$$C : \mathcal{F}_n \to \mathcal{P}^{\text{semi}}(\mathbb{Q}) \setminus \{0\} \quad \text{and} \quad U : \mathcal{F}_n \to \mathrm{GL}_n(\mathbb{Z}),$$

be any two maps such that $\pi_1({}^tUCU) = 0$ and $\pi_{12}({}^tUCU) = 0$. The following two conditions are equivalent.

(1) *$f = 0$.*
(2) *For all $s \in \mathcal{X}_{n_2}$ such that $\mathcal{T}[C,U](s) \leq \frac{k}{4\pi}$, we have we have $a_s = 0$.*

The C and the U functions can be selected with some freedom and each choice gives an estimate of the type we have discussed. The choice $C(\Omega) = I_n$, $U(\Omega) = I_n$ gives Siegel's Theorem. The choice $C(\Omega) = c\,{}^tc$ for some $c \in \mathbb{Z}^n \setminus \{0\}$ such that $m(Y^{-1}) = {}^tcY^{-1}c$ and $U(\Omega)$ such that ${}^tU(\Omega)c$ is of the form $(0, \ldots, 0, *) \in \mathbb{Z}^n \setminus \{0\}$ will give Eichler's Theorem. Other choices of C and U give the more general results 2.6, and 2.8. We do not examine many choices of $C : \mathcal{F}_n \to \mathcal{P}^{\text{semi}}(\mathbb{Q})$, however, because Theorem 2.4 itself has a generalization which allows maps $C : \mathcal{F}_n \to \mathcal{P}^{\text{semi}}(\mathbb{R})$ into real semidefinite matrices instead of rational semidefinite matrices.

The main result of the paper is Theorem 3.4 which gives estimates for fairly general choices of $C : \mathcal{F}_n \to \mathcal{P}^{\text{semi}}(\mathbb{R})$. The main condition on C in Theorem 3.4 is that the $\mathcal{T}$-transform $\mathcal{T}[C] : \mathcal{X}_{n_2} \to \mathbb{R}_{\geq 0}$ has finite spheres. This means that for any $B \in \mathbb{R}^+$ there are only a finite number of classes $[s]$ with $\mathcal{T}[C](s) < B$, see Definition 3.1. Theorem 3.4 gives greater flexibility in choosing a map C suited to an intended application, and represents an essential improvement upon Theorem 2.4.

We have a favorite choice of C in Theorem 3.4 , namely $C(\Omega) = Y$, which seems to give improved numerical upper bounds for $\dim S^k_n$. The details of this choice are spelled out in Corollary 3.8.

Corollary (3.8). *Let $f \in S^k_n$ have a Fourier expansion $f(\Omega) = \sum_{s>0} a_s e^{2\pi i \operatorname{tr}(s\Omega)}$. The following two conditions are equivalent.*

(1) *$f = 0$.*
(2) *For all $s \in \mathcal{X}_n$ such that $\inf_{v \in \mathrm{GL}_n(\mathbb{Z})} \inf_{\Omega \in \mathcal{F}_n} \operatorname{tr}({}^tvsvY) \leq n\frac{k}{4\pi}$, we have $a_s = 0$.*

The superiority of this estimate to known estimates is discussed at the end of section §3. Briefly, we can compare estimates by using the class function det and converting the estimates to the form $\det(s)^{1/n} \leq (\text{const})k$. From a computational point of view, the size of the constant coefficient of k is the entire issue. In this form of Siegel's Theorem the constant that appears is $\frac{2}{\sqrt{3}} c_n \frac{1}{4\pi}$ where c_n is Minkowski's constant [6, p.193]; whereas for Corollary 3.8 the constant that appears is $\frac{2}{\sqrt{3}} \mu_n \frac{1}{4\pi}$ where μ_n is Hermite's constant. Minkowski's constant is bigger than Hermite's constant and this makes a difference as k increases.

As an application of Corollary 3.8 we revisit a problem of Witt. For any Type II [2, p.48] lattice $\Lambda \subseteq \mathbb{R}^N$ we can associate a modular form $\vartheta_\Lambda \in M_n^{N/2}$ for each $n \geq 1$ called the *theta series* of Λ [9, p.48]. We can define ϑ_Λ for $\Omega \in \mathcal{H}_n$ by

$$\vartheta_\Lambda(\Omega) = \sum_{\ell_1,\dots,\ell_g \in \Lambda} \exp\left(i\pi \sum\nolimits_{j,k=1}^{g} \Omega_{jk}({}^t\ell_j\, \ell_k)\right).$$

The theta series ϑ_Λ is an isometry invariant of the lattice Λ and in $\mathbb{R}^{16}$ there are only two isometry classes of Type II lattices, given by $E_8 \oplus E_8$ and D_{16}^+ [2, pp.119, 120].

Theorem (Witt, Kneser, Igusa). *Let $n \in \mathbb{Z}^+$; we have*

$$\vartheta_{E_8 \oplus E_8} = \vartheta_{D_{16}^+} \text{ on } \mathcal{H}_n \iff 1 \leq n \leq 3.$$

In [12], Witt proved the above case $n = 2$ by computing a few Fourier coefficients but was unable to decide the case $n = 3$ due to the "ungeheueren Rechnung." As an application of Corollary 3.10, we give a straightforward proof of the difficult implication ($\Leftarrow$) of the above Theorem by computing just *one* Fourier coefficient, a_s for $s = \begin{bmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}$. We hope that others will use the new estimations in Theorem 2.4, Theorem 3.4, and Corollary 3.8 as a computational aid.

§2. Estimate Theorems.

Definition 2.1. For n_1, n_2 such that $n_1 + n_2 = n$ and $n_2 > 0$, and for any symmetric $n \times n$ matrix $A \in M_{n\times n}^{\mathrm{sym}}$, let π_1, π_2 and π_{12} be projection maps that decompose A into block form:

$$A = \begin{bmatrix} \pi_1(A) & \pi_{12}(A) \\ {}^t\pi_{12}(A) & \pi_2(A) \end{bmatrix}$$

where $\pi_1(A) \in M_{n_1\times n_1}^{\mathrm{sym}}$, $\pi_2(A) \in M_{n_2\times n_2}^{\mathrm{sym}}$ and $\pi_{12}(A) \in M_{n_1\times n_2}$.

The Fourier-Jacobi expansion of type (n_1, n_2) is the Fourier expansion of $f \in M_n^k$ as a function of $\pi_2(\Omega)$,

$$f(\Omega) = \sum_{s\geq 0} a_s(\pi_1(\Omega), \pi_{12}(\Omega)) e^{2\pi i \operatorname{tr}[s\,\pi_2(\Omega)]},$$

where the sum is over integral valued semidefinite $s \in \mathcal{X}_{n_2}$, and where the coefficients

$$a_s(\pi_1(\Omega), \pi_{12}(\Omega)) = \sum_{\substack{s_1, s_2: \\ s' = \left[\begin{smallmatrix} s_1 & s_2 \\ {}^ts_2 & s \end{smallmatrix}\right] \geq 0}} a_{s'} e^{2\pi i \operatorname{tr}[s_1 \pi_1(\Omega) + 2 s_2 \pi_{12}(\Omega)]}$$

are Jacobi forms, and where the $a_{s'}$ are the Fourier coefficients of f.

The following theorem of Siegel is the basis for many estimates of S_n^k. Also, remember that we always write $\Omega = X + iY$; that is, Y always refers to the imaginary part of Ω.

Theorem 2.2. *Let $f \in S_n^k$. Then $\phi(\Omega) = |\det(Y)^{\frac{k}{2}} f(\Omega)|$ attains its maximum at some point $\Omega_0 \in \mathcal{F}_n$.*

Proof. See [6, p.205].

We now define the $\mathcal{T}$-transform, which occurs naturally in the proof of Theorem 2.4.

Definition 2.3. For any two maps

$$C : \mathcal{F}_n \to \mathcal{P}_n^{\mathrm{semi}}(\mathbb{R}) \setminus \{0\} \quad \text{and} \quad U : \mathcal{F}_n \to \mathrm{GL}_n(\mathbb{Z}),$$

and $n_2 \leq n$, we define their transform $\mathcal{T}[C,U] : \mathcal{P}_{n_2}^{\mathrm{semi}}(\mathbb{R}) \to \mathbb{R}_{\geq 0}$ by

$$\mathcal{T}[C,U](s) = \inf_{\Omega \in \mathcal{F}_n} \frac{\mathrm{tr}[s\,\pi_2({}^t U(\Omega) C(\Omega) U(\Omega))]}{\mathrm{tr}[Y^{-1} C(\Omega)]},$$

and we define $\widehat{\mathcal{T}}[C,U] : \mathcal{P}_{n_2}^{\mathrm{semi}}(\mathbb{R}) \to \mathbb{R}_{\geq 0}$ by

$$\widehat{\mathcal{T}}[C,U](s) = \inf_{v \in \mathrm{GL}_n(\mathbb{Z})} \mathcal{T}[C,U]({}^t v s v).$$

In the simple case where $U = I$ we write $\mathcal{T}[C] = \mathcal{T}[C,I]$ and $\widehat{\mathcal{T}}[C] = \widehat{\mathcal{T}}[C,I]$.

Theorem 2.4. *Let $f \in S_n^k$ have a type (n_1, n_2) Fourier-Jacobi expansion*

$$f(\Omega) = \sum_{s>0} a_s(\pi_1(\Omega), \pi_{12}(\Omega)) e^{2\pi i \mathrm{tr}[s\,\pi_2(\Omega)]}.$$

Let

$$C : \mathcal{F}_n \to \mathcal{P}^{\mathrm{semi}}(\mathbb{Q}) \setminus \{0\} \quad \textit{and} \quad U : \mathcal{F}_n \to \mathrm{GL}_n(\mathbb{Z}),$$

be any two maps such that $\pi_1({}^t UCU) = 0$ and $\pi_{12}({}^t UCU) = 0$. The following two conditions are equivalent.

(1) $f = 0$.
(2) *For all $s \in \mathcal{X}_{n_2}$ such that $\mathcal{T}[C,U](s) \leq \frac{k}{4\pi}$, we have we have $a_s = 0$.*

Proof. Fix f, n_1, n_2, C, U. Condition (1) clearly implies condition (2). So assume condition (2); we will show that $f = 0$. We claim that without loss of generality, we may assume C actually maps into $\mathcal{P}_n^{\mathrm{semi}}(\mathbb{Z})$. Define $\tilde{C}(\Omega) = q(\Omega)C(\Omega) \in \mathcal{P}_n^{\mathrm{semi}}(\mathbb{Z}) \setminus \{0\}$, where $q(\Omega) \in \mathbb{Z}^+$ by letting $q(\Omega)$ be the smallest positive integer that clears the denominators of $C(\Omega)$. Since $\mathcal{T}[\tilde{C},U] = \mathcal{T}[C,U]$, condition (2) for C implies condition (2) for $\tilde{C}$. Furthermore, $\pi_1({}^t U\tilde{C}U) = 0$ and $\pi_{12}({}^t U\tilde{C}U) = 0$ are clearly true. Replacing C with $\tilde{C}$, we may indeed without loss of generality assume that $C : \mathcal{F}_n \to \mathcal{P}^{\mathrm{semi}}(\mathbb{Z}) \setminus \{0\}$.

By Theorem 2.2, $\det(Y)^{\frac{k}{2}}|f(\Omega)|$ attains a maximum value M at some $\Omega_0 = X_0 + iY_0 \in \mathcal{F}_n$. Let $V = U(\Omega_0)$. Observe that $|\det(Y)^{\frac{k}{2}} f(\Omega)|$ is invariant under $\Omega \mapsto {}^t V \Omega V$, so that it also attains a maximum at ${}^t V \Omega_0 V$. Let

$$\begin{aligned} \tilde{\Omega}_0 &= {}^t V \Omega_0 V, \\ \tilde{Y}_0 &= {}^t V Y_0 V, \\ T &= {}^t V\, C(\Omega_0)\, V. \end{aligned}$$

Note $\tilde{Y}_0 = \mathrm{Im}(\tilde{\Omega}_0)$. Note also $T \in \mathcal{P}_n^{\mathrm{semi}}(\mathbb{Z}) \setminus \{0\}$. Since $\mathcal{H}_n$ is open, there exists an $\epsilon > 0$ such that $\tilde{\Omega}_0 + \zeta T \in \mathcal{H}_n$ whenever $\mathrm{Im}\zeta \geq -\epsilon$. We define an analytic map

$$\begin{array}{rccc} Q: & \{\zeta \in \mathbb{C} : \mathrm{Im}\zeta \geq -\epsilon\} & \to & \mathcal{H}_n \\ & \zeta & \mapsto & \tilde{\Omega}_0 + \zeta T. \end{array}$$

We will investigate $f(Q(\zeta))$ as a function of ζ, with $\mathrm{Im}\zeta > -\epsilon$. Note that $\pi_1(T) = 0$ and $\pi_{12}(T) = 0$ since $T = {}^tU(\Omega_0)C(\Omega_0)U(\Omega_0)$. So $\pi_1(Q(\zeta)) = \pi_1(\tilde{\Omega}_0)$ and $\pi_{12}(Q(\zeta)) = \pi_{12}(\tilde{\Omega}_0)$ are independent of ζ. Therefore, we have

$$f(Q(\zeta)) = \sum_{s>0} b_s\, e^{2\pi i \mathrm{tr}[s\, \pi_2(\tilde{\Omega}_0 + \zeta T)]},$$

where we write b_s for the constants $a_s(\pi_1(\tilde{\Omega}_0), \pi_{12}(\tilde{\Omega}_0))$.

Note that $f(Q(\zeta))$ is invariant under $\zeta \mapsto \zeta + 1$ because $Q(\zeta + 1) = Q(\zeta) + T$ and T is integral. Let $q = e^{2\pi i \zeta}$. We may define the following function of q,

$$\begin{aligned} F(q) &= F(e^{2\pi i \zeta}) = f(Q(\zeta)) \\ &= \sum_{s>0} b_s\, e^{2\pi i \mathrm{tr}[s\, \pi_2(\tilde{\Omega}_0)]} q^{\mathrm{tr}[s\, \pi_2(T)]}, \end{aligned} \tag{2.5}$$

which is a priori analytic on the punctured disk $0 < |q| < e^{2\pi\epsilon}$. Let $\zeta = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$ with $\beta > 0$. We have

$$\begin{aligned} |F(q)| = |f(Q(\zeta))| &\le \sum_{s>0} |b_s|\, |e^{2\pi i \mathrm{tr}[s\, \pi_2(\tilde{\Omega}_0) + (\alpha + i\beta) s\, \pi_2(T)]}| \\ &\le \sum_{s>0} |b_s|\, |e^{2\pi i \mathrm{tr}[s\, \pi_2(\tilde{\Omega}_0)]}|\, |e^{-2\pi i \beta \mathrm{tr}[s\, \pi_2(T)]}| \\ &\le \sum_{s>0} |b_s|\, |e^{2\pi i \mathrm{tr}[s\, \pi_2(\tilde{\Omega}_0)]}| \end{aligned}$$

where in the last step we used the fact that since $\pi_2(T) \ge 0$, we have $\mathrm{tr}[s\, \pi_2(T)] \ge 0$. Since the Fourier-Jacobi series for $f(\tilde{\Omega}_0)$ converges absolutely, we have that the series $\sum_{s>0} |b_s|\, |e^{2\pi i \mathrm{tr}[s\, \pi_2(\tilde{\Omega}_0)]}|$ converges to some $L \in \mathbb{R}$. So $|f(Q(\zeta))| \le L$ as $\zeta \to i\infty$. Hence $|F(q)|$ is bounded as $q \to 0$, and so $F(q)$ is extendable to an analytic function at $q = 0$. So the power series (2.5) must necessarily be the Maclaurin series of $F(q)$.

Now, suppose by way of contradiction that f is not identically zero. So the order of $F(q)$ at $q = 0$ will be some nonnegative integer m. It is clear from the power series expression (2.5) that

$$m = \min_{s : b_s \ne 0} \mathrm{tr}[s\, \pi_2(T)] \ge \min_{s : a_s \ne 0} \mathrm{tr}[s\, \pi_2(T)].$$

Now, suppose $a_s \ne 0$ for some fixed s. Condition (2) implies that $T[C,U](s) > \frac{k}{4\pi}$. Since $\frac{\mathrm{tr}[s\, \pi_2({}^tU(\Omega_0)C(\Omega_0)U(\Omega_0))]}{\mathrm{tr}[Y_0^{-1}C(\Omega_0)]} \ge \mathcal{T}[C,U](s)$ in particular, we have

$$\frac{\mathrm{tr}[s\, \pi_2(T)]}{\mathrm{tr}[Y_0^{-1}C(\Omega_0)]} > \frac{k}{4\pi}.$$

So $\mathrm{tr}[s\, \pi_2(T)] > \frac{k}{4\pi}\mathrm{tr}[Y_0^{-1}C(\Omega_0)]$. Observe that $\mathrm{tr}(\tilde{Y}_0^{-1}T) = \mathrm{tr}[({}^tVY_0V)^{-1}({}^tVC(\Omega_0)V)] = \mathrm{tr}[V^{-1}Y_0^{-1}C(\Omega_0)V] = \mathrm{tr}[Y_0^{-1}C(\Omega_0)]$. Hence $\mathrm{tr}[s\, \pi_2(T)] > \frac{k}{4\pi}\mathrm{tr}(\tilde{Y}_0^{-1}T)$. Since $\mathrm{tr}[s\, \pi_2(T)] \in \mathbb{Z}$, this implies

$$\mathrm{tr}[s\, \pi_2(T)] \ge [\![\frac{k}{4\pi}\mathrm{tr}(\tilde{Y}_0^{-1}T)]\!] + 1.$$

Therefore, the order of $F(q)$ at $q = 0$ satisfies

$$m \geq [\![\frac{k}{4\pi}\mathrm{tr}(\tilde{Y}_0^{-1}T)]\!] + 1.$$

So defining the function $h(q)$ on $0 < |q| \leq e^{2\pi\epsilon}$ by

$$h(q) = \frac{F(q)}{q^m} = \frac{f(Q(\zeta))}{e^{2\pi i\zeta m}},$$

we obtain that this $h(q)$ extends to an analytic function at $q = 0$ also.

For each $0 < \eta < \epsilon$, we apply the Maximum Modulus Principle to $h(q)$ on the closed disc $|q| \leq e^{2\pi\eta}$ to obtain that $|h(q)|$ achieves a maximum on this closed disc at some boundary point $q_\eta = e^{2\pi i\zeta_\eta}$ with $\mathrm{Im}\zeta_\eta = -\eta$. In particular, since this closed disc contains 1, we have $|h(1)| \leq |h(q_\eta)|$. Then from $Q(0) = \tilde{\Omega}_0$, we obtain

$$\left|\frac{f(\tilde{\Omega}_0)}{1}\right| \leq \left|\frac{f(\tilde{\Omega}_0 + \zeta_\eta T)}{e^{2\pi i\zeta_\eta m}}\right|.$$

Now, continuing with our assumption that f is not identically zero, we have $M > 0$ and so $|f(\tilde{\Omega}_0)| > 0$. We can rewrite this inequality as

$$\left|\frac{1}{e^{2\pi i\zeta_\eta m}}\right| \geq \left|\frac{f(\tilde{\Omega}_0)}{f(\tilde{\Omega}_0 + \zeta_\eta T)}\right|.$$

Using $\left|\frac{1}{e^{2\pi i\zeta_\eta m}}\right| = \left|e^{-2\pi i\zeta_\eta m}\right| = e^{-2\pi\eta m}$, we have

$$e^{-2\pi\eta m} \geq \left|\frac{f(\tilde{\Omega}_0)}{f(\tilde{\Omega}_0 + \zeta_\eta T)}\right| = \frac{M\det(\tilde{Y}_0)^{-\frac{k}{2}}}{|f(\tilde{\Omega}_0 + \zeta_\eta T)|} \geq \frac{M\det(\tilde{Y}_0)^{-\frac{k}{2}}}{M\det(\tilde{Y}_0 - \eta T)^{-\frac{k}{2}}} = \det(I_n - \eta\tilde{Y}_0^{-1}T)^{\frac{k}{2}}.$$

Therefore, we have

$$\begin{aligned} -2\pi\eta m &\geq \ln\det(I_n - \eta\tilde{Y}_0^{-1}T)^{\frac{k}{2}}, \\ -2\pi\eta m &\geq \frac{k}{2}\ln(1 - \eta\mathrm{tr}(\tilde{Y}_0^{-1}T) + O(\eta^2)), \\ -2\pi\eta m &\geq \frac{k}{2}(-\eta\mathrm{tr}(\tilde{Y}_0^{-1}T) + O(\eta^2)). \end{aligned}$$

Therefore, for all η with $0 < \eta < \epsilon$, we have $m \leq \frac{k}{4\pi}\mathrm{tr}(\tilde{Y}_0^{-1}T) + O(\eta)$. Since this is true for all sufficiently small η, we obtain

$$m \leq \frac{k}{4\pi}\mathrm{tr}(\tilde{Y}_0^{-1}T).$$

This contradicts that $m \geq [\![\frac{k}{4\pi}\mathrm{tr}(\tilde{Y}_0^{-1}T)]\!] + 1$, and therefore we must have $f = 0$. $\square$

We now deduce some corollaries through various choices of C and U.

Corollary 2.6. *Let $f \in S_n^k$ have a type (n_1, n_2) Fourier-Jacobi expansion*

$$f(\Omega) = \sum_{s>0} a_s(\pi_1(\Omega), \pi_{12}(\Omega)) e^{2\pi i \operatorname{tr}[s\, \pi_2(\Omega)]}.$$

The following two conditions are equivalent.

(1) *$f = 0$.*

(2) *For all s such that $\operatorname{tr}(s) \le \dfrac{k}{4\pi} \sup_{\mathcal{F}_n} \operatorname{tr}[\pi_2(Y^{-1})]$, we have $a_s = 0$.*

Proof. Choose $C(\Omega) = \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix}$ and $U(\Omega) = I_n$ in Theorem 2.4. So $\mathcal{T}[C, U](s) = \inf_{\mathcal{F}_n} \frac{\operatorname{tr}(s)}{\operatorname{tr}(Y^{-1}C(\Omega))} = \inf_{\mathcal{F}_n} \frac{\operatorname{tr}(s)}{\operatorname{tr}(\pi_2(Y^{-1}))}$. Then condition (2) of Theorem 2.4 is for all s such that $\inf \frac{\operatorname{tr}(s)}{\operatorname{tr}(\pi_2(Y^{-1}))} \le \frac{k}{4\pi}$ we have $a_s = 0$. This is the same as condition (2) above. □

We may recover Siegel's Theorem from from this corollary.

Theorem 2.7 (Siegel). *Let $f \in S_n^k$ have Fourier expansion $f(\Omega) = \sum_{s>0} a_s e^{2\pi i \operatorname{tr}(s\Omega)}$. The following two conditions are equivalent.*

(1) *$f = 0$.*

(2) *For all s such that $\operatorname{tr}(s) \le \kappa_n \frac{k}{4\pi}$, we have $a_s = 0$.*

Proof. Since $\kappa_n = \sup_{\mathcal{F}_n} \operatorname{tr}(Y^{-1})$, this is exactly Corollary 2.6 with $n_1 = 0$, $n_2 = n$. □

Here is another corollary of Theorem 2.4.

Corollary 2.8. *Let $f \in S_n^k$ have a type (n_1, n_2) Fourier-Jacobi expansion*

$$f(\Omega) = \sum_{s>0} a_s(\pi_1(\Omega), \pi_{12}(\Omega)) e^{2\pi i \operatorname{tr}[s\, \pi_2(\Omega)]}.$$

The following two conditions are equivalent.

(1) *$f = 0$.*

(2) *For all s such that $m(s) \le \frac{k}{4\pi} \sup_{\mathcal{F}_n} m(Y^{-1})$, we have $a_s \equiv 0$.*

Proof. We will apply Theorem 2.4 with the following choices of C and U. For each $\Omega \in \mathcal{F}_n$, choose $C(\Omega) = c\, {}^t c$ such that ${}^t c Y^{-1} c = m(Y^{-1})$ and $U(\Omega) = {}^t u$ such that uc has 0 in the first n_1 coordinates. Then ${}^t U(\Omega) C(\Omega) U(\Omega) = (uc)\, {}^t(uc)$ satisfies the hypothesis of Theorem 2.4, since $\pi_1((uc)\, {}^t(uc)) = 0$ and $\pi_{12}((uc)\, {}^t(uc)) = 0$. We have

$$\mathcal{T}[C, U](s) = \inf_{\Omega \in \mathcal{F}_n} \frac{\operatorname{tr}[s\, \pi_2((uc)\, {}^t(uc))]}{\operatorname{tr}(Y^{-1} c\, {}^t c)} = \inf_{\Omega \in \mathcal{F}_n} \frac{\operatorname{tr}[s\, \pi_2((uc)\, {}^t(uc))]}{m(Y^{-1})} \ge \inf_{\mathcal{F}_n} \frac{m(s)}{m(Y^{-1})}.$$

Suppose f satisfies condition (2) above. For s such that $\mathcal{T}[C, U](s) \le \frac{k}{4\pi}$, we have $\inf_{\mathcal{F}_n} \frac{m(s)}{m(Y^{-1})} \le \frac{k}{4\pi}$ as well by the preceding calculation. Hence $m(s) \le \frac{k}{4\pi} \sup_{\mathcal{F}_n} m(Y^{-1})$, and so $a_s = 0$. Thus f satisfies condition (2) of Theorem 2.4 with the above choices of C and U, and so $f = 0$. □

We may recover Eichler's Theorem from this corollary.

Theorem 2.9 (Eichler). *Let $f \in S_n^k$ have the type $(n-1,1)$ Fourier-Jacobi expansion $f(\Omega) = \sum_{s=1}^{\infty} a_s(\pi_1(\Omega), \pi_{12}(\Omega)) e^{2\pi i s\, \pi_2(\Omega)}$. The following conditions are equivalent.*

(1) $f = 0$.
(2) *For $s \le \frac{2}{\sqrt{3}} \mu_n^2 \frac{k}{4\pi}$, we have $a_s = 0$.*

Proof. Let f satisfy condition (2) above. We will show that f satisfies condition (2) of Corollary 2.8 with $n_1 = n-1$ and $n_2 = 1$. Let $s \in \mathbb{Z}^+$ such that $m(s) \le \frac{k}{4\pi} \sup_{\mathcal{F}_n} m(Y^{-1})$; that is, $s \le \frac{k}{4\pi} \sup_{\mathcal{F}_n} m(Y^{-1})$. Since $m(Y^{-1}) \le \mu_n \det(Y^{-1})^{\frac{1}{n}} = \frac{\mu_n}{\det(Y)^{\frac{1}{n}}} \le \frac{\mu_n}{\frac{1}{\mu_n} m(Y)} \le \frac{2}{\sqrt{3}} \mu_n^2$, then $s \le \frac{2}{\sqrt{3}} \mu_n^2 \frac{k}{4\pi}$, and so $a_s = 0$. Here we have used $m(Y) \ge \frac{\sqrt{3}}{2}$ for all $\Omega \in \mathcal{F}_n$ [6, p.195]. So by Corollary 2.8, we have $f = 0$. □

§3. Main Theorem.

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we attempt to extend Theorem 2.4 to maps $C : \mathcal{F}_n \to \mathcal{P}_n^{\mathrm{semi}}(\mathbb{R}) \setminus \{0\}$. However, this extension requires placing some restriction on the map C. For computational purposes, the condition $\mathcal{T}[C,U](s) \le \frac{k}{4\pi}$ is more useful when it is satisfied by only a finite number of classes $[s]$ of integer valued positive definite quadratic forms. With this in mind, we make the following definition.

Definition 3.1. Let $\phi : \mathcal{X}_n \to \mathbb{R}_{\ge 0}$ be a map. For any $B > 0$, we call the set

$$\{[s] : s \in \mathcal{X}_n \text{ and } \phi(s) < B\}$$

a B-sphere of ϕ. We say that ϕ has finite spheres if for all $B > 0$, the B-sphere of ϕ is a finite set.

For a given map ϕ, it is sometimes necessary to consider an associated class function $\hat{\phi}$.

Lemma 3.2. *For a map $\phi : \mathcal{X}_n \to \mathbb{R}_{\ge 0}$, define $\hat{\phi} : \mathcal{X}_n \to \mathbb{R}_{\ge 0}$ by $\hat{\phi}(s) = \inf_{v \in \mathrm{GL}_n(\mathbb{Z})} \phi({}^t v s v)$. Then ϕ has finite spheres if and only if $\hat{\phi}$ has finite spheres.*

Proof. Since $\hat{\phi}(s) \le \phi(s)$, we have that the B-sphere of ϕ is contained in the B-sphere of $\hat{\phi}$. On the other hand, if $\hat{\phi}(s) < B$, then there exists a $v \in \mathrm{GL}_n(\mathbb{Z})$ such that $\phi({}^t v s v) < B$ and so $[s] = [{}^t v s v]$ will be in the B-sphere of ϕ. So the B-spheres are the same for ϕ and $\hat{\phi}$. □

The following is a simple sufficient condition on C and U that implies that $\mathcal{T}[C,U]$ has finite spheres.

Lemma 3.3. *Let $C : \mathcal{F}_n \to \mathcal{P}_n^{\mathrm{semi}}(\mathbb{R})$ and $U : \mathcal{F}_n \to \mathrm{GL}_n(\mathbb{Z})$ be maps. If*

$$\inf_{\Omega \in \mathcal{F}_n} \frac{\det[\pi_2({}^t U(\Omega) C(\Omega) U(\Omega))]^{\frac{1}{n_2}}}{\mathrm{tr}[Y^{-1} C(\Omega)]} > 0,$$

then $\mathcal{T}[C,U]$ has finite spheres.

Proof. Let $s \in \mathcal{X}_{n_2}$ and assume $\mathcal{T}[C,U](s) < B$. Let $\delta = \inf_{\Omega \in \mathcal{F}_n} \frac{\det[\pi_2({}^tU(\Omega)C(\Omega)U(\Omega))]^{\frac{1}{n_2}}}{\operatorname{tr}[Y^{-1}C(\Omega)]}$. The arithmetic-geometric inequality gives us

$$\begin{aligned} B &> \inf_{\Omega \in \mathcal{F}_n} \frac{\operatorname{tr}[s\pi_2({}^tU(\Omega)C(\Omega)U(\Omega))]}{\operatorname{tr}(Y^{-1}C(\Omega))} \\ &\geq \inf_{\mathcal{F}_n} \frac{n_2 \det(s)^{\frac{1}{n_2}} \det[\pi_2({}^tU(\Omega)C(\Omega)U(\Omega))]^{\frac{1}{n_2}}}{\operatorname{tr}[Y^{-1}C(\Omega)]} \\ &\geq n_2 \delta \det(s)^{\frac{1}{n_2}}. \end{aligned}$$

As a consequence, all $s \in \mathcal{X}_{n_2}$ such that $[s]$ is in a B-sphere of $\mathcal{T}[C,U]$ satisfy $\det(s) < \left(\frac{B}{n_2\delta}\right)^{n_2}$. Since the number of classes of integer valued positive definite quadratic forms with determinant less than a fixed bound is finite, we see that $\mathcal{T}[C,U]$ has finite spheres. $\square$

Theorem 3.4. *Let $f \in S_n^k$ have a type (n_1, n_2) Fourier-Jacobi expansion*

$$f(\Omega) = \sum_{s>0} a_s(\pi_1(\Omega), \pi_{12}(\Omega)) e^{2\pi i \operatorname{tr}[s\,\pi_2(\Omega)]}.$$

Let $C : \mathcal{F}_n \to \mathcal{P}^{\text{semi}}(\mathbb{R})$ be map such that $\pi_1(C) = 0$, $\pi_{12}(C) = 0$, $\pi_2(C) > 0$, and $\mathcal{T}[C]$ has finite spheres. The following two conditions are equivalent.

(1) *$f = 0$.*
(2) *For all $s \in \mathcal{X}_{n_2}$ such that $\widehat{\mathcal{T}}[C](s) \leq \frac{k}{4\pi}$, we have we have $a_s = 0$.*

Proof. We make use of Theorem 2.4 for $U(\Omega) = I$ and proceed by rational approximation. Fix C, f, k, n, n_1, n_2. Assume that condition (2) holds; we will prove $f = 0$. For each ρ with $0 < \rho < 1$, we define a map $R_\rho : \mathcal{F}_n \to \mathcal{P}_n^{\text{semi}}(\mathbb{Q})$ as follows. For any $\Omega \in \mathcal{F}_n$, the set $\mathcal{O}_\rho = \{T \in \mathcal{P}_{n_2}^{\text{semi}}(\mathbb{R}) : \rho\pi_2(C(\Omega)) < T < \pi_2(C(\Omega))\}$ is open in $\mathcal{P}_{n_2}(\mathbb{R})$, because it is the intersection in $M_{n_2 \times n_2}(\mathbb{R})$ of three open sets: $\mathcal{P}_{n_2}(\mathbb{R})$, $(\mathcal{P}_{n_2}(\mathbb{R}) + \rho\pi_2(C(\Omega)))$ and $(\pi_2(C(\Omega)) - \mathcal{P}_{n_2}(\mathbb{R}))$. We use the hypothesis that $\pi_2(C(\Omega)) > 0$ to ensure that $\frac{\rho+1}{2}\pi_2(C(\Omega))$ is in $\mathcal{O}_\rho$ and hence that $\mathcal{O}_\rho$ is nonempty. Since $\mathcal{P}_{n_2}(\mathbb{Q})$ is dense in $\mathcal{P}_{n_2}(\mathbb{R})$, we may choose $T \in \mathcal{O}_\rho \cap \mathcal{P}_{n_2}(\mathbb{Q})$ and define $R_\rho(\Omega) = \left[\begin{smallmatrix} 0 & 0 \\ 0 & T \end{smallmatrix}\right]$, thereby obtaining a never zero function R_ρ by the axiom of choice. Note that $\pi_1(R_\rho) = 0$ and $\pi_{12}(R_\rho) = 0$. We also set $R_1 = C$ since $\lim_{\rho \to 1^-} R_\rho(\Omega) = C(\Omega)$. Then for all $\rho \in [\frac{1}{2}, 1]$, we have $\rho C(\Omega) \leq R_\rho(\Omega) \leq C(\Omega)$.

We now show that the maps R_ρ have $\mathcal{T}[R_\rho]$ with uniformly finite spheres for $\rho \in [\frac{1}{2}, 1]$. For each $s \in \mathcal{P}_{n_2}(\mathbb{R})$, we have

$$\begin{aligned} \rho \frac{\operatorname{tr}[s\,\pi_2(C(\Omega))]}{\operatorname{tr}[Y^{-1}C(\Omega)]} &\leq \frac{\operatorname{tr}[s\,\pi_2(R_\rho(\Omega))]}{\operatorname{tr}[Y^{-1}C(\Omega)]} \\ &\leq \frac{\operatorname{tr}[s\,\pi_2(R_\rho(\Omega))]}{\operatorname{tr}[Y^{-1}R_\rho(\Omega)]} \leq \frac{\operatorname{tr}[s\,\pi_2(C(\Omega))]}{\operatorname{tr}[Y^{-1}R_\rho(\Omega)]} \leq \frac{1}{\rho} \frac{\operatorname{tr}[s\,\pi_2(C(\Omega))]}{\operatorname{tr}[Y^{-1}C(\Omega)]}. \end{aligned}$$

Taking infimums over $\Omega \in \mathcal{F}_n$ and then over the equivalence classes of s, we have

$$\rho\mathcal{T}[C](s) \leq \mathcal{T}[R_\rho](s) \leq \frac{1}{\rho}\mathcal{T}[C](s) \text{ and}$$

$$\rho\widehat{\mathcal{T}}[C](s) \leq \widehat{\mathcal{T}}[R_\rho](s) \leq \frac{1}{\rho}\widehat{\mathcal{T}}[C](s) \tag{3.5}$$

Define the set

$$\mathfrak{S}(B) = \bigcup_{\rho \in [\frac{1}{2},1]} \{[s] : \widehat{\mathcal{T}}[R_\rho](s) \leq B\},$$

and use the above inequality 3.5 to deduce from the cases $\rho = 1$ and $\rho = \frac{1}{2}$ the containments

$$\{[s] : \widehat{\mathcal{T}}[C](s) \leq B\} \subseteq \mathfrak{S}(B) \subseteq \{[s] : \widehat{\mathcal{T}}[C](s) \leq 2B\}. \tag{3.6}$$

Use the hypothesis that $\mathcal{T}[C]$ (and hence $\widehat{\mathcal{T}}[C]$) has finite spheres to see that $\mathfrak{S}(B)$ is a finite set, and use inequality 3.5 to choose a $\rho_0 < 1$ close enough to 1 so that

$$\{[s] \in \mathfrak{S}(\frac{k}{4\pi}) : \widehat{\mathcal{T}}[C](s) > \frac{k}{4\pi}\} \subseteq \{[s] \in \mathfrak{S}(\frac{k}{4\pi}) : \widehat{\mathcal{T}}[R_{\rho_0}](s) > \frac{k}{4\pi}\}. \tag{3.7}$$

For example, we may take ρ_0 as $\rho_0 = \frac{1}{2}[1 + \max\{\frac{k}{4\pi}\frac{1}{\mathcal{T}[C](s)}\}]$, where the maximum is over $[s]$ in $\{[s] \in \mathfrak{S}(\frac{k}{4\pi}) : \widehat{\mathcal{T}}[C](s) > \frac{k}{4\pi}\}$. Since $\widehat{\mathcal{T}}[C]$ and $\widehat{\mathcal{T}}[R_{\rho_0}]$ are class functions, when we take complements in equation 3.7 inside $\mathfrak{S}(\frac{k}{4\pi})$, we obtain

$$\{[s] \in \mathfrak{S}(\frac{k}{4\pi}) : \widehat{\mathcal{T}}[C](s) \leq \frac{k}{4\pi}\} \supseteq \{[s] \in \mathfrak{S}(\frac{k}{4\pi}) : \widehat{\mathcal{T}}[R_{\rho_0}](s) \leq \frac{k}{4\pi}\}.$$

Using 3.6, we also have $\{[s] : \widehat{\mathcal{T}}[C](s) \leq \frac{k}{4\pi}\} = \{[s] \in \mathfrak{S}(\frac{k}{4\pi}) : \widehat{\mathcal{T}}[C](s) \leq \frac{k}{4\pi}\}$. From the definition of $\mathfrak{S}(\frac{k}{4\pi})$, we have $\{[s] : \widehat{\mathcal{T}}[R_{\rho_0}](s) \leq \frac{k}{4\pi}\} = \{[s] \in \mathfrak{S}(\frac{k}{4\pi}) : \widehat{\mathcal{T}}[R_{\rho_0}](s) \leq \frac{k}{4\pi}\}$. Since $\widehat{\mathcal{T}}[R_{\rho_0}] = \inf_v \mathcal{T}[R_{\rho_0}]({}^t v s v)$, we have the containment $\{[s] : \mathcal{T}[R_{\rho_0}](s) \leq \frac{k}{4\pi}\} \subseteq \{[s] : \widehat{\mathcal{T}}[R_{\rho_0}](s) \leq \frac{k}{4\pi}\}$. Combining the equalities and containments, we have

$$\{[s] : \mathcal{T}[R_{\rho_0}](s) \leq \frac{k}{4\pi}\} \subseteq \{[s] : \widehat{\mathcal{T}}[C](s) \leq \frac{k}{4\pi}\}.$$

Since C satisfies condition (2) of this theorem, we see that $R_{\rho_0} : \mathcal{F}_n \to \mathcal{P}_n^{\mathrm{semi}}(\mathbb{Q}) \setminus \{0\}$ satisfies condition (2) of Theorem 2.4, and hence we have $f = 0$. $\square$

We have a favorite choice of C in the above theorem.

Corollary 3.8. *Let $f \in S_n^k$ have a Fourier expansion $f(\Omega) = \sum_{s>0} a_s e^{2\pi i \mathrm{tr}(s\Omega)}$. The following two conditions are equivalent.*

(1) *$f = 0$.*

(2) *For all $s \in \mathcal{X}_n$ such that $\inf_{v \in \mathrm{GL}_n(\mathbb{Z})} \inf_{\Omega \in \mathcal{F}_n} \mathrm{tr}({}^t v s v Y) \leq n\dfrac{k}{4\pi}$, we have $a_s = 0$.*

Proof. We are in the Fourier-Jacobi case of type $(0, n)$, so $n_1 = 0$ and $n_2 = n$. We show that $C : \mathcal{F}_n \to \mathcal{P}_n(\mathbb{R})$ defined by $C(\Omega) = Y$ satisfies the hypotheses of Theorem 3.4. We clearly have $\pi_1(C) = 0$, $\pi_{12}(C) = 0$ and $\pi_2(C) > 0$. To show that $\mathcal{T}[C]$ has finite spheres, we check the condition of Lemma 3.3. We have

$$\inf_{\Omega \in \mathcal{F}_n} \frac{\det(Y)^{\frac{1}{n}}}{\mathrm{tr}(Y^{-1}Y)} = \frac{1}{n} \inf_{\mathcal{F}_n} \det(Y)^{\frac{1}{n}} \geq \frac{1}{n} \inf_{\mathcal{F}_n} \frac{m(Y)}{\mu_n} \geq \frac{\frac{\sqrt{3}}{2}}{n\mu_n} > 0$$

since $\inf_{\Omega\in\mathcal{F}_n} m(Y) \geq \frac{\sqrt{3}}{2}$ [6, p.195]. So C indeed has finite spheres by Lemma 3.3. Since

$$\widehat{\mathcal{T}}[C](s) = \inf_{v\in\mathrm{GL}_n(\mathbb{Z})} \inf_{\Omega\in\mathcal{F}_n} \frac{\mathrm{tr}({}^t vsvY)}{\mathrm{tr}(Y^{-1}Y)}$$
$$= \frac{1}{n} \inf_{v\in\mathrm{GL}_n(\mathbb{Z})} \inf_{\Omega\in\mathcal{F}_n} \mathrm{tr}({}^t vsvY),$$

we have that condition (2) above is exactly the condition that $a_s = 0$ when $\widehat{\mathcal{T}}[C](s) \leq \frac{k}{4\pi}$. So the corollary follows from Theorem 3.4 applied to this C. $\square$

The Theorem of Siegel is actually true for $f \in M_n^k$ if semidefinite s are included in the estimate. We make a similar extension of Corollary 3.8.

Corollary 3.9. *Let $f \in M_n^k$ have a Fourier expansion $f(\Omega) = \sum_{s\geq 0} a_s e^{2\pi i \mathrm{tr}(s\Omega)}$. The following two conditions are equivalent.*

(1) *$f = 0$.*

(2) *For all $s \in \mathcal{X}_n^{\mathrm{semi}}$ such that $\inf_{v\in\mathrm{GL}_n(\mathbb{Z})} \inf_{\Omega\in\mathcal{F}_n} \mathrm{tr}({}^t vsvY) \leq n\frac{k}{4\pi}$, we have $a_s = 0$.*

Proof. Again, we show that condition (2) implies condition (1). In light of Corollary 3.8, we only need to show that an f satisfying the condition (2) here is necessarily a cusp form. We proceed to prove the corollary by induction on n. The conclusion is valid for $n = 1$ because $s = 0$ always satisfies the inequality in condition (2) and $a_0 = 0$ alone implies f is a cusp form.

For the induction step, let $n > 1$, and let f satisfy the hypothesis. The Fourier series for $f' = \Phi_n f$ is given by

$$f'(\Omega') = \sum_{s'\geq 0} a_{s'} e^{2\pi i \mathrm{tr}(s'\Omega')},$$

where $a_{s'} = a_s$ for $s = \left[\begin{smallmatrix} s' & 0 \\ 0 & 0 \end{smallmatrix}\right]$ (see [6, p203]). Consider any $s' \in \mathcal{X}_{n-1}^{\mathrm{semi}}$ satisfying $\inf_{v'\in\mathrm{GL}_{n-1}(\mathbb{Z})} \inf_{\Omega'\in\mathcal{F}_{n-1}} \mathrm{tr}({}^t v's'v'Y') \leq \frac{(n-1)k}{4\pi}$; we will show $a_{s'} = 0$. For any $\Omega' \in \mathcal{F}_{n-1}$, there exists a $\lambda \in \mathbb{R}^+$, sufficiently large, such that

$$iY = \begin{bmatrix} iY' & 0 \\ 0 & i\lambda \end{bmatrix} \in \mathcal{F}_n$$

(see [6, p.196]). Also, for any $v' \in \mathrm{GL}_{n-1}(\mathbb{Z})$, we have $\left[\begin{smallmatrix} v' & 0 \\ 0 & 1 \end{smallmatrix}\right] \in \mathrm{GL}_n(\mathbb{Z})$. Therefore, we have

$$\inf_{\Omega\in\mathcal{F}_n} \mathrm{tr}({}^t vsvY) \leq \inf_{\Omega'\in\mathcal{F}_{n-1}} \mathrm{tr}({}^t vsv\left[\begin{smallmatrix} Y' & 0 \\ 0 & \lambda \end{smallmatrix}\right])$$

and

$$\inf_{v\in\mathrm{GL}_n(\mathbb{Z})} \inf_{\Omega\in\mathcal{F}_n} \mathrm{tr}({}^t vsvY) \leq \inf_{v'\in\mathrm{GL}_{n-1}(\mathbb{Z})} \inf_{\Omega'\in\mathcal{F}_{n-1}} \mathrm{tr}(\left[\begin{smallmatrix} {}^t v' & 0 \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} s' & 0 \\ 0 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} v' & 0 \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} Y' & 0 \\ 0 & \lambda \end{smallmatrix}\right])$$
$$= \inf_{v'\in\mathrm{GL}_{n-1}(\mathbb{Z})} \inf_{\Omega'\in\mathcal{F}_{n-1}} \mathrm{tr}({}^t v's'vY') \leq \frac{(n-1)k}{4\pi} < n\frac{k}{4\pi}.$$

By the hypothesis of this corollary, we have $a_s = 0$, hence $a_{s'} = 0$. Since $a_{s'} = 0$ for all $s' \in \mathcal{X}_{n-1}^{\mathrm{semi}}$ satisfying $\inf_{v'\in\mathrm{GL}_{n-1}(\mathbb{Z})} \inf_{\Omega'\in\mathcal{F}_{n-1}} \mathrm{tr}({}^t v's'v'Y') \leq \frac{(n-1)k}{4\pi}$, we have $f' = 0$ by the induction hypothesis. This means that f is a cusp form, and so $f = 0$ by Corollary 3.8, completing the induction. $\square$

Conditions on determinants are more serviceable in conjunction with tables of quadratic forms so we state a slight reformulation of Corollary 3.8.

Corollary 3.10. *Let $f \in S_n^k$ have a Fourier expansion $f(\Omega) = \sum_{s>0} a_s e^{2\pi i \mathrm{tr}(s\Omega)}$. Suppose that $a_s = 0$ for all s satisfying both*

(i) $\det(s)^{\frac{1}{n}} \le \mu_n \frac{2}{\sqrt{3}} \frac{k}{4\pi}$, *and*

(ii) $\inf_{v \in \mathrm{GL}_n(\mathbb{Z})} \inf_{\Omega \in \mathcal{F}_n} \mathrm{tr}({}^t v s v Y) \le n \dfrac{k}{4\pi}$.

Then we have $f = 0$.

Furthermore, item (ii) *above implies item* (i).

Proof. We first show that item (ii) implies item (i) above. We have

$$\mathrm{tr}({}^t v s v Y) \ge n \det({}^t v s v)^{\frac{1}{n}} \det(Y)^{\frac{1}{n}} = n \det(s)^{\frac{1}{n}} \det(Y)^{\frac{1}{n}}.$$

Therefore, we have

$$\begin{aligned}\inf_{v \in \mathrm{GL}_n(\mathbb{Z})} \inf_{\Omega \in \mathcal{F}_n} \mathrm{tr}({}^t v s v Y) &\ge n \det(s)^{\frac{1}{n}} \inf_{\Omega \in \mathcal{F}_n} \det(Y)^{\frac{1}{n}} \\ &\ge n \det(s)^{\frac{1}{n}} \inf_{\Omega \in \mathcal{F}_n} \frac{m(Y)}{\mu_n} \ge \frac{n}{\mu_n} \frac{\sqrt{3}}{2} \det(s)^{\frac{1}{n}}.\end{aligned}$$

The main statement then follows from Corollary 3.8. $\square$

In concluding this section, we compare this new estimate with Siegel's estimate. Tables of quadratic forms are usually ordered by the value of the determinant. If Corollary 3.10 and Siegel's Theorem are converted to estimates on the determinant, then the bounds given by Siegel's Theorem are never better than the bounds given by Corollary 3.10. In order to convert Siegel's Theorem to a condition on the determinant, we calculate

$$\kappa_n \frac{k}{4\pi} \ge \mathrm{tr}(s) \ge n \det(s)^{\frac{1}{n}}, \tag{3.11}$$

whereas the conversion of Corollary 3.10 is

$$n \frac{k}{4\pi} \ge \inf_v \inf_\Omega \mathrm{tr}({}^t v s v Y) \ge n \det(s)^{\frac{1}{n}} \left(\inf_\Omega \det(Y)^{\frac{1}{n}} \right). \tag{3.12}$$

Accurate values of κ_n are not known for $n \ge 2$ and some upper bound must be used in practice. In any case, if a certain lower bound for κ_n is used in (3.11), the condition is identical to that in (3.12), so that quadratic form classes $[s]$ that satisfy (3.11) contain those that satisfy (3.12) regardless of the true value of κ_n. This lower bound for κ_n is given by

$$\kappa_n = \sup_{\Omega \in \mathcal{F}_n} \mathrm{tr}(Y^{-1}) \ge \sup_\Omega n \det(Y^{-1})^{\frac{1}{n}} = \frac{n}{\inf_\Omega \det(Y)^{\frac{1}{n}}}.$$

This show that, as determinant conditions, the estimate of Corollary 3.10 is at least as good as Siegel's Theorem.

In practice, moreover, we must use an upper bound for κ_n in (3.11). The best estimate known to us [6, p.197] is

$$\kappa_n \leq \frac{2n}{\sqrt{3}} c_n \leq \frac{2}{\sqrt{3}} n \mu_n^n,$$

where we have used the best known estimate $c_n \leq \mu_n^n$ on Minkowski's constant c_n [1]. In terms of a determinant condition, (3.11) gives

$$\frac{2}{\sqrt{3}} \mu_n^n \frac{k}{4\pi} \geq \det(s)^{\frac{1}{n}}. \tag{3.13}$$

In order to compare 3.13 with 3.12, we must estimate $\inf_\Omega \det(Y)^{\frac{1}{n}}$. Since accurate values of this constant are also unknown for $n \geq 2$, we must use the lower bound $\inf_\Omega \det(Y)^{\frac{1}{n}} \geq \inf_\Omega \frac{m(Y)}{\mu_n} = \frac{\sqrt{3}}{2} \frac{1}{\mu_n}$ used for item (i) of Corollary 3.10 and obtain said item,

$$\frac{2}{\sqrt{3}} \mu_n \frac{k}{4\pi} \geq \det(s)^{\frac{1}{n}}. \tag{3.14}$$

In conclusion, the difference between applying Siegel's Theorem via 3.13 and Corollary 3.10 via 3.14 is exactly the appearance of the smaller Hermite constant μ_n in 3.14 versus the larger Minkowski constant $c_n \leq \mu_n^n$ in 3.13. As mentioned in the introduction, the size of the coefficient of k in these estimates is the entire issue for computational purposes. In the last section, we give an example to illustrate the superiority of the new estimate given by Corollary 3.10.

§4. An Example.

The calculation of linear relations among theta series attached to Type II lattices was begun in [12] by Witt. From general estimates and the computation of three Fourier coefficients, Witt showed that the theta series of $E_8 \oplus E_8$ and D_{16}^+ agreed for degree $n = 2$. Witt also conjectured that these theta series also agreed for $n = 3$, but was unable to decide the problem due to the "monstrous calculations." Igusa [7] and Kneser [10] settled Witt's conjecture affirmatively using geometric and lattice-theoretic techniques, respectively. We will illustrate the computational advantage of Corollary 3.10 by giving a straightforward proof of $\vartheta_{E_8 \oplus E_8} = \vartheta_{D_{16}^+}$ in $n = 3$ that involves the computation of just one Fourier coefficient for $E_8 \oplus E_8$ and D_{16}^+. We begin with $n = 1$ to illustrate the use of Corollary 3.10.

For $n = 1$ and $k = 8$, Corollary 3.10 item (i) says that a cusp form in S_1^8 is zero if $a_s = 0$ for all $s \in \mathcal{X}_1 = \mathbb{Z}^+$ with $\det(s) \leq \mu_1 \frac{2}{\sqrt{3}} \frac{8}{4\pi}$. Using the value $\mu_1 = 1$, this inequality is

$$s \leq \frac{4}{\sqrt{3}\pi} \approx .735105.$$

Since there are no such s, we have $S_1^8 = 0$. In particular, $\vartheta_{E_8 \oplus E_8} - \vartheta_{D_{16}^+} = 0$ in $n = 1$.

Applying the Siegel map Φ_2, we see that $\vartheta_{E_8 \oplus E_8} - \vartheta_{D_{16}^+}$ is a cusp form in $n = 2$. Setting $n = 2$ and $k = 8$, Corollary 3.10 item (i) says that a cusp form is zero if its Fourier coefficients $a_s = 0$ for all $s \in \mathcal{X}_2$ with $\det(s)^{\frac{1}{2}} \leq \mu_2 \frac{2}{\sqrt{3}} \frac{8}{4\pi}$. Using the value $\mu_2 = \frac{2}{\sqrt{3}}$, this inequality is

$$\det(s) \leq \frac{64}{9\pi^2} \approx .720506.$$

The smallest possible determinant for $s \in \mathcal{X}_2$ is $\det \left[\begin{smallmatrix} 1 & 1/2 \\ 1/2 & 1 \end{smallmatrix}\right] = \frac{3}{4}$. So we have $S_2^8 = 0$, and in particular, $\vartheta_{E_8 \oplus E_8} - \vartheta_{D_{16}^+} = 0$ for $n = 2$ as well.

Applying the Φ_3 map we have $\vartheta_{E_8 \oplus E_8} - \vartheta_{D_{16}^+} \in S_3^8$. Setting $n = 3$ and $k = 8$, item (i) of Corollary 3.10 is that $a_s = 0$ for all $s \in \mathcal{X}_3$ with $\det(s)^{\frac{1}{3}} \leq \mu_3 \frac{2}{\sqrt{3}} \frac{8}{4\pi}$. Using the value $\mu_3 = \sqrt[3]{2}$, this inequality simplifies to

$$\det(s) \leq \frac{128}{\pi^3 3\sqrt{3}} \approx 0.794472.$$

Examination of Intrau's tables in [8] shows that there are only two classes of $[s] \in [\mathcal{X}_3]$ that satisfy the above inequality. Representatives of these two classes are:

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which have determinants .5 and .75, respectively. These consequently give the only classes that could satisfy the inequality in item (ii) of Corollary 3.10:

$$\inf_{v \in \mathrm{GL}_3(\mathbb{Z})} \inf_{\Omega \in \mathcal{F}_3} \operatorname{tr}({}^t v s v Y) \leq 3\frac{k}{4\pi} = \frac{6}{\pi} \approx 1.90986.$$

We can actually show that no matrix equivalent to B can satisfy this equality, thereby leaving only the Fourier coefficients corresponding to A for computation.

We write B as follows:

$$\begin{aligned} B = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} &= \tfrac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \tfrac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \tfrac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \sum_{i=1}^{4} \alpha_i \, p_i {}^t p_i, \end{aligned}$$

where $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{2}$ and $\alpha_4 = 1$, and $p_1 = \left[\begin{smallmatrix} 1 \\ 1 \\ 0 \end{smallmatrix}\right]$, $p_2 = \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right]$, $p_3 = \left[\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right]$ and $p_4 = \left[\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right]$. For any $\Omega = X + iY \in \mathcal{F}_3$ and $v \in \mathrm{GL}_3(\mathbb{Z})$, we have

$$\begin{aligned} \operatorname{tr}({}^t v B v Y) &= \sum_{i=1}^{4} \alpha_i \operatorname{tr}(({}^t v p_i)\, {}^t({}^t v p_i)\, Y) = \sum_{i=1}^{4} \alpha_i \operatorname{tr}({}^t({}^t v p_i)\, Y \,({}^t v p_i)) \geq \sum_{i=1}^{4} \alpha_i \, m(Y) \\ &\geq \sum_{i=1}^{4} \alpha_i \frac{\sqrt{3}}{2} = 2.5\frac{\sqrt{3}}{2} \approx 2.16506. \end{aligned}$$

Therefore $\inf_v \inf_\Omega \operatorname{tr}({}^t vBvY) \geq \frac{2.5\sqrt{3}}{2} > \frac{6}{\pi}$, and we can remove the class of B from consideration.

Consequently, $\vartheta_{E_8 \oplus E_8} = \vartheta_{D_{16}^+}$ in $n = 3$ if and only if their Fourier coefficients a_A are the same. Recall that to any integral lattice $\Lambda \subseteq \mathbb{R}^n$ we may define the *theta series* of Λ, $\vartheta_\Lambda : \mathcal{H}_n \to \mathbb{C}$ as follows: for $\Omega \in \mathcal{H}_n$ let

$$\vartheta_\Lambda(\Omega) = \sum_{\ell_1, \ldots, \ell_n \in \Lambda} \exp\left(i\pi \sum\nolimits_{j,k=1}^{n} \Omega_{jk}({}^t\ell_j\, \ell_k)\right).$$

So the coefficient a_s in the Fourier expansion of ϑ_Λ is the number of ways we can choose vectors $\ell_1, \ldots, \ell_n$ so that the dot products are exactly ${}^t\ell_i\, \ell_j = 2s_{ij}$. For the lattice $E_8 \oplus E_8$, there are 480 ways to choose ℓ_1 of length 2. For each of these there are 56 ways to choose ℓ_2 so that ${}^t\ell_1\, \ell_2 = 1$. For each choice of ℓ_1 and ℓ_2, noting that all such choices are equivalent by an isometry of $E_8 \oplus E_8$, there are 27 ways to choose ℓ_3 so that ${}^t\ell_3\, \ell_1 = 1$ and ${}^t\ell_3\, \ell_2 = 0$. Thus the coefficient a_A is $480 \cdot 56 \cdot 27 = 725760$. For the lattice D_{16}^+, a similar counting argument shows that the coefficient a_A is also $480 \cdot 56 \cdot 27 = 725760$. So in degree $n = 3$, $\vartheta_{E_8 \oplus E_8} - \vartheta_{D_{16}^+} = 0$. Thus we have shown that the theta series for $E_8 \oplus E_8$ and D_{16}^+ are the same in degrees $n \leq 3$. Even in this low degree, the use of Siegel's Theorem would require the computation of eight Fourier coefficients for each lattice.

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Accepted by Abhandlungen aus dem Mathematischen Seminar der Universitaet Hamburg, February 1996.

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