Online Appendix for *Optimal public debt indexation in advanced economies*

Patricia Gomez-Gonzalez *
## A Data sources, coverage, and additional IL debt statistics

<table>
<thead>
<tr>
<th>Country</th>
<th>Source</th>
<th>Coverage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Australia (AUS)</td>
<td>BIS Table C2</td>
<td>1995-2018</td>
</tr>
<tr>
<td>Canada (CAN)</td>
<td>BIS Table C2</td>
<td>1995-2018</td>
</tr>
<tr>
<td>France (FRA)</td>
<td>AFT - Agence France Trésor</td>
<td>1999-2018</td>
</tr>
<tr>
<td>Germany (DEU)</td>
<td>BIS Table C2</td>
<td>2006-2018</td>
</tr>
<tr>
<td>Iceland (ISL)</td>
<td>Bank of Iceland</td>
<td>1995-2018</td>
</tr>
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<td>Israel (ISR)</td>
<td>BIS Table C2</td>
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<tr>
<td>Italy (ITA)</td>
<td>Ministero dell’Economia e delle Finanze</td>
<td>2003-2018</td>
</tr>
<tr>
<td>Japan (JPN)</td>
<td>Japanese Ministry of Finance</td>
<td>2013-2018</td>
</tr>
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<td>Korea (KOR)</td>
<td>BIS Table C2</td>
<td>2007-2018</td>
</tr>
<tr>
<td>New Zealand (NZL)</td>
<td>Office of Debt Management</td>
<td>2000-2018</td>
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<td>Spain (ESP)</td>
<td>BIS Table C2</td>
<td>2014-2018</td>
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<tr>
<td>Sweden (SWE)</td>
<td>Swedish National Debt Office</td>
<td>1995-2018</td>
</tr>
<tr>
<td>United Kingdom (GBR)</td>
<td>BIS Table C2</td>
<td>2004-2018</td>
</tr>
<tr>
<td>United States (USA)</td>
<td>BIS Table C2</td>
<td>2007-2018</td>
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</tbody>
</table>

Table 1: Sources and coverage of IL debt for all countries in the sample.
<table>
<thead>
<tr>
<th>Country</th>
<th>Coverage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Australia</td>
<td>1960-2018</td>
</tr>
<tr>
<td>Canada</td>
<td>1961-2018</td>
</tr>
<tr>
<td>France</td>
<td>1960-2018</td>
</tr>
<tr>
<td>Germany</td>
<td>1960-2018</td>
</tr>
<tr>
<td>Iceland</td>
<td>1960-2018</td>
</tr>
<tr>
<td>Israel</td>
<td>1970-2018</td>
</tr>
<tr>
<td>Italy</td>
<td>1960-2018</td>
</tr>
<tr>
<td>Japan</td>
<td>1960-2018</td>
</tr>
<tr>
<td>Korea</td>
<td>1960-2018</td>
</tr>
<tr>
<td>New Zealand</td>
<td>1960-2018</td>
</tr>
<tr>
<td>Spain</td>
<td>1960-2018</td>
</tr>
<tr>
<td>Sweden</td>
<td>1960-2018</td>
</tr>
<tr>
<td>United Kingdom</td>
<td>1955-2018</td>
</tr>
<tr>
<td>United States</td>
<td>1955-2018</td>
</tr>
</tbody>
</table>

Table 2: Coverage of the OECD data for all countries in the sample.

Figure 1: IL debt share in advanced economies between 1995 and 2018. Sources: See Table 1 in this Online Appendix.
<table>
<thead>
<tr>
<th>Country</th>
<th>Year</th>
<th>Maturities issued</th>
<th>Inflation measure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Australia</td>
<td>1985</td>
<td>10 years or more</td>
<td>Domestic CPI</td>
</tr>
<tr>
<td>Canada</td>
<td>1991</td>
<td>30 years</td>
<td>Domestic CPI</td>
</tr>
<tr>
<td>France</td>
<td>1998</td>
<td>5 to 30 years</td>
<td>European and domestic CPI</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(2001 for European CPI)</td>
<td></td>
</tr>
<tr>
<td>Germany</td>
<td>2006</td>
<td>5, 10 to 30 years</td>
<td>European CPI</td>
</tr>
<tr>
<td>Iceland</td>
<td>1964</td>
<td>1 year or more</td>
<td>Domestic CPI</td>
</tr>
<tr>
<td>Israel</td>
<td>1955</td>
<td>2 to 30 years</td>
<td>Domestic CPI</td>
</tr>
<tr>
<td>Italy</td>
<td>2003</td>
<td>4, 6, 8 years</td>
<td>European and domestic CPI</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(2012 for Italian CPI)</td>
<td></td>
</tr>
<tr>
<td>Japan</td>
<td>2013</td>
<td>10 years</td>
<td>Domestic CPI</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(also 2004-2008)</td>
<td></td>
</tr>
<tr>
<td>Korea</td>
<td>2007</td>
<td>10 years</td>
<td>Domestic CPI</td>
</tr>
<tr>
<td>New Zealand</td>
<td>1977</td>
<td>1 year or more</td>
<td>Domestic CPI</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(5 to 25 years outstanding in 2021)</td>
<td></td>
</tr>
<tr>
<td>Spain</td>
<td>2014</td>
<td>4 to 15 years</td>
<td>European CPI</td>
</tr>
<tr>
<td>Sweden</td>
<td>1994</td>
<td>10 to 20 years outstanding in 2021</td>
<td></td>
</tr>
<tr>
<td>United Kingdom</td>
<td>1981</td>
<td>5, 10, 30 years</td>
<td>Domestic RPI</td>
</tr>
<tr>
<td>United States</td>
<td>1997</td>
<td>5, 10, 30 years</td>
<td>Domestic CPI</td>
</tr>
</tbody>
</table>

Table 3: Dates of the start of IL debt issuance. Sources for the start dates of IL debt issuance: Bank of International Settlements Table C2, McCray (1997), Thedeen (2004), Deacon et al. (2004), Kramer (2007), Appendix A in Fleckenstein (2013), and Japan’s Ministry of Finance. Sources for maturities issued and inflation measures used to index: each country’s Debt Management Offices. CPI stands for consumer price index and RPI stands for retail price index.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Number of lags (AIC)</th>
<th>$W_t$</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real GDP</td>
<td>0.57</td>
<td>12.31</td>
<td>1.00</td>
</tr>
<tr>
<td>Real government consumption</td>
<td>0.86</td>
<td>9.19</td>
<td>1.00</td>
</tr>
<tr>
<td>Inflation</td>
<td>0.47</td>
<td>-3.85</td>
<td>0.001</td>
</tr>
</tbody>
</table>

Table 4: Results of stationarity tests, where $W_t$ is the statistic in the unit root rests of Im et al. (2003) for panel data with heterogenous panels. The null hypothesis, $H_0$, is that all the panels contain unit roots. The alternative hypothesis, $H_a$, is that at least one panel is stationary. The number of lags is chosen using Akaike’s information criterion (AIC). Use of the Bayesian or Hannan-Quinn information criterion, including a trend and/or subtracting the cross-sectional averages from the series to mitigate the impact of cross-sectional dependence leaves the conclusions of the tests unchanged.
B Derivations for Section 4

Note that all equations referenced in this Online Appendix refer to the main body of the paper, unless stated otherwise.

B.1 Model with $v(S^NB) = 0$

To obtain equation (13) from the first order condition (12) the following steps are necessary:

$$E \left[ \left( G(Y_1) + \frac{S^NB}{P(Y_1)} + (1 - Q^N S^N)B \right) \left( \frac{B}{P(Y_1)} - Q^N B \right) \right] = 0$$

$$E \left[ (G(Y_1) + B) \left( \frac{1}{P(Y_1)} - Q^N \right) \right] + S^N BE \left[ \left( \frac{1}{P(Y_1)} - Q^N \right)^2 \right] = 0$$

$$E \left[ \frac{G(Y_1)}{P(Y_1)} \right] - Q^N E [G(Y_1)] + BE \left[ \frac{1}{P(Y_1)} \right] - Q^N B + S^N BE \left[ \left( \frac{1}{P(Y_1)} - Q^N \right)^2 \right] = 0$$

$$E \left[ \frac{G(Y_1)}{P(Y_1)} \right] - Q^N E [G(Y_1)] + S^N E \left[ \left( \frac{1}{P(Y_1)} \right)^2 \right] = 0$$

$$E \left[ \frac{G(Y_1)}{P(Y_1)} \right] - Q^N E [G(Y_1)] + S^N BVaR \left[ \frac{1}{P(Y_1)} \right] = 0 \quad (1)$$

where the definition of variance is used between the second and third equations. Because $E \left[ \frac{1}{P(Y_1)} \right] = Q^N$, the third and fourth terms on the left-hand side of the last equation cancel out.

To obtain equation (17) from the first order condition (11) the following steps are necessary:

$$G_0 - B = E \left[ \left( G(Y_1) + \frac{S^NB}{P(Y_1)} + (1 - Q^N S^N)B \right) \left( \frac{S^N}{P(Y_1)} + (1 - Q^N S^N) \right) \right]$$

$$\frac{G_0 - B}{B} = E \left[ \left( \frac{G(Y_1)}{B} + 1 + S^N \left( \frac{1}{P(Y_1)} - Q^N \right) \right) \left( 1 + S^N \left( \frac{1}{P(Y_1)} - Q^N \right) \right) \right]$$

$$\frac{G_0 - B}{B} = E \left[ \frac{G(Y_1)}{B} + 1 + S^N \left( \frac{1}{P(Y_1)} - Q^N \right) G(Y_1) + S^N \left( \frac{1}{P(Y_1)} - Q^N \right) \right]$$

$$+ E \left[ S^N \left( \frac{1}{P(Y_1)} - Q^N \right) + (S^N)^2 \left( \frac{1}{P(Y_1)} - Q^N \right)^2 \right]$$

$$\frac{G_0 - B}{B} = E \left[ \frac{G(Y_1)}{P(Y_1)} \right] + 1 + \frac{S^N}{B} E \left[ \frac{G(Y_1)}{P(Y_1)} \right] - \frac{S^N}{B} Q^N E(G_1) + (S^N)^2 Var \left[ \frac{1}{P(Y_1)} \right]$$

where, the last equation uses the facts that; first, $E \left( \frac{1}{P(Y_1)} \right) - Q^N = 0$ when holding nominal debt has no direct utility, and, second, $E \left( \left( \frac{1}{P(Y_1)} - Q^N \right)^2 \right) = Var \left( \frac{1}{P(Y_1)} \right) + E \left( \frac{1}{P(Y_1)} \right) - Q^N = Var \left( \frac{1}{P(Y_1)} \right)$. 
Applying the definition of covariance in equation (15) to the last equation above yields

\[
\frac{G_0 - B}{B} = E\left(\frac{G(Y_1)}{B}\right) + 1 + \frac{S^N}{B} \text{Cov}\left(G(Y_1), \frac{1}{P(Y_1)}\right) + \frac{S^N}{B} Q^N G_1 + \left(\frac{S^N}{B}\right)^2 \text{Var}\left(\frac{1}{P_1}\right)
\]

\[
\frac{G_0 - B}{B} = E\left(\frac{G(Y_1)}{B}\right) + 1 + \frac{S^N}{B} \text{Cov}\left(G(Y_1), \frac{1}{P(Y_1)}\right) + \left(\frac{S^N}{B}\right)^2 \text{Var}\left(\frac{1}{P_1}\right)
\]

Using the solution for \(S^N\) in the last equation above gives equation (17).

\[\text{B.2 Model with distortions on the tax rate}\]

This section presents an alternative modeling choice on the distortionary effects of taxes: imposing a quadratic cost on the tax rate. The tax base then becomes period’s 1 endowment. In this environment, the government’s budget constraint on date 1 equals:

\[
\tau_1 P_1 Y_1 = P_1 G(Y_1) + B^N + P_1 B^I
\]

and the government’s budget constraint on date 0 remains unchanged and equal to equation (2) in the main body of the paper.

Dividing equation (2) above by \(P_1 Y_1\) gives an expression for the tax rate, \(\tau_1\), which we then plug into the expression for \(C_1\) in equation (9):

\[
C_1 = Y_1 - G(Y_1) - \frac{1}{2} \left(\frac{G(Y_1)}{Y_1} + \frac{B^N}{P(Y_1) Y_1} + \frac{B^I}{Y_1}\right)^2
\]

\[C_0\] is given by equation (8).

Again, maximizing \(C_0 + E(C_1)\) is equivalent to minimizing the tax distortions. Thus, imposing the same change of variables as in the main body of the paper, the problem becomes:

\[
\min_{B,S^N} \frac{1}{2} \left(G_0 - B\right)^2 + \frac{1}{2} E \left[\frac{G(Y_1)}{Y_1} + \frac{S^N B}{P(Y_1) Y_1} + \left(1 - Q^N S^N\right) B^I\right]^2
\]

This problem’s first-order conditions for \(B\) and \(S^N\) are, respectively, given by:

\[
\tau_0 = E \left[\frac{\tau_1}{P(Y_1)} \left(\frac{S^N}{P(Y_1) Y_1} + \frac{1 - Q^N S^N}{Y_1}\right)\right]
\]

\[
BE \left[\frac{\tau_1}{P(Y_1) Y_1} \left(\frac{1}{P(Y_1) Y_1} - \frac{Q^N}{Y_1}\right)\right] = 0
\]
We start operating on equation (6) above:

\[
E \left[ \left( \frac{G(Y_1)}{Y_1} + \frac{S^N B}{P(Y_1)Y_1} \right) \left( \frac{1}{P(Y_1)Y_1} - \frac{Q^N}{Y_1} \right) \right] = 0
\]

\[
E \left[ \frac{1}{Y_1} (G(Y_1) + B) \left( \frac{1}{P(Y_1)} - \frac{Q^N}{Y_1} \right) \right] + S^N B E \left[ \left( \frac{1}{Y_1} \right)^2 \left( \frac{1}{P(Y_1)} - \frac{Q^N}{Y_1} \right)^2 \right] = 0
\]

\[
Cov \left( \frac{1}{Y_1} \frac{G_1}{P_1} - \frac{Q^N}{Y_1} \right) + E \left( \frac{1}{Y_1} \right) \left[ E \left( \frac{G_1}{P_1} \right) - \frac{Q^N}{Y_1} \right] + B Cov \left( \frac{1}{Y_1}, \frac{1}{P_1} - \frac{Q^N}{Y_1} \right)
\]

\[
+ S^N B \left\{ Cov \left( \frac{1}{Y_1^2}, \left( \frac{1}{P_1} - \frac{Q^N}{Y_1} \right)^2 \right) + E \left( \frac{1}{Y_1^2} \right) E \left( \frac{1}{P_1} - \frac{Q^N}{Y_1} \right)^2 \right\} = 0
\]

\[
Cov \left( \frac{1}{Y_1}, \frac{G_1}{P_1} - \frac{Q^N}{Y_1} \right) + E \left( \frac{1}{Y_1} \right) E \left( \frac{G_1}{P_1} \right) - E \left( \frac{1}{Y_1} \right) Q^N + B Cov \left( \frac{1}{Y_1}, \frac{1}{P_1} - \frac{Q^N}{Y_1} \right)
\]

\[
+ S^N B \left\{ Cov \left( \frac{1}{Y_1^2}, \left( \frac{1}{P_1} - \frac{Q^N}{Y_1} \right)^2 \right) + E \left( \frac{1}{Y_1^2} \right) Var \left( \frac{1}{P_1} \right) \right\} = 0
\]

where between the first and second equations we take common factor $\frac{1}{Y_1}$, between the second and third equations the definition of the expectation of a product is applied, and between the third and fourth equations the definition of the variance is used.

Note that, in the last equation, although new relationships with $Y_1$ appear, the hedging motive and the variance of inflation force remain. The hedging motive is in the second summand, and the variance of (the inverse of) inflation is in the last term of the last line. Both enter with the same signs and in the same way as in equation (6) in the previous subsection.
B.3 Model with \( v(S^N B) > 0 \)

Equation (22) can be simplified by noting that the second summand on the right-hand side equals \( v'(S^N B) S^N \) by equation (23). Plugging the expressions for \( \tau_0 \) and \( \tau_1 \), equation (22) becomes:

\[
\begin{align*}
G_0 - B &= E \left( G(Y_1) + \frac{S^N B}{P_1(Y_1)} + B - E \left( \frac{1}{P_1} \right) S^N B - v'(S^N B) S^N B \right) \\
\frac{G_0 - B}{B} &= E \left( \frac{G(Y_1)}{B} \right) + S^N E \left( \frac{1}{P_1} \right) + 1 - S^N E \left( \frac{1}{P_1} \right) - v'(S^N B) S^N \\
G_0 - B &= E \left( G(Y_1) + B - v'(S^N B) S^N B \right) \\
B &= \frac{G_0 - E \left( G(Y_1) \right)}{2 - v'(S^N B) S^N}
\end{align*}
\]

Plugging the expression for \( \tau_1 \) into equation (23), we obtain:

\[
\begin{align*}
\frac{v'(S^N B)}{B} &= E \left\{ \left[ \frac{G(Y_1)}{B} + 1 + S^N \left( \frac{1}{P(Y_1)} - E \left( \frac{1}{P(Y_1)} \right) - v'(S^N B) \right) \right] \right. \\
&\quad \left\{ \frac{1}{P(Y_1)} - E \left( \frac{1}{P(Y_1)} \right) - v'(S^N B) - v''(S^N B) S^N B \right\} \} \\
\frac{v'(S^N B)}{B} &= \frac{1}{B} E \left( \frac{G(Y_1)}{P(Y_1)} \right) - E \left( \frac{G_1}{B} \right) E \left( \frac{1}{P(Y_1)} \right) - \frac{E(G_1)}{B} \left( v'(S^N B) + v''(S^N B) S^N B \right) \\
&\quad - v'(S^N B) - v''(S^N B) S^N B + S^N E \left( \frac{1}{P(Y_1)} - E \left( \frac{1}{P(Y_1)} \right) - v'(S^N B) \right) \right. \\
&\quad + S^N \left( E \left( \frac{1}{P(Y_1)} \right) - E \left( \frac{1}{P(Y_1)} \right) - v'(S^N B) \right) \left( - v''(S^N B) S^N B \right) \\
v'(S^N B) &= Cov \left( G(Y_1), \frac{1}{P(Y_1)} \right) - (E(G_1) + B) \left( v'(S^N B) + v''(S^N B) S^N B \right) \\
&\quad + S^N B Var \left( \frac{1}{P(Y_1)} \right) - S^N B v'(S^N B) + S^N B v'(S^N B) v''(S^N B) S^N B \\
v'(S^N B) &= Cov \left( G(Y_1), \frac{1}{P(Y_1)} \right) - (E(G_1) + B) \left( v'(S^N B) + v''(S^N B) S^N B \right) \\
&\quad + S^N B Var \left( \frac{1}{P(Y_1)} \right) + S^N B v'(S^N B) \left( v''(S^N B) S^N B - 1 \right)
\end{align*}
\]

where we have applied the distributive property of the multiplication, simplified the equation, and applied the definitions of variance and covariance.

Finally, using equation (7) above to substitute for \( E \left( G(Y_1) \right) + B \) in the last equation above, we obtain the following condition for \( S^N \):

\[
v'(S^N B) + \tau_0 \left( v'(S^N B) + v''(S^N B) S^N B \right) + S^N B v'(S^N B) (1 + v'(S^N B)) = \right. \\
Cov \left( G(Y_1), \frac{1}{P(Y_1)} \right) + S^N B Var \left( \frac{1}{P(Y_1)} \right)
\]
B.4 Welfare analysis derivations

Denoting the welfare for the model without liquidity services of nominal debt as \( W \), we have:

\[
W = C_0 + E(C_1) = C_0 + E(Y_1 - G_1) - \frac{1}{2} E \left[ G_1 + B + S^N B \left( \frac{1}{P_1} - Q^N \right) \right]^2
\]

\[ (8) \]

\[
= 1 - G_0 - \frac{1}{2} \left( G_0^2 + B^2 - 2G_0 B \right) + E(Y_1 - G_1)
\]

\[ (9) \]

Ignoring the terms that are independent of \( B \) and \( S^N \), which will be the same for both models, and using the definition of the expectation of a product (equation 15) and the fact that \( E(X^2) = Var(X) + E(X)^2 \) yields

\[
W = G_0 B - \frac{B^2}{2} - \frac{B^2}{2} - E(G_1) B - \left( \frac{S^N B}{2} \right)^2 \frac{1}{P_1} - \frac{S^N B}{2} Cov \left( G_1, \frac{1}{P_1} \right) - \frac{S^N B}{2} \gamma (S^N B) \gamma = (10)
\]

\[
G_0 B - B^2 - E(G_1) B
\]

\[ (11) \]

where between the two equations we use the expression for the optimal \( S^N B \) in equation (18).

Turning now to the model with liquidity services of nominal debt and denoting the welfare in that version of the model as \( W_\gamma \), yields, from equation [9] above, the following:

\[
W_\gamma = G_0 B_\gamma - \frac{B_\gamma^2}{2} - \frac{B_\gamma^2}{2} - E(G_1) B_\gamma - \frac{\left( S^N B_\gamma \right)^2}{2} \left[ Var \left( \frac{1}{P_1} \right) - (v'(S^N B_\gamma))^2 \right]
\]

\[ (12) \]

\[
- \frac{S^N B_\gamma}{2} Cov \left( G_1, \frac{1}{P_1} \right) + \frac{S^N B_\gamma}{2} E(G_1) v'(S^N B) + \frac{S^N B_\gamma}{2} B_\gamma v'(S^N B) = (13)
\]

\[
G_0 B_\gamma - B_\gamma^2 - E(G_1) B_\gamma - \frac{S^N B_\gamma}{2} v'(S^N B_\gamma) + \frac{\left( S^N B_\gamma \right)^2}{2} (v'(S^N B_\gamma))^2
\]

\[ (14) \]

\[
+ \frac{S^N B_\gamma}{2} E(G_1) v'(S^N B) + \frac{S^N B_\gamma^2}{2} v'(S^N B)
\]

\[ (15) \]

where between the first and second equations, we use the first order condition for \( S^N \) and abstract from the lower tax distortions (since they are normally considered of smaller order): \( v'(S^N \gamma B_\gamma) = Cov \left( G_1, B_\gamma \right) + S^N \gamma B_\gamma Var \left( \frac{1}{P_1} \right) \). Taking \( \frac{S^N B_\gamma}{2} v'(S^N \gamma B_\gamma) \) as common factor yields:

\[
W_\gamma = G_0 B_\gamma - B_\gamma^2 - E(G_1) B_\gamma - \frac{S^N B_\gamma}{2} v'(S^N B_\gamma) \left[ 1 - S^N \gamma B_\gamma v'(S^N \gamma B_\gamma) - E(G_1) - B_\gamma \right]
\]

\[ (16) \]
C Additional scatter plots

Figure 2: IL debt share against inflation behavior, excluding Israel and Iceland.
Figure 3: IL debt share against inflation behavior, using the inflation index used. See Table 3 in this Online Appendix for index used. Sources: World Bank Databank, Eurostat for the European CPI, and UK’s Office of National Statistics for the UK’s RPI.
Figure 4: IL debt share against inflation behavior, using the inflation index used. See Table 3 in this Online Appendix for index used. Sources: World Bank Databank, Eurostat for the European CPI, and UK’s Office of National Statistics for the UK’s RPI.
References


Thedeen, E. (2004), ‘Ten years with inflation-linked bonds - a new asset class has been established’, Speech in FIM Asset Management Ltd.