Online Appendix for *Optimal public debt indexation in advanced economies*

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A Data sources, coverage, and additional IL debt statistics

IL debt: coverage and sources		
Country	Source	Coverage
Australia (AUS)	BIS Table C2	1995-2018
Canada (CAN)	BIS Table C2	1995-2018
France (FRA)	AFT - Agence France Trésor	1999-2018
Germany (DEU)	BIS Table C2	2006-2018
Iceland (ISL)	Bank of Iceland	1995-2018
Israel (ISR)	BIS Table C2	1995-2018
Italy (ITA)	Ministero dell 'Economia e delle Finanze	2003-2018
Japan (JPN)	Japanese Ministry of Finance	2013-2018
Korea (KOR)	BIS Table C2	2007-2018
New Zealand (NZL)	Office of Debt Management	2000-2018
Spain (ESP)	BIS Table C2	2014-2018
Sweden (SWE)	Swedish National Debt Office	1995-2018
United Kingdom (GBR)	BIS Table C2	2004-2018
United States (USA)	BIS Table C2	2007-2018

Table 1: Sources and coverage of IL debt for all countries in the sample.

OECD data: coverage	
Country	Coverage
Australia	1960-2018
Canada	1961-2018
France	1960-2018
Germany	1960-2018
Iceland	1960-2018
Israel	1970-2018
Italy	1960-2018
Japan	1960-2018
Korea	1960-2018
New Zealand	1960-2018
Spain	1960-2018
Sweden	1960-2018
United Kingdom	1955-2018
United States	1955-2018

Table 2: Coverage of the OECD data for all countries in the sample.

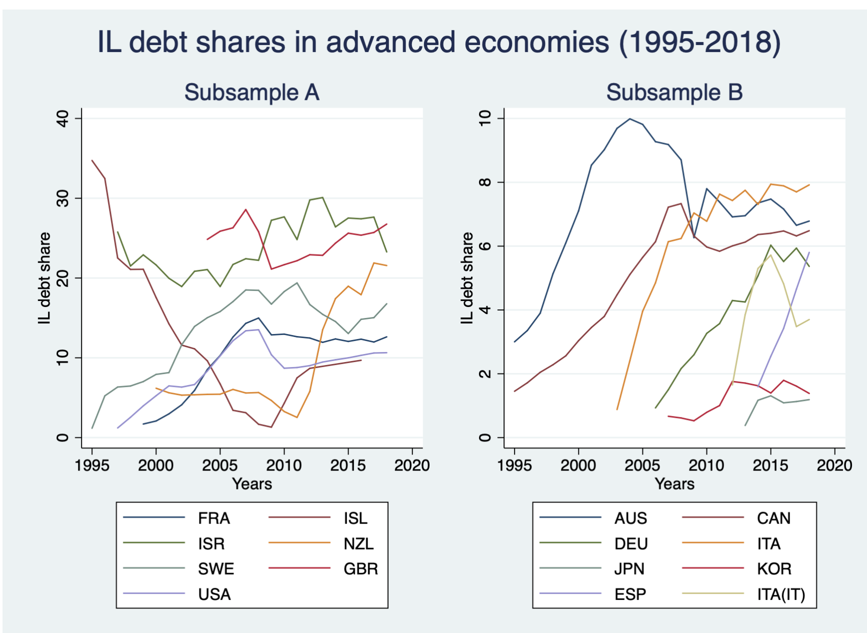

Figure 1: IL debt share in advanced economies between 1995 and 2018. Sources: See Table 1 in this Online Appendix.

Dates of IL debt issuance start and maturities issued			
Country	Year	Maturities issued	Inflation measure
Australia	1985	10 years or more	Domestic CPI
Canada	1991	30 years	Domestic CPI
France	1998 (2001 for European CPI)	5 to 30 years	European and domestic CPI
Germany	2006	5, 10 to 30 years	European CPI
Iceland	1964	1 year or more	Domestic CPI
Israel	1955	2 to 30 years	Domestic CPI
Italy	2003 (2012 for Italian CPI)	4, 6, 8 years	European and domestic CPI
Japan	2013 (also 2004-2008)	10 years	Domestic CPI
Korea	2007	10 years	Domestic CPI
New Zealand	1977	1 year or more (5 to 25 years outstanding in 2021)	Domestic CPI
Spain	2014	4 to 15 years	European CPI
Sweden	1994	10 to 20 years outstanding in 2021	Domestic CPI
United Kingdom	1981	5, 10, 30 years	Domestic RPI
United States	1997	5, 10, 30 years	Domestic CPI

Table 3: Dates of the start of IL debt issuance. Sources for the start dates of IL debt issuance: Bank of International Settlements Table C2, McCray (1997), Thedeen (2004), Deacon et al. (2004), Kramer (2007), Appendix A in Fleckenstein (2013), and Japan's Ministry of Finance. Sources for maturities issued and inflation measures used to index: each country's Debt Management Offices. CPI stands for consumer price index and RPI stands for retail price index.

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Results of Stationarity Tests			
Variable	Number of lags (AIC)	W_t	p-value
Real GDP	0.57	12.31	1.00
Real government consumption	0.86	9.19	1.00
Inflation	0.47	-3.85	0.001

Table 4: Results of stationarity tests, where W_t is the statistic in the unit root rests of Im et al. (2003) for panel data with heterogenous panels. The null hypothesis, H_0, is that all the panels contain unit roots. The alternative hypothesis, H_a, is that at least one panel is stationary. The number of lags is chosen using Akaike's information criterion (AIC). Use of the Bayesian or Hannan-Quinn information criterion, including a trend and/or subtracting the cross-sectional averages from the series to mitigate the impact of cross-sectional dependence leaves the conclusions of the tests unchanged.

B Derivations for Section 4

Note that all equations referenced in this Online Appendix refer to the main body of the paper, unless stated otherwise.

B.1 Model with $v(S^N B) = 0$

To obtain equation (13) from the first order condition (12) the following steps are necessary:

$$\begin{aligned}
E\left[\left(G(Y_1) + \frac{S^N B}{P(Y_1)} + (1 - Q^N S^N)B\right)\left(\frac{B}{P(Y_1)} - Q^N B\right)\right] &= 0 \\
E\left[(G(Y_1) + B)\left(\frac{1}{P(Y_1)} - Q^N\right)\right] + S^N B E\left[\left(\frac{1}{P(Y_1)} - Q^N\right)^2\right] &= 0 \\
E\left[\frac{G(Y_1)}{P(Y_1)}\right] - Q^N E\left[G(Y_1)\right] + BE\left[\frac{1}{P(Y_1)}\right] - Q^N B + S^N B E\left[\left(\frac{1}{P(Y_1)} - Q^N\right)^2\right] &= 0 \\
E\left[\frac{G(Y_1)}{P(Y_1)}\right] - Q^N E\left[G(Y_1)\right] + S^N B E\left[\left(\frac{1}{P(Y_1)} - Q^N\right)^2\right] &= 0 \\
E\left[\frac{G(Y_1)}{P(Y_1)}\right] - Q^N E\left[G(Y_1)\right] + S^N B Var\left[\frac{1}{P(Y_1)}\right] &= 0 \quad (1)
\end{aligned}$$

where the definition of variance is used between the second and third equations. Because $E\left[\frac{1}{P(Y_1)}\right] = Q^N$, the third and fourth terms on the left-hand side of the last equation cancel out.

To obtain equation (17) from the first order condition (11) the following steps are necessary:

$$\begin{aligned}
G_0 - B &= E\left[\left(G(Y_1) + \frac{S^N B}{P(Y_1)} + (1 - Q^N S^N)B\right)\left(\frac{S^N}{P(Y_1)} + (1 - Q^N S^N)\right)\right] \\
\frac{G_0 - B}{B} &= E\left[\left(\frac{G(Y_1)}{B} + 1 + S^N\left(\frac{1}{P(Y_1)} - Q^N\right)\right)\left(1 + S^N\left(\frac{1}{P(Y_1)} - Q^N\right)\right)\right] \\
\frac{G_0 - B}{B} &= E\left[\frac{G(Y_1)}{B} + 1 + S^N\left(\frac{1}{P(Y_1)} - Q^N\right) G(Y_1) + S^N\left(\frac{1}{P(Y_1)} - Q^N\right)\right] \\
&\quad + E\left[S^N\left(\frac{1}{P(Y_1)} - Q^N\right) + (S^N)^2\left(\frac{1}{P(Y_1)} - Q^N\right)^2\right] \\
\frac{G_0 - B}{B} &= \frac{E(G_1)}{B} + 1 + \frac{S^N}{B} E\left[\frac{G(Y_1)}{P(Y_1)}\right] - \frac{S^N}{B} Q^N E(G_1) + (S^N)^2 Var\left(\frac{1}{P_1}\right)
\end{aligned}$$

where, the last equation uses the facts that; first, $E\left(\frac{1}{P_1}\right) - Q^N = 0$ when holding nominal debt has no direct utility, and, second, $E\left(\left(\frac{1}{P_1} - Q^N\right)^2\right) = Var\left(\frac{1}{P_1}\right) + E\left(\frac{1}{P_1}\right) - Q^N = Var\left(\frac{1}{P_1}\right)$.

Applying the definition of covariance in equation (15) to the last equation above yields

$$\begin{aligned}\frac{G_0 - B}{B} &= \frac{E(G_1)}{B} + 1 + \frac{S^N}{B} Cov\left(G(Y_1), \frac{1}{P(Y_1)}\right) + \frac{S^N}{B}\bar{G}_1 Q^N - \frac{S^N}{B} Q^N \bar{G}_1 \\ &\quad + (S^N)^2 Var\left(\frac{1}{P_1}\right) \\ \frac{G_0 - B}{B} &= \frac{E(G_1)}{B} + 1 + \frac{S^N}{B} Cov\left(G(Y_1), \frac{1}{P(Y_1)}\right) + (S^N)^2 Var\left(\frac{1}{P_1}\right)\end{aligned}$$

Using the solution for S^N in the last equation above gives equation (17).

B.2 Model with distortions on the tax rate

This section presents an alternative modeling choice on the distortionary effects of taxes: imposing a quadratic cost on the tax rate. The tax base then becomes period's 1 endowment. In this environment, the government's budget constraint on date 1 equals:

$$\tau_1 P_1 Y_1 = P_1 G(Y_1) + B^N + P_1 B^I \tag{2}$$

and the government's budget constraint on date 0 remains unchanged and equal to equation (2) in the main body of the paper.

Dividing equation (2) above by $P_1 Y_1$ gives an expression for the tax rate, τ_1, which we then plug into the expression for C_1 in equation (9):

$$C_1 = Y_1 - G(Y_1) - \frac{1}{2}\left(\frac{G(Y_1)}{Y_1} + \frac{B^N}{P(Y_1)Y_1} + \frac{B^I}{Y_1}\right)^2 \tag{3}$$

C_0 is given by equation (8).

Again, maximizing $C_0 + E(C_1)$ is equivalent to minimizing the tax distortions. Thus, imposing the same change of variables as in the main body of the paper, the problem becomes:

$$min_{B,S^N} \quad \frac{1}{2}(G_0 - B)^2 + \frac{1}{2}E\left[\frac{G(Y_1)}{Y_1} + \frac{S^N B}{P(Y_1)Y_1} + (1 - Q^N S^N)\frac{B}{Y_1}\right]^2 \tag{4}$$

This problem's first-order conditions for B and S^N are, respectively, given by:

$$\tau_0 = E\left[\frac{\tau_1}{P(Y_1)}\left(\frac{S^N}{P(Y_1)Y_1} + \frac{1 - Q^N S^N}{Y_1}\right)\right] \tag{5}$$

$$BE\left[\frac{\tau_1}{P(Y_1)Y_1}\left(\frac{1}{P(Y_1)Y_1} - \frac{Q^N}{Y_1}\right)\right] = 0 \tag{6}$$

We start operating on equation (6) above:

$$
\begin{aligned}
E\left[\left(\frac{G(Y_1)}{Y_1}+\frac{S^N B}{P(Y_1)Y_1}+(1-Q^N S^N)\frac{B}{Y_1}\right)\left(\frac{1}{P(Y_1)Y_1}-\frac{Q^N}{Y_1}\right)\right] &= 0 \\
E\left[\frac{1}{Y_1}\left(G(Y_1)+B\right)\left(\frac{1}{P(Y_1)}-Q^N\right)\right]+S^N BE\left[\left(\frac{1}{Y_1}\right)^2\left(\frac{1}{P(Y_1)}-Q^N\right)^2\right] &= 0 \\
Cov\left(\frac{1}{Y_1},\frac{G_1}{P_1}-Q^N\right)+E\left(\frac{1}{Y_1}\right)\left[E\left(\frac{G_1}{P_1}\right)-Q^N\right]+BCov\left(\frac{1}{Y_1},\frac{1}{P_1}-Q^N\right) & \\
+S^N B\left\{Cov\left(\frac{1}{Y_1^2},\left(\frac{1}{P_1}-Q^N\right)^2\right)+E\left(\frac{1}{Y_1^2}\right)E\left(\frac{1}{P_1}-Q^N\right)^2\right\} &= 0 \\
Cov\left(\frac{1}{Y_1},\frac{G_1}{P_1}-Q^N\right)+E\left(\frac{1}{Y_1}\right)\left[E\left(\frac{G_1}{P_1}\right)-Q^N\right]+BCov\left(\frac{1}{Y_1},\frac{1}{P_1}-Q^N\right) & \\
+S^N B\left\{Cov\left(\frac{1}{Y_1^2},\left(\frac{1}{P_1}-Q^N\right)^2\right)+E\left(\frac{1}{Y_1^2}\right)Var\left(\frac{1}{P_1}\right)\right\} &= 0 \\
Cov\left(\frac{1}{Y_1},\frac{G_1}{P_1}-Q^N\right)+E\left(\frac{1}{Y_1}\right)\underbrace{E\left(\frac{G_1}{P_1}\right)}_{\text{Hedging}}-E\left(\frac{1}{Y_1}\right)Q^N+BCov\left(\frac{1}{Y_1},\frac{1}{P_1}-Q^N\right) & \\
+S^N B\left\{Cov\left(\frac{1}{Y_1^2},\left(\frac{1}{P_1}-Q^N\right)^2\right)+E\left(\frac{1}{Y_1^2}\right)\underbrace{Var\left(\frac{1}{P_1}\right)}_{\text{Variance}}\right\} &= 0
\end{aligned}
$$

where between the first and second equations we take common factor $\frac{1}{Y_1}$, between the second and third equations the definition of the expectation of a product is applied, and between the third and fourth equations the definition of the variance is used.

Note that, in the last equation, although new relationships with Y_1 appear, the hedging motive and the variance of inflation force remain. The hedging motive is in the second summand, and the variance of (the inverse of) inflation is in the last term of the last line. Both enter with the same signs and in the same way as in equation 1 in the previous subsection.

B.3 Model with $v(S^N B) > 0$

Equation (22) can be simplified by noting that the second summand on the right-hand side equals $v'(S^N B)S^N$ by equation (23). Plugging the expressions for τ_0 and τ_1, equation (22) becomes:

$$\begin{aligned}
G_0 - B &= E\left(G(Y_1) + \frac{S^N B}{P_1(Y_1)} + B - E\left(\frac{1}{P_1}\right) S^N B - v'(S^N B)S^N B\right) \\
\frac{G_0 - B}{B} &= \frac{E\left(G(Y_1)\right.}{B} + S^N E\left(\frac{1}{P_1}\right) + 1 - S^N E\left(\frac{1}{P_1}\right) - v'(S^N B)S^N \\
G_0 - B &= E\left(G(Y_1) + B - v'(S^N B)S^N B\right. \qquad (7) \\
B &= \frac{G_0 - E\left(G(Y_1)\right)}{2 - v'(S^N B)S^N}
\end{aligned}$$

Plugging the expression for τ_1 into equation (23), we obtain:

$$\begin{aligned}
\frac{v'(S^N B)}{B} &= E\{\left[\frac{G(Y_1)}{B} + 1 + S^N\left(\frac{1}{P(Y_1)} - E\left(\frac{1}{P(Y_1)}\right) - v'(S^N B)\right)\right] \\
&\quad \left[\frac{1}{P(Y_1)} - E\left(\frac{1}{P(Y_1)}\right) - v'(S^N B) - v''(S^N B)S^N B\right]\} \\
\frac{v'(S^N B)}{B} &= \frac{1}{B}E\left(\frac{G(Y_1)}{P(Y_1)}\right) - \frac{E(G_1)}{B}E\left(\frac{1}{P(Y_1)}\right) - \frac{E(G_1)}{B}\left(v'(S^N B) + v''(S^N B)S^N B\right) \\
&\quad - v'(S^N B) - v''(S^N B)S^N B + S^N E\left(\frac{1}{P(Y_1)} - E\left(\frac{1}{P(Y_1)}\right) - v'(S^N B)\right)^2 \\
&\quad + S^N\left(E\left(\frac{1}{P(Y_1)}\right) - E\left(\frac{1}{P(Y_1)}\right) - v'(S^N B)\right)\left(-v''(S^N B)S^N B\right) \\
v'(S^N B) &= Cov\left(G(Y_1), \frac{1}{P(Y_1)}\right) - \left(E(G_1) + B\right)\left(v'(S^N B) + v''(S^N B)S^N B\right) \\
&\quad + S^N B Var\left(\frac{1}{P(Y_1)}\right) - S^N B v'(S^N B) + S^N B v'(S^N B) v''(S^N B)S^N B \\
v'(S^N B) &= Cov\left(G(Y_1), \frac{1}{P(Y_1)}\right) - \left(E(G_1) + B\right)\left(v'(S^N B) + v''(S^N B)S^N B\right) \\
&\quad + S^N B Var\left(\frac{1}{P(Y_1)}\right) + S^N B v'(S^N B)\left(v''(S^N B)S^N B - 1\right)
\end{aligned}$$

where we have applied the distributive property of the multiplication, simplified the equation, and applied the definitions of variance and covariance.

Finally, using equation (7) above to substitute for $E\left(G(Y_1)\right) + B$ in the last equation above, we obtain the following condition for S^N:

$$\begin{aligned}
v'(S^N B) + \tau_0\left(v'(S^N B) + v''(S^N B)S^N B\right) + S^N B v'(S^N B)\left(1 + v'(S^N B)\right) &= \\
Cov\left(G(Y_1), \frac{1}{P(Y_1)}\right) + S^N B Var\left(\frac{1}{P(Y_1)}\right)&
\end{aligned}$$

B.4 Welfare analysis derivations

Denoting the welfare for the model without liquidity services of nominal debt as W, we have:

$$W = C_0 + E(C_1) = C_0 + E(Y_1 - G_1) - \frac{1}{2}E\left[G_1 + B + S^N B\left(\frac{1}{P_1} - Q^N\right)\right]^2 \tag{8}$$

$$= 1 - G_0 - \frac{1}{2}\left(G_0^2 + B^2 - 2G_0 B\right) + E(Y_1 - G_1)$$
$$-\frac{E}{2}\left[(G_1 + B)^2 + (S^N B)^2\left(\frac{1}{P_1} - Q^N\right)^2 + S^N B(G_1 + B)\left(\frac{1}{P_1} - Q^N\right)\right] \tag{9}$$

Ignoring the terms that are independent of B and S^N, which will be the same for both models, and using the definition of the expectation of a product (equation 15) and the fact that $E(X^2) = Var(X) + E(X)^2$ yields

$$W = G_0 B - \frac{B^2}{2} - \frac{B^2}{2} - E(G_1)B - \frac{(S^N B)^2}{2}Var\left(\frac{1}{P_1}\right) - \frac{S^N B}{2}Cov\left(G_1, \frac{1}{P_1}\right) \tag{10}$$

$$G_0 B - B^2 - E(G_1)B \tag{11}$$

where between the two equations we use the expression for the optimal $S^N B$ in equation (18).

Turning now to the model with liquidity services of nominal debt and denoting the welfare in that version of the model as W_γ yields, from equation 9 above, the following:

$$W_\gamma = G_0 B_\gamma - \frac{B_\gamma^2}{2} - \frac{B_\gamma^2}{2} - E(G_1)B_\gamma - \frac{(S_\gamma^N B_\gamma)^2}{2}\left[Var\left(\frac{1}{P_1}\right) - (v'(S_\gamma^N B_\gamma))^2\right] \tag{12}$$

$$-\frac{S_\gamma^N B_\gamma}{2}Cov\left(G, \frac{1}{P_1}\right) + \frac{S_\gamma^N B_\gamma}{2}E(G_1)v'(S^N B) + \frac{S_\gamma^N B_\gamma}{2}B_\gamma v'(S^N B) = \tag{13}$$

$$G_0 B_\gamma - B_\gamma^2 - E(G_1)B_\gamma - \frac{S_\gamma^N B_\gamma}{2}v'(S_\gamma^N B_\gamma) + \frac{(S_\gamma^N B_\gamma)^2}{2}(v'(S_\gamma^N B_\gamma))^2 \tag{14}$$

$$+\frac{S_\gamma^N B_\gamma}{2}E(G_1)v'(S^N B) + \frac{S_\gamma^N B_\gamma^2}{2}v'(S^N B) \tag{15}$$

where between the first and second equations, we use the first order condition for S^N and abstract from the lower tax distortions (since they are normally considered of smaller order): $v'(S_\gamma^N B_\gamma) = Cov\left(G, \frac{1}{P_1}\right) + S_\gamma^N B_\gamma Var\left(\frac{1}{P_1}\right)$. Taking $\frac{S_\gamma^N B_\gamma}{2}v'(S_\gamma^N B_\gamma)$ as common factor yields:

$$W_\gamma = G_0 B_\gamma - B_\gamma^2 - E(G_1)B_\gamma - \frac{S_\gamma^N B_\gamma}{2}v'(S_\gamma^N B_\gamma)\left[1 - S_\gamma^N B_\gamma v'(S_\gamma^N B_\gamma) - E(G_1) - B_\gamma\right] \tag{16}$$

C Additional scatter plots

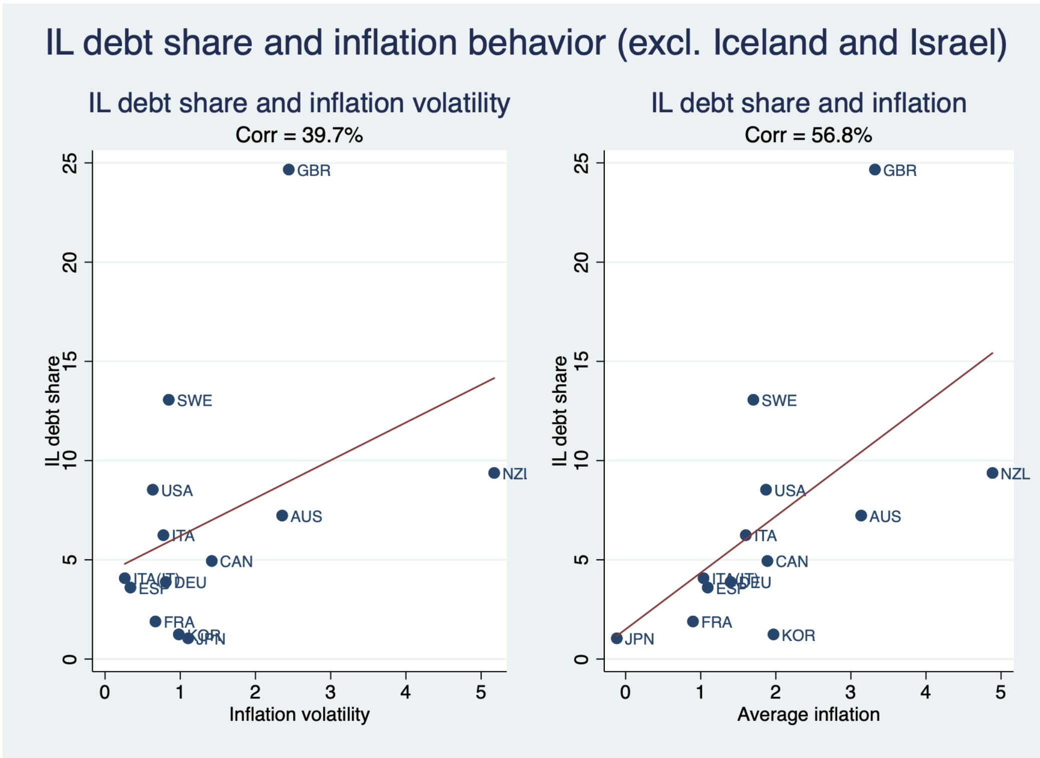

Figure 2: IL debt share against inflation behavior, excluding Israel and Iceland.

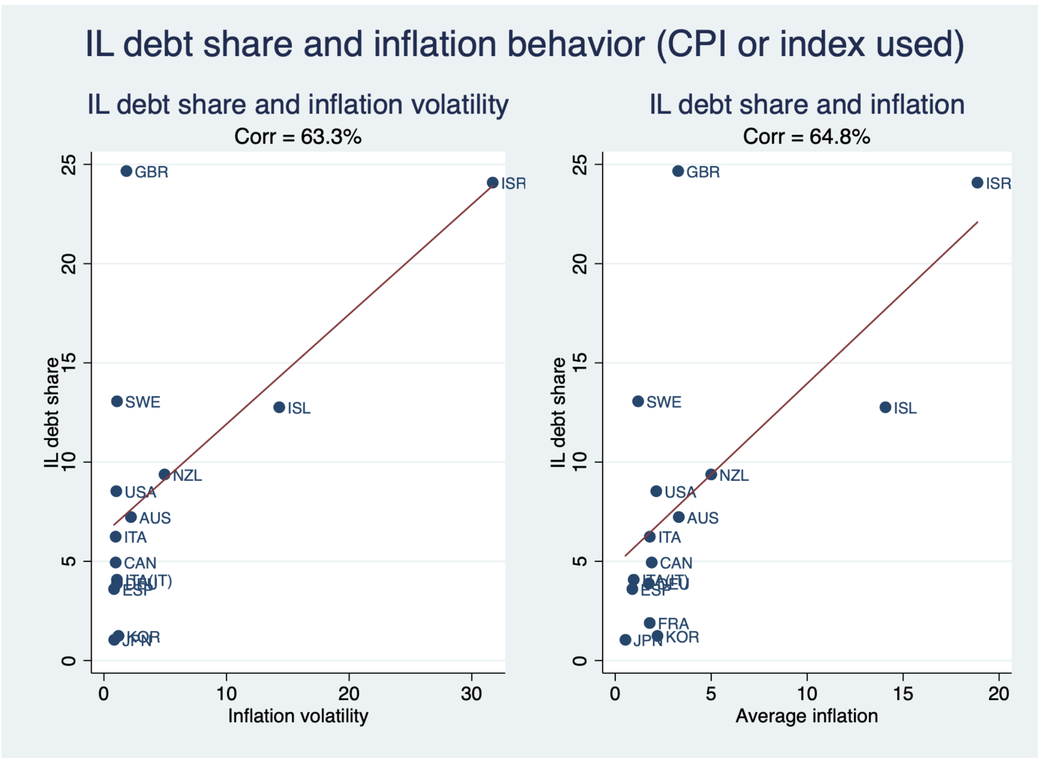

Figure 3: IL debt share against inflation behavior, using the inflation index used. See Table 3 in this Online Appendix for index used. Sources: World Bank Databank, Eurostat for the European CPI, and UK's Office of National Statistics for the UK's RPI.

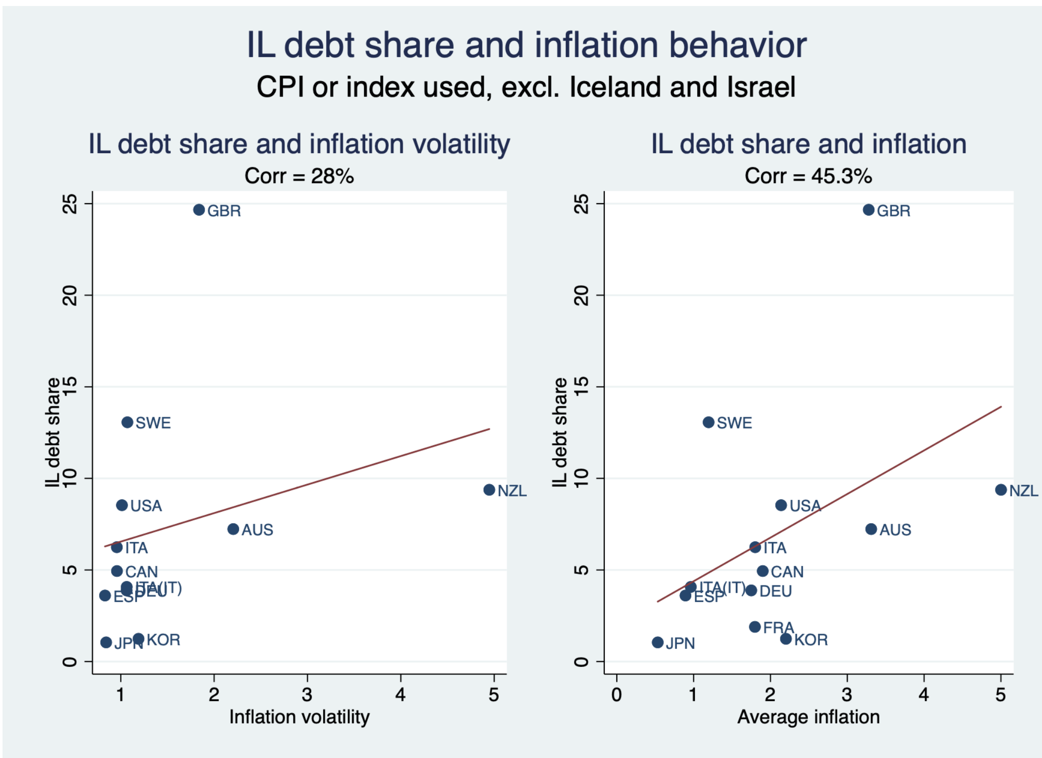

Figure 4: IL debt share against inflation behavior, using the inflation index used. See Table 3 in this Online Appendix for index used. Sources: World Bank Databank, Eurostat for the European CPI, and UK's Office of National Statistics for the UK's RPI.

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