A compactness theorem for Yamabe metrics

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A well-known corollary of Aubin's work on the Yamabe problem [Aub76a] is the fact that, in a conformal class other than the conformal class of the round sphere, the set of Yamabe metrics is compact.

In these notes we give an exposition of a slight generalization (well-known to experts, but hard to find a published reference for): in a compact set of conformal classes not containing the conformal class of the round sphere, the set of Yamabe metrics is compact.

1 Standard inequalities

We will need versions of various standard inequalities in which the dependence of the constant on the metric is explicit.

Proposition 1.1 (Sobolev inequalities)**.** *Let n, m, i, λ be given.*

1. *Let $q < p < n$ be given. Then there exists $C = C(n, p, q, m, i, \lambda)$,*
2. *Let $p > n$ be given. Then there exists $C = C(n, p, m, i, \lambda)$,*

such that for each

- *complete Riemannian n-manifold (M, g) with injectivity radius at least i and $Ric \geq \lambda g$*
- *smooth function u on M,*

we have

1. *(if $p < n$)*
$$||u||_{\mathcal{W}^{m,\frac{nq}{n-q}},g} \leq C||u||_{\mathcal{W}^{m+1,p},g}.$$

2. *(if $p > n$)*

$$||u||_{\mathcal{C}^m,g} \leq C||u||_{\mathcal{W}^{m+1,p},g}.$$

At least to first order, a sharp version of this Sobolev inequality is also available. It is due to Aubin [Aub76b]. For the form we state here see [Heb96] Theorem 4.6.

For $1 < p < n$, let

$$S(n,p) := n\left(\frac{n-p}{p-1}\right)^{p-1}\left(\frac{\Gamma(n/p)\Gamma(n+1-n/p)\omega_{n-1}}{\Gamma(n+1)}\right)^{p/n},$$

where ω_{n-1} is the volume of the unit $(n-1)$-sphere, so that $S(n,p)$ is the greatest constant such that for compactly supported smooth functions $u \in \mathcal{C}^\infty(\mathbb{R}^n)$,

$$\left(\int_{\mathbb{R}^n} u^{\frac{np}{n-p}}\right)^{\frac{n-p}{n}} \leq \frac{1}{S(n,p)}\int_{\mathbb{R}^n}|du|^p.$$

Proposition 1.2 (Aubin's sharp Sobolev inequality)**.** *Let n, $p < n$, ϵ, i, λ be given. Then there exists $C = C(n,p,\epsilon,i,\lambda)$, such that if (M,g) is a complete Riemannian n-manifold with injectivity radius at least i and $Ric \geq \lambda g$, and u a smooth function on M, then*

$$\left(\int_M u^{\frac{np}{n-p}}\right)^{\frac{n-p}{n}} \leq \frac{1+\epsilon}{S(n,p)}\int_M |du|^p + C\int_M u^p.$$

Proposition 1.3 (L^p estimate)**.** *Let n, p, m, i, A be given. Then there exists $C = C(n,p,m,i,A)$, such that if (M,g) is a complete Riemannian n-manifold with injectivity radius at least i, $||Ric||_{\mathcal{C}^m,g} \leq A$, and u a smooth function on M, then*

$$||u||_{\mathcal{W}^{m+2,p},g} \leq C[||\Delta_g u||_{\mathcal{W}^{m,p},g} + ||u||_{p,g}].$$

Proposition 1.4 (Harnack inequality)**.** *Let n, i, λ, B, D be given. Then there exists $C = C(n,i,\lambda,B,D)$, such that for each*

- *compact Riemannian n-manifold (M,g) with injectivity radius at least i, Ricci curvature $Ric \geq \lambda g$, and diameter at most D,*
- *smooth function u on M with $u \geq 0$ and $|\Delta_g u| \leq B|u|$*

we have

$$\sup_M u \leq C \inf_M u.$$

2 L^r bound on constant-scalar-curvature metrics

We now prove versions of standard bounds on metrics with (small) constant scalar curvature, in which the dependence of the bounds on the metric is made explicit. For the first, an L^r bound, we adapt the presentation of [LP87] Theorem 4.4.

First observe that if $p = 2$ then the constant $S(n,p)$ of the sharp Sobolev inequality Proposition 1.2 has,

$$S(n,2) = n(n-2)\left(\frac{\Gamma(\frac{n}{2})\Gamma(\frac{n}{2}+1)\omega_{n-1}}{\Gamma(n+1)}\right)^{2/n}.$$

By the Gamma duplication identity $\Gamma(\frac{n}{2}+1)\Gamma(\frac{n+1}{2}) = 2^{-n}\sqrt{\pi}\Gamma(n+1)$, and the sphere volume recursion formula $\Gamma(\frac{n+1}{2})\omega_n = \sqrt{\pi}\Gamma(\frac{n}{2})\omega_{n-1}$, we obtain

$$S(n,2) = \tfrac{n(n-2)}{4}\omega_n^{\frac{2}{n}}.$$

By convention we define

$$\Lambda := 4\frac{n-1}{n-2}S(n,2) = n(n-1)\omega_n^{\frac{2}{n}}.$$

Proposition 2.1. *Suppose given i, A, B, η. There exists $r > \frac{2n}{n-2}$, $C > 0$ dependent only on n, i, A, B, η, such that for each*

- *smooth metric g on M such that $Vol(M,g) \leq A$, $injrad(M,g) \geq i$, $Ric(g) \geq -Bg$, $|R(g)| \leq nB$,*
- *smooth positive function φ on M such that $\varphi^{\frac{4}{n-2}}g$ has volume 1 and constant scalar curvature $\lambda \leq \Lambda - \eta$,*

we have the uniform bound

$$||\varphi||_{L^r,g} \leq C.$$

Proof. Let g and φ be as given. Thus the function φ satisfies the scalar curvature equation

$$-4\tfrac{n-1}{n-2}\Delta_g\varphi + R(g)\varphi = \lambda\varphi^{\frac{n+2}{n-2}}.$$

Multiplying by $\varphi^{1+2\delta}$ and integrating (implicitly with respect to dVol_g), we obtain, by Stokes' theorem, the identity

$$\begin{aligned} 4\left(\tfrac{n-1}{n-2}\right)\left(\tfrac{1+2\delta}{(1+\delta)^2}\right)\int_M |d(\varphi^{1+\delta})|_g^2 &= -4\tfrac{n-1}{n-2}\int_M \varphi^{1+2\delta}\Delta_g\varphi \\ &= I([g])\int_M \varphi^{\frac{2n}{n-2}+2\delta} - \int_M R(g)\varphi^{2+2\delta}. \end{aligned}$$

Thus, by our bound on $R(g)$, for some $C(\delta)$

$$4\left(\tfrac{n-1}{n-2}\right)\int_M |d(\varphi^{1+\delta})|^2_{g'} \leq \lambda\frac{(1+\delta)^2}{1+2\delta}\int_M \varphi^{\frac{2n}{n-2}+2\delta} + C\int_M \varphi^{2(1+\delta)}. \tag{1}$$

By the sharp Sobolev inequality Proposition 1.2 applied to the function $\varphi^{1+\delta}$, since we have bounds on $\mathrm{injrad}(M,g)$ and $\inf_M \mathrm{Ric}(g)$, there exists $C = C(n,\epsilon)$ such that

$$\left(\int_M \varphi^{\frac{2n}{n-2}(1+\delta)}\right)^{\frac{n-2}{n}} \leq 4(1+\epsilon)\left(\tfrac{n-1}{n-2}\right)\Lambda^{-1}\int_M |d(\varphi^{1+\delta})|^2_g + C\int_M \varphi^{2(1+\delta)}. \tag{2}$$

By Hölder's inequality, we have

$$\begin{aligned}\int_M \varphi^{\frac{2n}{n-2}+2\delta} &\leq \left(\int_M \varphi^{\frac{2n}{n-2}(1+\delta)}\right)^{\frac{n-2}{n}}\left(\int_M \varphi^{\frac{2n}{n-2}}\right)^{\frac{2}{n}} \\ &= \left(\int_M \varphi^{\frac{2n}{n-2}(1+\delta)}\right)^{\frac{n-2}{n}}, &(3)\\ \int_M \varphi^{2(1+\delta)} &\leq \left(\int_M 1\right)^{1-(1+\delta)\frac{n-2}{2n}}\left(\int_M \varphi^{\frac{2n}{n-2}}\right)^{(1+\delta)\frac{n-2}{n}} \\ &= C(\delta), &(4)\end{aligned}$$

where in each case the second identity is from the condition

$$\int_M \varphi^{\frac{2n}{n-2}} = \mathrm{Vol}(\varphi^{\frac{4}{n-2}}g) = 1,$$

and where the second inequality also uses our bound on $\mathrm{Vol}(M,g)$.

Combining the inequalities (1), (2), (3), (4), we see that for some $C = C(n,\epsilon,\delta)$,

$$\left(\int_M \varphi^{\frac{2n}{n-2}(1+\delta)}\right)^{\frac{n-2}{n}} \leq (1+\epsilon)\frac{(1+\delta)^2}{1+2\delta}\frac{\lambda}{\Lambda}\left(\int_M \varphi^{\frac{2n}{n-2}(1+\delta)}\right)^{\frac{n-2}{n}} + C.$$

Rearranging and using our bound $\Lambda - \lambda \geq \eta$,

$$\left[1-(1+\epsilon)\frac{(1+\delta)^2}{1+2\delta}\left(1-\frac{\eta}{\Lambda}\right)\right]\left(\int_M \varphi^{\frac{2n}{n-2}(1+\delta)}\right)^{\frac{n-2}{n}} \leq C.$$

There exist $\epsilon > 0$, $\delta > 0$, sufficiently small that

$$(1+\epsilon)\frac{(1+\delta)^2}{1+2\delta}\left(1-\frac{\eta}{\Lambda}\right) < 1.$$

Therefore for this ϵ, δ and for $r := \frac{2n}{n-2}(1+\delta) > \frac{2n}{n-2}$,

$$\left(\int_M \varphi^r \mathrm{dVol}_g\right)^{\frac{1}{r}} \leq \left[1 - (1+\epsilon)\frac{(1+\delta)^2}{1+2\delta}\left(1 - \frac{\eta}{\Lambda}\right)\right]^{-1} C \leq C.$$

□

3 Higher-order bounds on constant-scalar-curvature metrics

In this section we follow the arguments of [LP87] Lemma 4.1, with slight modifications to yield sharper bounds.

Lemma 3.1. *For each*

- *smooth metric g on M*
- *smooth positive function φ on M such that $\varphi^{\frac{4}{n-2}}g$ has volume 1 and constant scalar curvature λ,*

we have, $\lambda \geq -\max_M[-R(g)]\, Vol(M,g)^{2/n}$.

Proof. Multiplying the condition on the scalar curvature of $\varphi^{\frac{4}{n-2}}g$,

$$-4\tfrac{n-1}{n-2}\Delta_g\varphi + R(g)\varphi = \lambda\varphi^{\frac{n+2}{n-2}},$$

by φ, integrating, and using Stokes' theorem, we obtain,

$$4\tfrac{n-1}{n-2}\int_M |d\varphi|_g^2 + \int_M R(g)\varphi^2 = \lambda\int_M \varphi^{\frac{2n}{n-2}}.$$

Rearrange and apply Hölder's inequality:

$$(-\lambda)\int_M \varphi^{\frac{2n}{n-2}} \leq \max_M[-R(g)]\int_M \varphi^2 \leq \max_M[-R(g)]\,\mathrm{Vol}(M,g)^{2/n}\left(\int_M \varphi^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}.$$

Since $\varphi^{\frac{4}{n-2}}g$ has volume 1, $\int_M \varphi^{\frac{2n}{n-2}} = 1$. The result follows. □

Proposition 3.2. *Suppose given $p > 1$, i, A, B, η. There exists $C > 0$ dependent only on n, p, i, A, B, η, such that for each*

- *smooth metric g on M such that $Vol(M,g) \leq A$, injrad$(M,g) \geq i$, $|Ric(g)| \leq B$,*

- *smooth positive function φ on M such that $\varphi^{\frac{4}{n-2}}g$ has volume 1 and constant scalar curvature $\lambda \leq \Lambda - \eta$,*

we have the uniform bound

$$||\varphi||_{\mathcal{W}^{2,p},g} \leq C.$$

Proof. We prove this by an induction-type argument on p. In this proof C denotes a constant dependent on n, i, A, B, η; and C([other quantities]) denotes a constant dependent on those quantities and on the other quantities specified. We will show that for each $p > 1$,

$$||\varphi||_{\mathcal{W}^{2,p},g} \leq C(p).$$

Let $S = \max(\Lambda - \eta, nBA^{2/n})$. By Lemma 3.1, $|\lambda| \leq S$.

For each $p \geq 1$, by our bounds on $\mathrm{Vol}(M, g)$ and $\mathrm{Ric}(g)$ and by Hölder's inequality, the scalar curvature equation

$$-4\tfrac{n-1}{n-2}\Delta_g\varphi + R(g)\varphi = \lambda\varphi^{\frac{n+2}{n-2}}.$$

implies the uniform bounds

$$\begin{aligned}
||\varphi||_{p,g} &\leq \mathrm{Vol}(M,g)^{\frac{4}{(n+2)p}}||\varphi||_{\frac{n+2}{n-2}p,g} \\
&\leq C||\varphi||_{\frac{n+2}{n-2}p,g}; \\
||\Delta_g\varphi||_{p,g} &\leq \frac{n-2}{4(n-1)}\left[||R(g)||_{\infty,g}||\varphi||_{p,g} + |\lambda|||\varphi^{\frac{n+2}{n-2}}||_{p,g}\right] \\
&\leq \frac{n-2}{4(n-1)}\left[||R(g)||_{\infty,g}\mathrm{Vol}(M,g)^{\frac{4}{(n+2)p}}||\varphi||_{\frac{n+2}{n-2}p,g} + S||\varphi||^{\frac{n+2}{n-2}}_{\frac{n+2}{n-2}p,g}\right] \\
&\leq C\left(||\varphi||_{\frac{n+2}{n-2}p,g}\right).
\end{aligned}$$

Therefore for each $p > 1$, by the L^p elliptic estimate Proposition 1.3 (which is available by our bounds on $\mathrm{injrad}(M, g)$ and $|\mathrm{Ric}(g)|$),

$$||\varphi||_{\mathcal{W}^{2,p},g} \leq C(p)\left[||\Delta_g\varphi||_{p,g} + ||\varphi||_{p,g}\right] \leq C\left(p, ||\varphi||_{\frac{n+2}{n-2}p,g}\right).$$

By Proposition 2.1 there exist $r > \frac{2n}{n-2}$ and C, both with the appropriate dependence, such that

$$||\varphi||_{r,g} \leq C.$$

So, as a base case,

$$||\varphi||_{\mathcal{W}^{2,\frac{n-2}{n+2}r},g} \leq C\left(\tfrac{n-2}{n+2}r\right).$$

Moreover, by the Sobolev inequality Proposition 1.1 (which is available by our bounds on $\mathrm{injrad}(M, g)$ and $\inf_M \mathrm{Ric}(g)$), we have,

1. if $q < n/2$, then for each p satisfying

$$1 < p < \left(\frac{n-2}{n+2}\right)\left(\frac{nq}{n-2q}\right),$$

we have

$$||\varphi||_{\mathcal{W}^{2,p},g} \leq C\left(p, ||\varphi||_{\frac{n+2}{n-2}p,g}\right) \leq C\left(p, ||\varphi||_{\mathcal{W}^{2,q},g}\right);$$

2. if $q > n/2$, then for each $p > 1$ (arbitrarily large), we have

$$||\varphi||_{\mathcal{W}^{2,p},g} \leq C\left(p, ||\varphi||_{\frac{n+2}{n-2}p,g}\right) \leq C\left(p, ||\varphi||_{\mathcal{C}^0,g}\right) \leq C\left(p, ||\varphi||_{\mathcal{W}^{2,q},g}\right);$$

Thus it suffices to observe that if $r > \frac{2n}{n-2}$ then the sequence

$$\frac{n-2}{n+2}r, \left(\frac{n-2}{n+2}\right)\left(\frac{n\left(\frac{n-2}{n+2}r\right)}{n-2\left(\frac{n-2}{n+2}r\right)}\right), \ldots$$

1. stays greater than 1; and,
2. eventually becomes greater than $n/2$.

□

Lemma 3.3. *Suppose given i, D, B, η. There exists $C > 0$ dependent only on n, i, D, B, η, such that for each*

- *smooth metric g on M such that diam$(M, g) \leq D$, injrad$(M, g) \geq i$, $|Ric(g)| \leq B$,*
- *smooth positive function φ on M such that $\varphi^{\frac{4}{n-2}}g$ has volume 1 and constant scalar curvature $\lambda \leq \Lambda - \eta$,*

we have the uniform bounds

$$||\varphi||_{\mathcal{C}^0} \leq C, \quad ||\varphi^{-1}||_{\mathcal{C}^0} \leq C.$$

Proof. In this proof

- C denotes constants dependent on n, i, D, B, η,
- g is a smooth metric on M such that $\operatorname{diam}(M, g) \leq D$, $\operatorname{injrad}(M, g) \geq i$, $|\operatorname{Ric}(g)| \leq B$,

- φ is a smooth positive function on M such that $\varphi^{\frac{4}{n-2}} g$ has volume 1 and constant scalar curvature $\lambda \leq \Lambda - \eta$.

Diameter and Ricci bounds imply a volume bound, so the hypotheses of Proposition 3.2 are satisfied. By the same argument as in Proposition 3.2, $|\lambda| \leq C$.

For the first inequality, choose $p > \frac{n}{2}$. By the Sobolev inequality Proposition 1.1 and by Proposition 3.2,

$$||\varphi||_{\mathcal{C}^0} \leq C||\varphi||_{\mathcal{W}^{2,p},g} \leq C(p).$$

This moreover shows

$$||\tfrac{n-2}{4(n-1)}\left(R(g) - \lambda\varphi^{\frac{4}{n-2}}\right)||_{\mathcal{C}^0} \leq C,$$

so by the Yamabe equation

$$-4\tfrac{n-1}{n-2}\Delta_g\varphi + R(g)\varphi = \lambda\varphi^{\frac{n+2}{n-2}}.$$

we obtain C with only the given dependence such that $|\Delta_g\varphi| \leq C|\varphi|$. Thus the Harnack inequality Proposition 1.4 (applicable due to our bounds on $\operatorname{diam}(M,g)$, $\operatorname{injrad}(M,g)$ and $\operatorname{Ric}(g)$) implies

$$||\varphi^{-1}||_{\mathcal{C}^0} \leq C||\varphi||_{\mathcal{C}^0}^{-1}.$$

We conclude by using Hölder's inequality and the volume condition $||\varphi||_{\frac{2n}{n-2},g} = 1$ to bound $||\varphi||_{\mathcal{C}^0}$ from below:

$$||\varphi||_{\mathcal{C}^0}^{-1} \leq \operatorname{Vol}(M,g)^{\frac{n-2}{2n}}||\varphi||_{\frac{2n}{n-2},g}^{-1} \leq C.$$

□

Theorem 3.4. *Suppose given $m \geq 2$, $p > n$, i, D, B, η. There exists $C > 0$ dependent only on n, m, p, i, D, B, η, such that for each*

- *smooth metric g on M such that $diam(M,g) \leq D$, $injrad(M,g) \geq i$, $||Ric(g)||_{\mathcal{C}^{m-2},g} \leq B$,*

- *smooth positive function φ on M such that $\varphi^{\frac{4}{n-2}} g$ has volume 1 and constant scalar curvature $\lambda \leq \Lambda - \eta$,*

we have the uniform bound

$$||\varphi||_{\mathcal{W}^{m,p},g} \leq C.$$

Proof. We prove this by induction on m. In this proof

- C denotes a constant dependent on n, i, D, B, η,
- $C([\text{other quantities}])$ denotes a constant dependent on those quantities and on the [other quantities] specified,
- g denotes a smooth metric on M such that $\mathrm{diam}(M,g) \leq D$, $\mathrm{injrad}(M,g) \geq i$,
- φ is a smooth positive function on M such that $\varphi^{\frac{4}{n-2}}g$ has volume 1 and constant scalar curvature $\lambda \leq \Lambda - \eta$.

By the same argument as in Proposition 3.2, $|\lambda| \leq C$. We will show that for each $m \geq 2$ and $p > n$, if $||\mathrm{Ric}(g)||_{\mathcal{C}^{m-2}} \leq B$ then

$$||\varphi||_{\mathcal{W}^{m,p},g} \leq C(m,p).$$

The $m = 2$ case is Proposition 3.2. (Bounds on diameter and Ricci curvature imply a bound on volume, so the assumptions of Proposition 3.2 are strictly stronger than our assumptions here.)

Suppose the result is known for $m-1$.

If now $||\mathrm{Ric}(g)||_{\mathcal{C}^{m-2},g} \leq B$, then by our bounds on $\mathrm{Vol}(M,g)$ and $||R(g)||_{\mathcal{C}^{m-2},g}$ and by Hölder's inequality, the scalar curvature equation

$$-4\tfrac{n-1}{n-2}\Delta_g\varphi + R(g)\varphi = \lambda\varphi^{\frac{n+2}{n-2}}.$$

implies the uniform bound

$$\begin{aligned}
||\Delta_g\varphi||_{\mathcal{W}^{m-2,p},g} &\leq \frac{n-2}{4(n-1)}C(m)\left[||R(g)||_{\mathcal{C}^{m-2},g}||\varphi||_{\mathcal{W}^{m-2,p},g} + |\lambda|||\varphi^{\frac{n+2}{n-2}}||_{\mathcal{W}^{m-2,p},g}\right] \\
&\leq C(m)||\varphi||_{\mathcal{W}^{m-2,p},g} \\
&\quad + C(m)\sum_{i=0}^{m-2}\sum_{a_1+\cdots+a_r=i}||\varphi^{\frac{n+2}{n-2}-r}\nabla^{a_1}\varphi\cdots\nabla^{a_r}\varphi||_{p,g} \\
&\leq C(m)||\varphi||_{\mathcal{W}^{m-2,p},g} \\
&\quad + C(m)\mathrm{Vol}(M,g)^{\frac{1}{p}}\sum_{i=0}^{m-2}\sum_{a_1+\cdots+a_r=i}||\varphi^{\frac{n+2}{n-2}-r}\nabla^{a_1}\varphi\cdots\nabla^{a_r}\varphi||_{\mathcal{C}^0,g} \\
&\leq C(m)||\varphi||_{\mathcal{W}^{m-2,p},g} \\
&\quad + C(m)\sum_{i=0}^{m-2}\sum_{a_1+\cdots+a_r=i}||\varphi^{\frac{n+2}{n-2}-r}||_{\mathcal{C}^0}||\nabla^{a_1}\varphi||_{\mathcal{C}^0,g}\cdots||\nabla^{a_r}\varphi||_{\mathcal{C}^0,g}.
\end{aligned}$$

Since, for each r,

$$||\varphi^{\frac{n+2}{n-2}-r}||_{\mathcal{C}^0} \leq \max\left(||\varphi||_{\mathcal{C}^0}^{\frac{n+2}{n-2}-r}, ||\varphi^{-1}||_{\mathcal{C}^0}^{r-\frac{n+2}{n-2}}\right),$$

this yields,

$$\begin{aligned} ||\Delta_g\varphi||_{\mathcal{W}^{m-2,p},g} &\leq C\left(m, ||\varphi||_{\mathcal{W}^{m-2,p},g}, ||\varphi||_{\mathcal{C}^{m-2},g}, ||\varphi^{-1}||_{\mathcal{C}^0}\right) \\ &\leq C(m,p), \end{aligned}$$

where the final inequality comes from combining

- Lemma 3.3 that $||\varphi^{-1}||_{\mathcal{C}^0} \leq C$;
- the Sobolev inequality Proposition 1.1 that $||\varphi||_{\mathcal{C}^{m-2},g} \leq ||\varphi||_{\mathcal{W}^{m-1,p},g}$;
- the inductive hypothesis $||\varphi||_{\mathcal{W}^{m-1,p},g} \leq C(m-1,p)$.

So, by the L^p elliptic estimate Proposition 1.3 (applicable by our bounds on $\mathrm{injrad}(M,g)$ and $||\mathrm{Ric}(g)||_{\mathcal{C}^{m-2},g}$),

$$||\varphi||_{\mathcal{W}^{m,p},g} \leq C\left[||\Delta_g\varphi||_{\mathcal{W}^{m-2,p},g} + ||\varphi||_{p,g}\right] \leq C.$$

□

4 Compactness

Lemma 4.1. *I is upper semi-continuous.*

Proof. It is the infimum of a continuous functional. □

Theorem 4.2. *Let $m \geq 3$. Let (M,c) be a conformal manifold with $I(c) < \Lambda$. Let (c_k) be a sequence of smooth conformal classes on M which $\mathcal{C}^m$-converges to c, and let (g_k) be volume-1 Yamabe metrics for the classes (c_k).*

Then there exists a subsequence (g_{k_i}) which $\mathcal{C}^{m-1}$-converges to a volume-1 Yamabe metric for c.

Proof of Theorem 4.2. Choose $p > n$, and choose representatives g_k of the classes c_k with $g_k \to g$ in the $\mathcal{C}^m$ topology. For sufficiently large k,

1. $I(c_k) \leq \frac{1}{2}[I(c) + \Lambda]$ (by Lemma 4.1);
2. there are uniform bound on the expressions

$$\mathrm{diam}(M, g_k) \quad \mathrm{injrad}(M, g_k), \quad ||\mathrm{Ric}(g_k)||_{\mathcal{C}^{m-2},g_k};$$

3. there exists a uniform C such that for k sufficiently large,

$$C^{-1}||\cdot||_{\mathcal{W}^{m+1,p},g_k} \leq ||\cdot||_{\mathcal{W}^{m+1,p},g} \leq C||\cdot||_{\mathcal{W}^{m+1,p},g_k}.$$

Therefore, by Proposition 3.4, if φ_k are smooth positive functions on M such that $\varphi_k^{\frac{4}{n-2}} g_k$ are volume-1 Yamabe metrics for c_k, then we have the uniform bound $||\varphi_k||_{\mathcal{W}^{m,p},g} \leq C$, hence by Morrey's inequality the uniform bound $||\varphi_k||_{\mathcal{C}^{m-1,1-\frac{n}{p}},g} \leq C$, and so there exists a subsequence (k_i) such that φ_{k_i} $\mathcal{C}^{m-1}$-converges.

It remains to be checked that the limit, φ, makes $\varphi^{\frac{4}{n-2}} g$ a Yamabe metric for c. Indeed, $\varphi_k^{\frac{4}{n-2}} g_k$ all have volume 1 and constant scalar curvature $I(c_k)$. So their $\mathcal{C}^{m-1}$-limit $\varphi^{\frac{4}{n-2}} g$ has volume 1, and constant scalar curvature $\lim_{k\to\infty} I(c_k)$, which is $\leq I(c)$ by Lemma 4.1; this is a contradiction unless equality holds and g is a Yamabe metric. □

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