Explicit constants for Riemannian inequalities

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Abstract

We prove versions of various standard inequalities in which the dependence of the constant on the metric is explicit.

1 Technical results

Definition. Let (M, g) be a smooth Riemannian n-manifold and let $x \in M$. Given $Q > 1$, $k \in \mathbb{N}$, and $p > n$, the (Q, k, p)*-harmonic radius at* x, $r_H(Q, k, p)(x)$, is the supremum of reals r such that, on the geodesic ball $B_x(r)$ of center x and radius r, there is a harmonic co-ordinate chart such that if g_{ij} are the components of g in these co-ordinates, then

1. $Q^{-1}\delta_{ij} \leq g_{ij} \leq Q\delta_{ij}$ as bilinear forms;
2. $\sum_{1\leq|\beta|\leq k} r^{|\beta|-n/p}||\partial_\beta g_{ij}||_{L^p} \leq Q - 1$.

The (Q, k, p)*-harmonic radius of* M is

$$r_H(Q, k, p)(M) := \inf_{x\in M} r_H(Q, k, p)(x).$$

Theorem 1.1 ([HH97], Theorem 11)**.** *Let $n \in \mathbb{N}$, $Q > 1$, $p > n$, $i > 0$. Suppose (M, g) is a Riemannian n-manifold with injrad$(M, g) \geq i$.*

1. *Let $\lambda \in \mathbb{R}$. There exists $C = C(n, Q, p, i, \lambda)$, such that if*

   $$Ric \geq \lambda g,$$

   then the harmonic radius $r_H(Q, 1, p)(M)$ is $\geq C$.

2. *Let $k \geq 2$, and let $(C(j))_{0\leq j\leq k-2}$ be positive constants. There exists $C = C(n, Q, p, i, (C(j))_{0\leq j\leq k-2})$, such that if for each $0 \leq j \leq k - 2$ we have*

   $$|\nabla^j Ric| \leq C(j),$$

   then the harmonic radius $r_H(Q, k, p)(M)$ is $\geq C$.

Lemma 1.2 ([Heb96], Lemma 1.6). *Let $n \in \mathbb{N}$, let $\lambda \in \mathbb{R}$, and let $r \geq \rho > 0$. Let (M, g) be a complete Riemannian n-manifold with $Ric \geq \lambda g$. Then there exist $N = N(n, \lambda, \rho, r)$, and an (at most) countable set (x_i) of points in M, such that*

1. *the family $(B(x_i, \rho))$ covers M;*
2. *each point in M is contained in at most $N(n, \lambda, \rho, r)$ balls of the family $(B(x_i, r))$.*

Let $U \subseteq \mathbb{R}^n$ be an open set, g a Riemannian metric on u, ∇ its Levi-Civita connection, and Γ the Christoffel symbols of g on the co-ordinate patch U. We write D for the (Euclidean) derivative on U.

In the following we denote by S a multilinear map

$$S : T^{a_1}_{b_1}(\mathbb{R}^n) \times T^{a_2}_{b_2}(\mathbb{R}^n) \times \ldots \times T^{a_r}_{b_r}(\mathbb{R}^n) \to T^a_b(\mathbb{R}^n),$$

composed of sums of traces (so $a - b = \sum a_i - \sum b_i$). It is to be understood that the particular map S, and the a_i's, b_i's, a and b determining its domain and range, may vary from use to use and from line to line, but are independent of the choice of g, and of the choice of A (in (1)) or of u (in (2)).

Lemma 1.3. 1. *Let $k \in \mathbb{N}$. For all covariant tensors $A : U \to T^k(\mathbb{R}^n)$,*

$$\nabla A = DA + S(\Gamma, A).$$

2. *Let $m \in \mathbb{N}$. For all functions $u : U \to \mathbb{R}$,*

$$(\nabla)^m u = D^m u + \sum_{k=1}^{m-1} \sum_{r=1}^{m-k} \sum_{\substack{a_1 \geq \cdots \geq a_r \geq 0, \\ a_1 + \cdots + a_r = m-k-r}} S(D^{a_1}\Gamma, \ldots D^{a_r}\Gamma, D^k u).$$

Proof. 1. For any covariant tensor A,

$$(\nabla A)(\partial_j, \partial_{i_1}, \ldots \partial_{i_k}) = \partial_j \left(A(\partial_{i_1}, \ldots \partial_{i_k})\right) - \sum_r \Gamma^a_{j i_r} A(\partial_{i_1}, \ldots \partial_a, \ldots \partial_{i_k})$$

2. For $m = 0, 1$ these are the identities

$$u = u, \quad \nabla u = du.$$

Thenceforth we proceed by induction. Suppose this is known for some m. Then

$$(\nabla)^{m+1}u = \nabla(D^m u) + \sum_{k=1}^{m-1}\sum_{r=1}^{m-k}\sum_{\substack{a_1\geq\cdots\geq a_r\geq 0,\\ a_1+\cdots+a_r=m-k-r}} \nabla\left(S(D^{a_1}\Gamma,\dots D^{a_r}\Gamma, D^k u)\right),$$

and we may calculate

$$\begin{aligned}
\nabla(D^m u) &= D^{m+1}u + S(\Gamma, D^m u),\\
\nabla\left(S(D^{a_1}\Gamma,\dots D^{a_r}\Gamma, D^k u)\right) &= \sum_{i=1}^{r} S(D^{a_1}\Gamma,\dots D^{a_i+1},\dots D^{a_r}\Gamma, D^k u)\\
&\quad + S(D^{a_1}\Gamma,\dots D^{a_r}\Gamma, D^{k+1}u)\\
&\quad + S(D^{a_1}\Gamma,\dots D^{a_r}\Gamma, \Gamma, D^k u).
\end{aligned}$$

The result follows.

□

2 Sobolev estimates

Theorem 2.1 (Sobolev estimate, Gilbarg-Trudinger [GT01] 7.10 & 7.25)**.** *For each n, $p \neq n$, and bounded domain V with $\mathcal{C}^1$ boundary, there exists $C = C(n,p,V)$, such that for all functions $u \in \mathcal{W}^{1,p}(V)$, we have*

1. *if $p < n$,*
$$||u||_{\frac{np}{n-p},V} \leq C||u||_{\mathcal{W}^{1,p};V}.$$

2. *if $p > n$,*
$$\sup_V |u| \leq C||u||_{\mathcal{W}^{1,p};V}$$

Lemma 2.2 (Local Sobolev estimate)**.** *Let n, m, i, λ be given.*

1. *Let $q < p < n$ be given. Then there exists $r = r(n,p,q,m,i,\lambda) < i$ and $C = C(n,p,q,m,i,\lambda)$,*

2. *Let $p > n$ be given. Then there exists $r = r(n,p,m,i,\lambda) < i$ and $C = C(n,p,m,i,\lambda)$,*

such that for each

- *complete Riemannian n-manifold (M,g) with injectivity radius at least i and $Ric \geq \lambda g$*

- *point $x \in M$*
- *smooth covariant m-tensor A on $B(x,r)$,*

we have

1. *(if $p < n$)*

$$||A||^p_{\frac{nq}{n-q},g,B(x,r)} \leq C\left[||\nabla A||^p_{p,g,B(x,r)} + ||A||^p_{p,g,B(x,r)}\right].$$

2. *(if $p > n$)*

$$\left(\sup_{B(x,r)} |A|_g\right)^p \leq C\left[||\nabla A||^p_{p,g,B(x,r)} + ||A||^p_{p,g,B(x,r)}\right].$$

Proof. If $p > n$, let $q := \frac{1}{2}(n+p)$. Then, either way, let

$$s = \frac{pq}{p-q},$$

and choose r to be less than i and less than the $C(n, Q := 1, s, i, \lambda)$ of Theorem 1.1 (1), so that the harmonic radius $r_H(Q := 1, 1, s)$ is greater than r. We therefore have uniform bounds in terms of n, i, λ, p, and (if $p < n$) q on the co-ordinate norms $||g||_{s,B(x,r)}$, $||g^{-1}||_{s,B(x,r)}$, $||\Gamma||_{s,B(x,r)}$.

Write B for $B(x,r)$ throughout. Let u be a function on B, with $u \in \mathcal{W}^{m+1,p}_{loc}(B) \cap L^p(B)$.

By our bounds on the components of the tensor g, the metrics g and g_{eucl} are comparable, so the norms $||\cdot||_{p,g,B}$ and $||\cdot||_{p,B}$ are comparable, the norms $||\cdot||_{\frac{nq}{n-q}p,g,B}$ and $||\cdot||_{\frac{nq}{n-q},B}$ are comparable, and the pointwise tensor norms $|\cdot|_g$ and $|\cdot|$ are comparable. It therefore it suffices to prove the inequalities with the latter norms.

Applying the Sobolev inequality Theorem 2.1 co-ordinatewise and combining, we have $C = C(n,m,q)$, such that

1. (if $p < n$)
$$||A||^q_{\frac{nq}{n-q},B} \leq C\left[||DA||^q_{q,B} + ||A||^q_{q,B}\right].$$
2. (if $p > n$)
$$\left(\sup_B |A|\right)^q \leq C\left[||DA||^q_{q,B} + ||A||^q_{q,B}\right].$$

By Lemma 1.3 (1) and the power means inequalities,

$$\begin{aligned}\left[||DA||^q_{q,B}+||A||^q_{q,B}\right]^{\frac{p}{q}} &\leq 2^{\frac{p}{q}-1}\left[||DA||^p_{q,B}+||A||^p_{q,B}\right]\\ &\leq 2^{\frac{p}{q}-1}\left[(||\nabla A||_{q,B}+||S(\Gamma,A)||_{q,B})^p+||A||^p_{q,B}\right]\\ &\leq 2^{\frac{p}{q}-1}\left[2^{p-1}\left(||\nabla A||^p_{q,B}+||S(\Gamma,A)||^p_{q,B}\right)+||A||^p_{q,B}\right].\end{aligned}$$

By Hölder's inequality, for $C=C(n,m)$,

$$\begin{aligned}||\nabla A||_{q,B} &\leq C||1||_{s,B}||\nabla A||_{p,B}\\ ||S(\Gamma,A)||_{q,B} &\leq C||\Gamma||_{s,B}||A||_{p,B}\\ ||A||_{q,B} &\leq C||1||_{s,B}||A||_{p,B}\end{aligned}$$

The terms other than A, ∇A in these right-hand sides are controlled by construction. The result follows.

□

Proposition 2.3 (Sobolev inequalities)**.** *Let n, m, i, λ be given.*

1. *Let $q<p<n$ be given. Then there exists $C=C(n,p,q,m,i,\lambda)$,*
2. *Let $p>n$ be given. Then there exists $C=C(n,p,m,i,\lambda)$,*

such that for each

- *complete Riemannian n-manifold (M,g) with injectivity radius at least i and $Ric\geq\lambda g$*
- *smooth function u on M,*

we have

1. *(if $p<n$)*
$$||u||_{\mathcal{W}^{m,\frac{nq}{n-q}},g}\leq C||u||_{\mathcal{W}^{m+1,p},g}.$$
2. *(if $p>n$)*
$$||u||_{\mathcal{C}^m,g}\leq C||u||_{\mathcal{W}^{m+1,p},g}.$$

Proof. Choose r and C from the local Sobolev estimate Lemma 2.2.

By Lemma 1.2, there exist $N=N(n,A,r,r)$ and an (at most) countable set (x_α) of points in M, such that each point in M is contained in at least one and at most N balls of the family $(B(x_\alpha,r))$. Let χ_α be the characteristic function of $B(x_\alpha,r)$.

By Minkowski's inequality (that is, the triangle inequality), we have,

$$\begin{aligned} ||A||^p_{\frac{nq}{n-q}} &\le ||\sum_\alpha \chi_\alpha |A|^p||_{\frac{nq}{p(n-q)}} \\ &\le \sum_\alpha ||\chi_\alpha |A|^p||_{\frac{nq}{p(n-q)}} \\ &= \sum_\alpha ||A||^p_{\frac{nq}{n-q},B_\alpha} \end{aligned}$$

By the local Sobolev inequality Lemma 2.2,

$$||A||^p_{\frac{nq}{n-q},B_\alpha} \le C\left[\int_{B_\alpha} |\nabla A|^p + \int_{B_\alpha} |A|^p\right]$$

So, since each point is in at most N of the B_α's,

$$\begin{aligned} ||A||^p_{\frac{nq}{n-q}} &\le C\sum_\alpha \left[\int_{B_\alpha} |\nabla A|^p + \int_{B_\alpha} |A|^p\right] \\ &\le NC\left[\int_M |\nabla A|^p + \int_M |A|^p\right]. \end{aligned}$$

Similarly, if $p > n$, by the local Sobolev inequality Lemma 2.2,

$$\begin{aligned} \left(\sup_M |A|_g\right)^p &= \max_\alpha \left(\sup_{B_\alpha} |A|_g\right)^p \\ &\le C\max_\alpha \left[\int_{B_\alpha} |\nabla A|^p + \int_{B_\alpha} |A|^p\right] \\ &\le C\left[\int_M |\nabla A|^p + \int_M |A|^p\right]. \end{aligned}$$

Now, applying these inequalities simultaneously to the covariant tensors $A = \nabla^i u$, for each $0 \le i \le m$, and summing, we obtain, as required,

1. (if $p < n$)

$$\begin{aligned} \sum_{i=0}^m ||\nabla^i u||^{\frac{nq}{n-q}}_{\frac{nq}{n-q},g} &\le C\left(\sum_{i=0}^m ||\nabla^i u||^p_{\frac{nq}{n-q},g}\right)^{\frac{nq}{p(n-q)}} \\ &\le C\left(\sum_{i=0}^{m+1} ||\nabla^i u||^p_{p,g}\right)^{\frac{nq}{p(n-q)}}. \end{aligned}$$

2. (if $p > n$)

$$\begin{aligned}\sum_{i=0}^{m} \sup_M |\nabla^i u|_g &\le C\left(\sum_{i=0}^{m}\left(\sup_M |\nabla^i u|_g\right)^p\right)^{\frac{1}{p}} \\ &\le C\left(\sum_{i=0}^{m+1} ||\nabla^i u||_{p,g}^p\right)^{1/p}.\end{aligned}$$

□

3 Elliptic estimates

Theorem 3.1 (L^p estimates, Gilbarg-Trudinger [GT01] 9.11, modified)**.** *For each n, p, m, U, $V \subset\subset U$, λ, Λ, $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ increasing, there exists $C = C(n, p, U, V, \lambda, \Lambda, \mu)$, such that if*

$$Lu := a^{ij}\partial_i\partial_j u + b^i\partial_i u$$

satisfies

- *For all $x \in U$ and $\xi \in \mathbb{R}^n$, $a^{ij}(x)\xi_i\xi_j \ge \lambda|\xi|^2$;*
- $||a^{ij}||_{m,\infty}, ||b^i||_{m,\infty} \le \Lambda$*;*
- *For all $x, y \in U$, $a^{ij}(x) - a^{ij}(y) \le \mu(|x - y|)$*

then for all functions $u \in \mathcal{W}^{m+2,p}_{loc}(U) \cap L^p(U)$,

$$||u||_{m+2,p;V} \le C(||Lu||_{m,p;U} + ||u||_{p;U}).$$

Lemma 3.2 (Local elliptic estimate)**.** *Let n, p, m, i, A be given. Then there exists $r = r(n, p, m, i, A) < i$ and $C = C(n, p, m, i, A)$, such that for each*

- *complete Riemannian n-manifold (M, g) with injectivity radius at least i and $||Ric||_{\mathcal{C}^m,g} \le A$,*
- *point $x \in M$,*
- *smooth function u on $B(x, r)$,*

we have

$$\sum_{i=0}^{m+2} ||(\nabla)^i u||^p_{p;g,B(x,r/2)} \leq C \left[\left(\sum_{i=0}^{m} ||(\nabla)^i \Delta_g u||^p_{p;g,B(x,r)} \right) + ||u||^p_{p;g,B(x,r)} \right]$$

Proof. Let $q > n$ be arbitrary. Choose r to be $< i$ and less than or equal to the $C(n, Q := 1, q, i, (A)_{0 \leq i \leq m})$ of Theorem 1.1 (2), so that the harmonic radius $r_H(Q := 1, m+2, q)$ is at least r. By the Sobolev estimate Theorem 2.1 (2), we therefore have uniform bounds in terms of n, m, i, A on $||g||_{\infty,B(x,r)}$, $||g^{-1}||_{\infty,B(x,r)}$, $||D(g^{-1})||_{\infty,B(x,r)}$, $||\Gamma||_{m,\infty,B(x,r)}$.

For shorthand we write B_1 for $B(x, r/2)$ and B_2 for $B(x, r)$. Let u be a function on B_2, with $u \in \mathcal{W}^{m+2,p}_{loc}(B_2) \cap L^p(B_2)$.

Since (by our bounds on the components of the tensor g) the metrics g and g_{eucl} are comparable, the norms $|| \cdot ||_{p,g,B_1}$ and $|| \cdot ||_{p,B_1}$ are comparable and the norms $|| \cdot ||_{p,g,B_2}$ and $|| \cdot ||_{p,B_2}$ are comparable. It therefore suffices to prove the inequality with the latter norms.

By Lemma 1.3 (2),

$$\sum_{i=0}^{m+2} ||(\nabla)^i u||^p_{p;B_1} \leq C \left(\sum_{i=0}^{m} ||D^i \Gamma||_{\infty;B_1} \right) \sum_{i=0}^{m+2} ||D^i u||^p_{p;B_1}$$

$$\sum_{i=0}^{m} ||D^i \Delta_g u||^p_{p;B_2} \leq C \left(\sum_{i=0}^{m-2} ||D^i \Gamma||_{\infty;B_2} \right) \sum_{i=0}^{m} ||(\nabla)^i \Delta_g u||^p_{p;B_2}$$

(where if $m - 2 < 0$ the second inequality is simply an identity).

Also, applying the L^p estimate of Theorem 3.1 with the operator

$$Lu = \Delta_g u = g^{ij} \partial_i \partial_j u + g^{ij} \Gamma^k_{ij} \partial_k u$$

shows: there exists $C = C(n, p, m, r, ||g||_{m,\infty,B_2}, ||g^{-1}||_{m,\infty,B_2}, ||\Gamma||_{m,\infty,B_2})$ such that

$$\begin{aligned} \sum_{i=0}^{m+2} ||D^i u||^p_{p;B_1} &\leq C \left[\left(\sum_{i=0}^{m} ||D^i \Delta_g u||^p_{p;B_2} \right)^{\frac{1}{p}} + ||u||_{p,B_2} \right]^p \\ &\leq 2^{p-1} C \left(\sum_{i=0}^{m} ||D^i \Delta_g u||^p_{p;B_2} + ||u||^p_{p,B_2} \right). \end{aligned}$$

Combining these three inequalities gives the result. □

Proposition 3.3 (L^p estimate)**.** *Let n, p, m, i, A be given. Then there exists $C = C(n, p, m, i, A)$, such that if (M, g) is a complete Riemannian n-manifold with injectivity radius at least i, $||Ric||_{\mathcal{C}^m,g} \leq A$, and u a smooth function on M, then*

$$||u||_{\mathcal{W}^{m+2,p},g} \leq C[||\Delta_g u||_{\mathcal{W}^{m,p},g} + ||u||_{p,g}].$$

Proof. Choose r and C (dependent on n, p, m, i, A) as in the local elliptic estimate Lemma 3.2.

By Lemma 1.2 (setting $\rho = \frac{1}{2}r$), there exists $N = N(n, A, \frac{1}{2}r, r)$ and an (at most) countable set (x_α) of points in M, such that

1. the family $(B(x_\alpha, \frac{1}{2}r))$ covers M;
2. each point in M is contained in at most N balls of the family $(B(x_i, r))$.

So, by the local elliptic estimate Lemma 3.2,

$$\begin{aligned}
||u||^p_{\mathcal{W}^{m+2,p},g} &\leq \sum_{i=0}^{m+2} \sum_{\alpha} ||(\nabla)^i u||^p_{p;g,B(x_\alpha,r/2)} \\
&\leq C \sum_{\alpha} \left[\left(\sum_{i=0}^{m} ||(\nabla)^i \Delta_g u||^p_{p;g,B(x_\alpha,r)} \right) + ||u||^p_{p;g,B(x_\alpha,r)} \right] \\
&\leq NC[||\Delta_g u||^p_{\mathcal{W}^{m,p},g} + ||u||^p_{p,g}] \\
&\leq 2NC[||\Delta_g u||_{\mathcal{W}^{m,p},g} + ||u||_{p,g}]^p.
\end{aligned}$$

□

4 Moser's Harnack inequality

Theorem 4.1 (Harnack inequality, Gilbarg-Trudinger [GT01] 8.21)**.** *For each n, r, λ, Λ, there exists $C = C(n, r, \lambda, \Lambda)$, such that if the operator L on $\mathcal{W}^{1,2}(B_{4r})$),*

$$Lu := \partial_i(a^{ij}\partial_j u) + cu,$$

satisfies

- *For all $x \in U$ and $\xi \in \mathbb{R}^n$, $a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$;*
- *$||a^{ij}||_\infty, ||c||_\infty \leq \Lambda$;*

then for all functions $u \in \mathcal{W}^{1,2}(B_{4r})$ *with* $u \geq 0$ *and* $Lu = 0$,

$$\sup_{B_r} u \leq C \inf_{B_r} u.$$

Proposition 4.2 (Harnack inequality)**.** *Let* n, i, λ, B, D *be given. Then there exists* $C = C(n, i, \lambda, B, D)$, *such that for each*

- *compact Riemannian* n*-manifold* (M, g) *with injectivity radius at least* i, *Ricci curvature* $Ric \geq \lambda g$, *and diameter at most* D,
- *smooth function* u *on* M *with* $u \geq 0$ *and* $|\Delta_g u| \leq B|u|$

we have

$$\sup_{M} u \leq C \inf_{M} u.$$

Proof. Let $p > n$ be arbitrary; choose $4r$ to be $< i$ and less than or equal to the $C(n, Q := 1, p, i, \lambda)$ of Theorem 1.1 (1), so that the harmonic radius $r_H(Q := 1, 1, p)$ is at least $4r$. By the Sobolev estimate Theorem 2.1 (2), we therefore have uniform bounds in terms of n, i, λ on, for each $x \in M$, the co-ordinate norms $||g||_{\infty, B(x,4r)}$, $||g^{-1}||_{\infty, B(x,4r)}$.

Define a measurable function c on M by,

$$c(x) = \begin{cases} 0, & \text{if } u(x) = 0 \\ -\frac{\sqrt{|g|}(\Delta_g u)(x)}{u(x)}, & \text{if } u(x) \neq 0. \end{cases}$$

This function satisfies the bound $||c||_\infty \leq \sqrt{n!}||g||^{n/2}_{\infty, B(x,4r)} B < \infty$. By construction $\sqrt{|g|}\Delta_g u + cu = 0$. Applying the Harnack estimate of Theorem 4.1 with the operator

$$Lu = (\sqrt{|g|}\Delta_g + c)u = \partial_i(\sqrt{|g|}g^{ij}\partial_j u) + cu,$$

we deduce that for $C = C(n, i, \lambda, B)$, for each $x \in M$,

$$\sup_{B(x,r)} u \leq C \inf_{B(x,r)} u.$$

Applying Lemma 1.2 with $(\rho, r) = (r, D)$, we obtain an integer $N = N(n, \lambda, r, D)$ such that M may be covered by a set of at most N radius-r balls. Let $(B_\alpha)_{\alpha \in \mathfrak{A}}$ be such a covering.

For any two balls B_α, B_β in the set, there exists a sequence $\alpha_0 := \alpha, \alpha_1, \dots \alpha_l := \beta$, with $l \le |\mathfrak{A}| - 1$, such that each pair B_{α_i}, $B_{\alpha_{i+1}}$ of adjacent balls in the sequence intersects. Thus, for each $0 \le i \le l-1$,

$$\inf_{B_{\alpha_i}} u \le \inf_{B_{\alpha_i} \cap B_{\alpha_{i+1}}} u \le \sup_{B_{\alpha_{i+1}}} u.$$

Therefore, by induction,

$$\sup_{B_\alpha} u \le C^N \inf_{B_\beta} u.$$

Since this holds for all $\alpha, \beta \in \mathfrak{A}$, and $(B_\alpha)_{\alpha \in \mathfrak{A}}$ cover M, we conclude

$$\sup_M u \le C^N \inf_M u.$$

□

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