Moduli spaces of Einstein metrics

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Abstract

We discuss the space of Einstein metrics, up to diffeomorphism, on a compact manifold. In particular we mention some heuristics on its dimension and some theorems on its compactness.

1 Introduction

Let M be a compact smooth manifold. Recall some of the basic objects of Riemannian geometry:

- a *Riemannian metric* is a section g of T^2M which is everywhere symmetric and positive definite;

and to a Riemannian metric g are naturally associated

- its *Levi-Civita connection*, which is a map
$$\nabla : \Gamma(TM) \to \Gamma(T^*M \otimes TM);$$
- several objects measuring *curvature*, which turn out to be tensorial; in particular
  - the *Riemannian curvature*, a section Rm of T^4M, defined by
$$Rm(X,Y,Z,W) = g(W, \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z);$$

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– the *Ricci curvature*, a section Ric of T^2M, defined by

$$Ric(X,Y) = \mathrm{Tr}_g[Rm(\cdot, X, \cdot, Y)];$$

– the *scalar curvature*, a function R on M, defined by

$$R = \mathrm{Tr}_g Ric.$$

To each of these curvature tensors corresponds a notion of "good metric:"

	Riemannian	Ricci	scalar
notion of "good metric"	constant [sectional] curvature metric	Einstein metric	constant scalar curvature metric
definition of notion	$\exists \lambda \in \mathbb{R}$, such that everywhere $Rm = \lambda\, g \circ g$[1]	$\exists \lambda \in \mathbb{R}$, such that everywhere $Ric = \lambda g$	$\exists \lambda \in \mathbb{R}$, such that everywhere $R = \lambda$
rigidity of notion	Most manifolds have none: *Killing-Hopf theorem* says a complete (M,g) has constant curvature if and only if it is some quotient $(\widetilde{M},\widetilde{g})/\Gamma$, where $(\widetilde{M},\widetilde{g})$ is (up to scaling) one of {sphere-with-round-metric, Euclidean space, hyperbolic space} and Γ is a discrete freely-acting group of isometries.	Heuristic: "a few" on "most" manifolds?	All manifolds have an ∞-dim'l space of them: *Yamabe problem* (solved by Schoen, 1984) says every (M,g) compact admits some $\varphi \in \mathcal{C}^\infty(M)$ such that $e^\varphi g$ has constant scalar curvature.

As is clear from the final row, the appealing feature (or, at least, one of the appealing features!) of Einstein metrics is that they are neither "too rigid" nor "too abundant" for a manifold's space of Einstein metrics to be interesting.

[1] The *Kulkarni-Nomizu product*:

$$(a \circ b)(X,Y,Z,W) = a(X,Z)b(Y,W) + b(X,Z)a(Y,W) - a(X,W)b(Y,Z) - b(X,W)a(Y,Z).$$

Two freedoms to note:

- $\mathbb{R}^+$ acts on [metrics-on-M]: Let μ be a positive real. Then μg is Einstein if and only if g is.
- Diff(M) acts on [metrics-on-M]: Let φ be a diffeomorphism. Then $\varphi^* g$ is Einstein if and only if g is.

Definition. *The* moduli space *of Einstein metrics on M, denoted $\mathcal{E}(M)$, is the quotient*

$$\{\textit{Einstein metrics on } M\}/\textit{Diff}(M).$$

We have not specified a topology on this moduli space. In fact we will vary this as it suits us. Some possibilities are:

- the restriction of the *Gromov-Hausdorff metric* (a natural metric on $\{$compact metric spaces$\}$) to $\mathcal{E}(M)$.
  This is a very weak topology.
- the topologies whose convergence are: $(M, g_k) \to (M, g)$ if there exist diffeos (φ_k) of M such that ${\varphi_k}^* g_k \to g$ in $\mathcal{C}^{k,\alpha}$ or $\mathcal{W}^{k,p}$ or $\ldots$.
  These are finer topologies but very "incomplete" – they doesn't capture some interesting limits.

2 Examples

2.1 Oriented surfaces

In dimension 2, all three notions of "good metric" turn out to coincide. When the surface M is oriented, the following are equivalent:

1. Einstein metrics (up to diffeomorphism and rescaling) on M;
2. conformal equivalence classes of Riemannian metrics (up to diffeomorphism) on M;
3. complex structures (up to diffeomorphism) on M, a.k.a. Riemann surfaces of M's genus;
4. smooth irreducible algebraic curves of M's genus.

(These equivalences are nontrivial but of course well-known. For the two that concern us here,

$1 \leftrightarrow 2$ is given by the map

$$g \mapsto [g],$$

and the proof that this is a bijection is the Uniformization Theorem;

$2 \leftrightarrow 3$ is given by the map

$$[dx_j{}^2 + dy_j{}^2] \hookleftarrow (U_j, x_j + iy_j)_{j \in J}$$

and the proof that this is a bijection is the Korn-Lichtenstein theorem on existence of isothermal co-ordinates.)

So the moduli space of Einstein metrics on the oriented surface Σ_g of genus g is the same (up to scaling) as the moduli space of Riemann surfaces of genus g.

Hence we have a fairly explicit description of the $\mathcal{E}(\Sigma_g)$'s:

$$\mathcal{E}(\Sigma_g) \cong \mathbb{R}^+ \times [\text{quotient of } \mathbb{R}^{\sigma(g)} \text{ by a discrete group}].$$

Here σ is the integer function defined by

g	0	1	≥ 2
$\sigma(g)$	0	2	$6g-6$

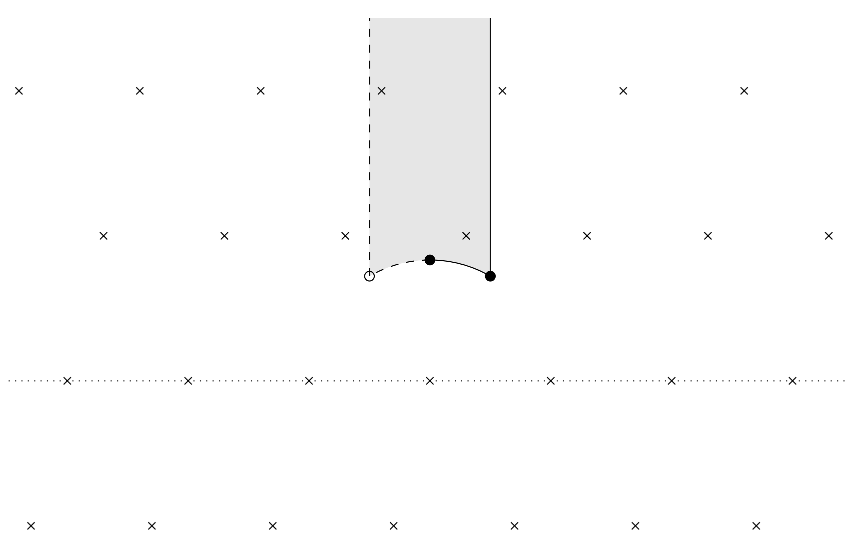
,

and the $\mathbb{R}^+$ factor corresponds to rescalings of the same metric. In particular, for $g = 0$ (the sphere),

$$\mathcal{E}(\Sigma_g) = \mathbb{R}^+ = \{\text{multiples of } g_{round}\},$$

and for $g = 1$ (the torus),

$$\mathcal{E}(\Sigma_g) = \mathbb{R}^+ \times \mathbb{H}^2/SL(2, \mathbb{Z}).$$



2.2 4-manifolds with $\mathcal{E}(M)$ empty

Let (M, g) be a compact Riemannian 4-manifold. The Chern-Gauss-Bonnet theorem states:

$$\begin{aligned} 8\pi^2\chi(M) &= \int_M \mathrm{Pfaff}(Rm) \\ &= \int_M |W|^2 - \frac{1}{2}\int_M |Ric - \frac{1}{4}Rg|^2 + \frac{1}{24}\int_M R^2. \end{aligned}$$

(Here W is the *Weyl curvature tensor*, which can be defined in terms of Rm, Ric and R by a long formula which we won't write out here.)

If g is an Einstein metric, with $Ric = \lambda g$, then $R = \mathrm{Tr}_g(\lambda g) = 4\lambda$, so the second integrand completely vanishes:

$$Ric - \frac{1}{4}Rg = \lambda g - \frac{1}{4}\cdot 4\lambda g = 0.$$

Hence the Chern-Gauss-Bonnet formula implies

$$8\pi^2\chi(M) = \int_M |W|^2 + \frac{1}{24}\int_M R^2 \geq 0.$$

Therefore a 4-manifold M with negative Euler characteristic can admit no Einstein metric. For instance, for $g \geq 2$ the manifold $S^2 \times \Sigma_g$ has

$$\chi(S^2 \times \Sigma_g) = \chi(S^2)\chi(\Sigma_g) = 2\cdot(2-2g) < 0,$$

so $\mathcal{E}(S^2 \times \Sigma_g)$ is empty.

2.3 Some miscellaneous Einstein metrics

For motivation, here are some well-known metrics and families of metrics which are Einstein (although whether they represent all of $\mathcal{E}$([underlying manifold]) or not may be hard or unknown):

- the constant-[sectional]-curvature metrics;
- the Fubini-Study metric on $\mathbb{CP}^n$;
- various Einstein *Kähler metrics*:

  Theorem (Yau, 1976)**.** *On every complex manifold (M, J) such that $c_1(M, J) = 0$, in every Kähler class, there is a Kähler metric which is Einstein with $\lambda = 0$.*

(Hence one gets lots of Einstein metrics on the smooth manifold M: one for each point in the bundle

$$\coprod_{\text{complex structures } J \text{ on } M \text{ with } c_1(M,J)=0} \{\text{Kähler classes of } (M,J)\},$$

although some of these Einstein metrics turn out to be the same.)

Theorem (Aubin, Yau, 1976)**.** *On every complex manifold (M,J) such that $-c_1(M,J)$ is Kähler, there is a Kähler metric (in $-c_1(M,J)$) which is Einstein with $\lambda=-1$.*

3 Dimension

Let M be a compact smooth manifold. Then each $\mathcal{E}_\lambda(M)$ is "finite-dimensional" in the following sense:

Theorem. *Let g be a λ-Einstein metric on M. Then there exists a finite-dimensional subspace V_g of $\Gamma(T^2M)$, such that for each λ-Einstein variation (g_t) of g, there is a 1-parameter family (φ_t) of diffeos of M with*

$$\frac{d}{dt}|_{t=0}\ [{\varphi_t}^* g_t] \in V_g.$$

(Hence the full moduli space

$$\mathcal{E}(M) = \coprod_{\lambda\in\mathbb{R}} \mathcal{E}_\lambda(M)$$

is also in some sense finite-dimensional.)

Proof. Let (g_t) be a λ-Einstein variation of g. There exists a 1-parameter family (φ_t) of diffeos of M with $\varphi_0 = \mathrm{Id}$ and such that for all t,

$$\mathrm{div}_g({\varphi_t}^* g_t) = 0.$$

For such a family, "differentiating" the equation

$$Ric_{{\varphi_t}^* g_t} = \lambda\ {\varphi_t}^* g_t$$

with respect to t turns out to demonstrate that

$$h := \frac{d}{dt}|_{t=0}\ [{\varphi_t}^* g_t]$$

satisfies the equation

$$\Delta h = \lambda h,$$

where Δ is the *Lichnerowicz operator* on symmetric sections of T^2M:

$$\Delta h = \mathrm{Tr}_g(\nabla\nabla h) + [\text{terms (involving } g\text{'s curvature) linear in } h]$$

Therefore for every λ-Einstein variation (g_t) of g, there is a 1-parameter family (φ_t) of diffeos of M with

$$\frac{d}{dt}|_{t=0}\ [{\varphi_t}^* g_t] \in V_g := \ker(\Delta - \lambda).$$

The operator $\Delta - \lambda$ has symbol

$$g^{ij}\partial_i\partial_j \otimes \mathrm{Id},$$

so is elliptic; therefore

$$V_g = \ker(\Delta - \lambda)$$

is finite-dimensional. □

In fact we can calculate the index of the operator $\Delta - \lambda$. It has the same symbol as the rough Laplacian $\mathrm{Tr}_g(\nabla\nabla h)$. The rough Laplacian is self-adjoint with respect to the inner product induced by g on symmetric sections of T^2M, and so has index 0. Therefore $\Delta - \lambda$ also has index 0.

4 Compactness

Problem. *"How noncompact" is $\mathcal{E}(M)$?*

Equivalently, given a sequence (g_k) of Einstein metrics, what can be said about convergence of subsequences?

There are two simple ways in which there might fail to be any convergent (say, in the category of compact metric spaces) subsequence at all:

1. **Noncompact limits** A "natural" (say, pointed Gromov-Hausdorff) limit of a sequence (g_k) of Einstein metrics on a compact manifold M might be a noncompact space.

   Example. *There are sequences of flat tori $\mathbb{R}^2/\mathbb{Z}^2$ which converge to the flat cylinder $\mathbb{R}^2/\mathbb{Z}$.*

   For this reason, one typically considers only sequences (g_k) with a uniform upper diameter bound.

2. Dropping dimension A sequence (g_k) of Einstein metrics on a compact manifold M might converge to a space of "dimension" less than that of M.

Example. *There are sequences of flat tori* $\mathbb{R}^2/\mathbb{Z}^2$ *which converge to the circle* $\mathbb{R}/\mathbb{Z}$.

For this reason, one typically considers only sequences (g_k) with a uniform lower volume bound.

In two and three dimensions this is all that can go wrong:

Theorem. *Let M be a compact surface or 3-manifold. Then for any fixed positive D and v, the moduli space*

$$\{g \in \mathcal{E}(M) : \mathit{diam}(g) \leq D, \mathit{vol}(g) \geq v\}$$

is compact.

But in higher dimensions, more intriguing things can happen.

3. Orbifold singularities A sequence (g_k) of Einstein metrics on a compact manifold M might converge to an *Einstein orbifold* – that is, an orbifold, together with an Einstein metric on the complement of the singular points, which extends smoothly across these points on smooth covers of their neighbourhoods.

Non-compact toy example. *There are sequences of Einstein metrics on the 4-manifold TS^2 which converge to the flat orbifold $\mathbb{R}^4/\pm$.*

Topological motivation:

$$TS^2/[\textit{contracting-the-zero-section}] \;\cong\; (\mathbb{R}^+ \times SO(3)) \cup \{0\} \;\cong\; \mathbb{R}^4/\pm.$$

Example. *There is a compact 4-manifold M which has a 40-dimensional space of complex structures all with $c_1 = 0$, called the* K3 surfaces. *By Yau's theorem (Section 2.3), M carries a large family of Einstein Kähler metrics.*

There exists an open subset $U \subseteq M$ diffeomorphic to TS^2, and a sequence of M's Einstein Kähler metrics, such that the sequence

- *converges smoothly outside U;*
- *"looks like" a neighbourhood of the Eguchi-Hanson toy example inside U.*

This sequence of Einstein metrics has an Einstein orbifold limit, with a singular point whose neighbourhood is homeomorphic to $\mathbb{R}^4/\pm$.

Theorem (Anderson, Bando-Kasue-Nakajima, ..., 1989-90)**.** *Let M be a compact 4-manifold. Then for any fixed positive D and v, the moduli space*

$$\{g \in \mathcal{E}(M) : \mathit{diam}(g) \leq D, \mathit{vol}(g) \geq v\}$$

is compact after adding in some possible Einstein orbifold limits.

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