The Arnold Chord Conjecture

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Contents

Chapter 1

Introduction

This essay concerns the behaviour, in contact and symplectic manifolds, of certain structure-preserving flows with respect to certain submanifolds. Our two main results are the following:

Theorem 1.1 (Arnold chord conjecture)**.** *Let (N, α) be a compact simply-connected contact-type hypersurface of $\mathbb{R}^{2n}$. Then for each compact Legendrian submanifold l of (N, α), some integral curve of α's Reeb vector field runs from l to l.*

Theorem 1.2. *Let X_{F_t} be a compactly supported time-dependent Hamiltonian vector field on $\mathbb{R}^{2n}$ of Hofer norm less than σ. Let L be a compact Lagrangian submanifold of $\mathbb{R}^{2n}$, such that no disc with boundary on L has positive symplectic area less than σ. Then some length-1 integral curve of X_{F_t} runs from L to L.*

Versions of both appear in Vladimir Arnold's classic and influential set [Arn86] of conjectures in symplectic topology, inspired by "nebulous ideas" on analogies between the category of symplectic manifolds and the category of manifolds.

Theorem 1.1 was proved in 2001 by Mohnke [Moh01]. His beautiful (and very short) argument deduces the theorem from Theorem 1.2 by considering l's extensions, within a fixed neighbourhood of N in $\mathbb{R}^{2n}$, to closed Lagrangian submanifolds of $\mathbb{R}^{2n}$. We present this argument in this essay as Proposition 3.1.7 and Theorem 6.2.1.

Theorem 1.2 has a more intricate history. The '$\sigma = \infty$' version

Theorem 1.3 (Arnold conjecture)**.** *Let X_{F_t} be a compactly supported time-dependent Hamiltonian vector field on $\mathbb{R}^{2n}$. Let L be a compact Lagrangian submanifold of $\mathbb{R}^{2n}$, such that every disc with boundary on L has zero symplectic area. Then some length-1 integral curve of X_{F_t} runs from L to L.*

was proved by Gromov, in the foundational paper [Gro85] in which he introduced techniques of J-holomorphic curves to symplectic geometry. After

Hofer's introduction of the Hofer norm, Polterovich [Pol93] built on Gromov's ideas to prove the weakening of Theorem 1.2 in which the bound σ needed on symplectic areas of discs is increased to 2σ.

At roughly the same time, Floer [Flo88] established a new approach to Theorem 1.3 and related problems. His method was to consider strips with edges in L satisfying versions of the PDE

$$\partial_s u + J(\partial_t u - X_{F_t}|_u) = 0.$$

Such strips tend to concentrate around integral curves of X_{F_t}, providing a way of discovering such integral curves. Later Chekanov [Che98], building on Floer's ideas, established the full Theorem 1.2.

In fact Floer's proof of Theorem 1.3, and Chekanov's proof following Floer of Theorem 1.2, use a much more powerful and general framework, and deduce a refinement of the theorems which we do not need for deducing the chord conjecture Theorem 1.1. The framework is *Lagrangian Floer homology*, which roughly speaking is a homology theory generated by the length-1 integral curves of X_{F_t} from L to L. The consequent refinement of Theorems 1.3 and 1.2 is, 'generically,' the much stronger lower bound $\dim H_*(L, \mathbb{Z}_2)$ for the numbers of such integral curves – in fact, the bound originally conjectured by Arnold.

On the other hand, the monograph [MS04] proves the basic Theorem 1.3 (their Theorem 9.2.14) using Floer's equation and analytical content, but without explicitly setting up Lagrangian Floer homology. It is this argument (suitably sharpened), rather than Chekanov's original presentation, that we use in Section 6.1 to prove Theorem 1.2.

The plan of this essay is as follows. Chapter 2 reviews necessary background in contact and symplectic geometry. Chapter 3 is a detailed introduction to problems concerning Reeb and Hamiltonian flows' relation with Legendrian and Lagrangian submanifolds. It includes a number of examples and a further discussion of the literature.

Chapters 4 and 5 are the essay's technical core. The target of these chapters is Proposition 5.3.1. We state it here for convenience:

Proposition 1.4. *Let (M, ω, J) be a tame compact symplectic manifold, and L a compact Lagrangian submanifold of M, such that no sphere in M or disc in M with boundary on L has positive symplectic area less than σ. Then for each Hamiltonian form H on $D^2 \times (M, L)$ whose curvature satisfies*

$$\int_\Sigma \sup_{p \in M} R_H|_{(\cdot, p)} < \sigma,$$

each $w \in \partial D^2$ and each $p \in L$, there is a map $u : (D^2, \partial D^2, w) \to (M, L, p)$ such that for all $z \in D^2$,

$$\overline{\partial}_J u|_z + X_H^{0,1}|_{(z, u(z))} = 0.$$

Observe that this proposition asserts the existence of discs with boundary on L which satisfy perturbations of certain PDE, so long as the perturbations are small in relation to the symplectic areas of spheres and discs in M.

There are two main components to the proof of Proposition 5.3.1. The first, discussed in Chapter 4, is a very general pair of theorems, valid in any almost-complex manifold, which describes the moduli space of such discs for a typical perturbation of the *Cauchy-Riemann equations*. The proof of these theorems uses the *Sard-Smale theorem* and a version of the *Riemann-Roch theorem*. Roughly speaking, the theorems say: 'generically,' this moduli space is a manifold, and depends smoothly on the perturbation.

The second component, developed in Chapter 5, is *Gromov's compactness theorem*, which severely prescribes the ways in which a sequence of solutions in a symplectic manifold to a perturbation of the Cauchy-Riemann equations can fail to have a convergent subsequence. The power of combining this result (on *compactness* of moduli spaces of solutions) with those of Chapter 4 is that we can conclude that the moduli spaces for different perturbation terms are all compact, and cobordant to each other. Results on existence of solutions for the zero perturbation then immediately imply existence of solutions in general.

The conclusion of the essay, Chapter 6, uses Proposition 5.3.1 to deduce Theorem 1.2, and thence Theorem 1.1, as previously described.

We make one major simplification throughout: we prove Theorems 1.1 and 1.2 for $\mathbb{R}^{2n}$ (as we have stated them here), rather than for the class of *tame geometrically bounded* symplectic manifolds as seen in the versions of these theorems in the literature.

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Notation

- Let A and B be manifolds. We write

  $$pr_1 : A \times B \to A, \quad pr_2 : A \times B \to B,$$

  for the projections from $A \times B$ onto its co-ordinates. For a vector bundle E over A, we write ${pr_1}^*E$ for the pullback of E under pr_1 to a bundle over $A \times B$. Likewise for a section η of E, we write ${pr_1}^*\eta$ for η's pullback to a section of ${pr_1}^*E$. (By contrast, $\pi_1, \pi_2, \ldots$ denote the first, second, ... homotopy groups.

- Let A and B be manifolds, and E and F vector bundles over A and B respectively. We write $E \boxtimes F$ for the vector bundle

$$pr_1{}^*E \otimes pr_2{}^*F$$

over $A \times B$. We will sometimes also write, for instance, $E \boxtimes \mathbb{R}$, to denote the bundle $pr_1{}^*E$; the idea is that $\mathbb{R}$ stands for the trivial bundle over B.

- Let X_1, X_2, Y_1, Y_2 be topological spaces, with $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$. A map

$$f : (X_1, Y_1) \to (X_2, Y_2)$$

is a continuous function $f : X_1 \to X_2$ such that $f(Y_1) \subseteq Y_2$.

- In a number of related senses defined throughout the essay, a tilde $\widetilde{u}$ denotes the graph of a map u.

- Let Σ be a Riemann surface. Then $\Lambda^{i,j}\Sigma$ denotes its bundle of (i, j)-forms.

Chapter 2

Contact and symplectic geometry

In this chapter we briefly review some basic definitions, facts and examples in contact and symplectic geometry. Our presentation derives from the texts [CdS01], [MS98] and [Gei08] and the notes [Etn03].

2.1 Contact manifolds

Let N be a manifold of odd dimension $2n-1$.

Definition. *A* contact form *on N is a nonzero 1-form α, such that the $(2n-1)$-form $\alpha \wedge (d\alpha)^{n-1}$ is nonvanishing.*

(Equivalently, such that $d\alpha$ has kernel of dimension 1, and

$$\ker(d\alpha) \cap \ker \alpha = (0).)$$

If α is contact, then for each smooth function $f : N \to \mathbb{R}^+$, the form $f\alpha$ is also contact: for

$$f\alpha \wedge [d(f\alpha)]^{n-1} = f\alpha \wedge (fd\alpha + df \wedge \alpha)^{n-1} = f^n \left[\alpha \wedge (d\alpha)^{n-1}\right].$$

A *contact structure* on N is an equivalence class of contact forms on N, where $\alpha \sim \beta$ if $\beta = f\alpha$ for some smooth $f : N \to \mathbb{R}^+$. A *contact manifold* is a manifold equipped with a contact structure.

Remark. *Strictly speaking, it is standard to define a contact structure on a manifold N to be a $(2n-2)$-plane distribution on N which locally is the kernel of a contact form as we have defined them. The two concepts are equivalent if the distribution is co-orientable. For simplicity, we restrict to this case throughout the essay.*

Example 2.1.1 (Standard contact form on $\mathbb{R}^{2n-1}$). *Consider the 1-form*

$$\alpha = dz + \sum_{i=1}^{2n-1} x^j dy^j$$

on $\mathbb{R}^{2n-1}$. *We have*

$$\begin{aligned} d\alpha &= \sum_{i=1}^{2n-1} dx^j \wedge dy^j; \\ \alpha \wedge (d\alpha)^{n-1} &= dz \wedge (dx^1 \wedge dy^1) \wedge \cdots \wedge (dx^{n-1} \wedge dy^{n-1}), \end{aligned}$$

the standard volume form, so α *is contact.*

Example 2.1.2 (Standard overtwisted form on $\mathbb{R}^3$). *Consider the 1-form on* $\mathbb{R}^3$ *defined in cylindrical co-ordinates by*

$$\alpha = (\cos r)dz + (r \sin r)d\theta.$$

We have

$$\begin{aligned} d\alpha &= -(\sin r)dr \wedge dz + [r\cos r + \sin r]\, dr \wedge d\theta; \\ \alpha \wedge d\alpha &= (\cos r)\,[r\cos r + \sin r]\, dz \wedge dr \wedge d\theta - (r \sin r)(\sin r) d\theta \wedge dr \wedge dz \\ &= \left[1 + \frac{\sin r}{r}\right] (r dr) \wedge d\theta \wedge dz. \end{aligned}$$

Since $1 + \frac{\sin r}{r}$ *is* ≥ 2 *for nonnegative* r, *we conclude* $\alpha \wedge d\alpha$ *is nonvanishing. Thus* α *is contact.*

Henceforth in this section fix a contact form α on N.

Definition. *A* Legendrian submanifold *of* (N, α) *is a submanifold* l *of* N, *of dimension* $n-1$, *such that* α *vanishes on* Tl.

Example 2.1.3 (Legendrian submanifolds of standard $\mathbb{R}^3$). *The following discussion is from [Etn03]. Recall from Example 2.1.1 the standard (not-overtwisted) contact structure*

$$\alpha = dz + xdy$$

on $\mathbb{R}^3$. *We can construct a Legendrian submanifold of* $(\mathbb{R}^3, \alpha)$ *as follows: Pick a closed curve* γ *in* $\mathbb{R}^2$ *which satisfies the following conditions:*

1. γ *is smooth except for a finite number of points, 'cusps,' at which* γ*'s slope tends to 0 from above from one direction and from below from the other direction, but* γ*'s 'direction' reverses.*

2. γ *is never tangent to vertical.*

3. *At points where γ crosses itself, the slopes of the different branches at the crossing points are all distinct.*

Now identify $\mathbb{R}^2$ with the yz-plane (y horizontal, z vertical). Construct a closed loop l in $\mathbb{R}^3$ whose projection to yz is γ, and whose x-co-ordinate is $-dz/dy$. This gives a continuous map into $\mathbb{R}^3$ by conditions (1) and (2), and by condition (3) does not self-intersect. Since $x = -dz/dy$, the submanifold l is Legendrian.

Conversely, it is clear by projection onto the yz-plane that every compact Legendrian submanifold of $(\mathbb{R}^3, \alpha)$ arises uniquely in this way.

It is clear that equivalent contact forms α, $f\alpha$ have the same Legendrian submanifolds. Thus we can talk of Legendrian submanifolds as associated with a contact structure rather than with a particular contact form.

Definition. *The* Reeb vector field *of α is the vector field Y on N such that $Y \in \ker d\alpha$ and $\alpha(Y) = 1$.*

(Since $d\alpha$ has kernel of dimension 1, and

$$\ker(d\alpha) \cap \ker \alpha = (0),$$

there indeed exists a unique such vector field.)

Lemma 2.1.4. *The flow along the Reeb vector field of (N, α) preserves α.*

2.2 Symplectic manifolds

Let M be a manifold of even dimension $2n$.

Definition. *A symplectic form on M is a closed 2-form ω, such that the $2n$-form ω^n is nonvanishing.*

(Equivalently, such that ω has zero kernel.)

A *symplectic manifold* is a manifold equipped with a symplectic form. A *symplectomorphism* of a symplectic manifold is a diffeomorphism of the manifold under which the symplectic form is preserved.

Example 2.2.1 (Standard symplectic form on $\mathbb{R}^{2n}$). *Consider the closed 2-form*

$$\omega = \sum_{i=1}^{2n-1} dx^j \wedge dy^j$$

on $\mathbb{R}^{2n}$. We have

$$\omega^n = (dx^1 \wedge dy^1) \wedge \cdots \wedge (dx^n \wedge dy^n),$$

the standard volume form, so α is contact.

Example 2.2.2 (Cotangent bundles)**.** *Let W be a smooth n-manifold. Choose co-ordinates (x^i) on a neighbourhood of W, and corresponding co-ordinates $((x^i),(\xi^i))$ on the lift of that neighbourhood to T^*W. That is, (ξ^i) are the co-ordinate functions such that the covector $\xi = \sum_{i=1}^n \xi^i dx^i$ at a point $(x^1,\dots x^n)$ on W has co-ordinate expression $(x^1,\dots x^n,\xi^1,\dots \xi^n)$.*

Consider the 1-form

$$\alpha = -\sum_{i=1}^{n} \xi^i dx^i,$$

on T^*W (that is, it is a section of $T^*(T^*W)$.) This 1-form is independent of the defining choice (x^i) of co-ordinates. The closed 2-form

$$d\alpha = \sum_{i=1}^{n} dx^i \wedge d\xi^i$$

on T^*W is therefore also independent of choice of co-ordinates, and has zero kenel. We call $d\alpha$ the canonical symplectic form *on T^*W.*

Example 2.2.3 (Products)**.** *Let (M_1,ω_1) and (M_2,ω_2) be symplectic manifolds. Then*

$$(M_1 \times M_2, {pr_1}^*\omega_1 - {pr_2}^*\omega_2)$$

is a symplectic manifold.

We now introduce some concepts of symplectic geometry.

Definition. *A* Lagrangian submanifold *of (M,ω) is a submanifold L of M, of dimension n, such that ω vanishes on TL.*

(Since ω has zero kernel, there indeed exists a unique such vector field.)

Example 2.2.4. *All 1-dimensional submanifolds of $\mathbb{R}^2$ are Lagrangian.*

Example 2.2.5. *For each closed 1-form η on a manifold W, the graph of η is a Lagrangian submanifold of the cotangent bundle T^*W with its canonical symplectic structure.*

Also for each point $p \in W$, the fibre T_p^*W is a Lagrangian submanifold of T^*W.

Example 2.2.6. *Let (M,ω) be a symplectic manifold. Then the graph of a diffeomorphism $\phi : M \to M$ is a Lagrangian submanifold of*

$$(M_1 \times M_2, {pr_1}^*\omega_1 - {pr_2}^*\omega_2)$$

if and only if ϕ is a symplectomorphism. In particular the diagonal Δ is a Lagrangian submanifold.

Definition. *The* symplectic area *of an immersed two-dimensional submanifold of* (M, ω),

$$u : \Sigma \to M,$$

is the integral

$$\int_\Sigma u^* \omega.$$

Since ω *is closed, this is by Stokes' theorem homology invariant.*

Definition. *Let* $H \in \mathcal{C}^\infty(M)$ *be a smooth function on* (M, ω)*. The* Hamiltonian vector field *of* H *is the vector field* X_H *such that*

$$\omega(X_H, \cdot) = dH.$$

We will be interested in flows of Hamiltonian vector fields on symplectic manifolds, and also in a slightly more general class of flows:

Definition. *A* smooth time-dependent vector field *on a manifold* M *is a smooth section of the pullback bundle* $pr_2{}^*TM$ *over* $I \times M$*, where* I *(the 'time interval') is an interval of* $\mathbb{R}$ *containing 0.*

For instance, a smooth function $H : I \times M \to \mathbb{R}$ on a symplectic manifold (M, ω) defines a *time-dependent Hamiltonian vector field* X_{H_t}.

Proposition 2.2.7 (Flows of time-dependent vector fields)**.** *Let* $X_t : I \to \mathcal{C}^\infty(M)$ *be a smooth time-dependent vector field on a manifold* M*, and let* $p \in M$*. Then there exists an interval* $J \subseteq I$ *containing 0, on which there is a unique curve* $\gamma : J \to M$ *with* $\gamma(0) = p$ *and for each* $t \in J$,

$$\dot\gamma(t) = X_t|_{\gamma(t)}.$$

Proof. Apply the standard theorem on flows of (time-independent) vector fields to the manifold $I \times M$ and the vector field defined as the section $(1, X_t)$ of $TI \boxtimes TM \cong T(I \times M)$. □

Lemma 2.2.8. *A flow along a time-dependent Hamiltonian vector field is a symplectomorphism.*

2.3 Examples and constructions

Lemma 2.3.1. *Let* α *be a contact form on a manifold* N*. Then the closed 2-form* $d(e^s\alpha)$ *on* $N \times \mathbb{R}$ *has zero kernel.*

Proof.

$$[d(e^s\alpha)]^n = e^{ns}(ds \wedge \alpha + d\alpha)^n = e^{ns}\left[ds \wedge \alpha \wedge (d\alpha)^{n-1}\right].$$

This is nonvanishing since α is contact. □

We call the symplectic manifold $(N \times \mathbb{R}, d(e^s\alpha))$ thus constructed the *symplectization* of N.

In some sense conversely, we have the following concept.

Definition. *A hypersurface N of a symplectic manifold (M, ω) is* of contact type*, if there exists a vector field X defined on a neighbourhood of N in M, which is tranverse to N and satisfies $d(\iota_X\omega) = \omega$.*

This is a global condition: locally we can always construct such an X, by picking a vector field X with this property which is transverse to N at $p \in N$, then restricting to a neighbourhood of p on which X remains transverse.

Lemma 2.3.2. *Let N be a contact-type hypersurface of a symplectic manifold (M, ω), and let X be a vector field on a neighbourhood of N in M, which is tranverse to N and satisfies $d(\iota_X\omega) = \omega$. Then the restriction of $\iota_X\omega$ to N is a contact form on N.*

Proof. Let $\alpha = \iota_X\omega$. Then

$$\alpha \wedge (d\alpha)^{n-1} = \iota_X\omega \wedge \omega^{n-1} = \iota_X\omega^n,$$

and since ω^n is nonvanishing and X is transverse to N, this has nonvanishing restriction to N. □

Let us make precise the sense in which the two concepts are converse:

Example 2.3.3. *Let (N, α) be a manifold with contact form. Then*

$$d\left(\iota_{\partial_s}\left[e^s(d\alpha + ds \wedge \alpha)\right]\right) = d(e^s\alpha),$$

so N is a contact-type hypersurface of (N, α)'s symplectization; the induced contact form $\iota_{\partial_s} d(e^s\alpha)$ is just α.

Lemma 2.3.4. *Let N be a contact-type hypersurface of a symplectic manifold (M, ω). Then there exists a neighbourhood of N in M which is symplectomorphic, via a symplectomorphism fixing N, to a neighbourhood of (N, α)'s symplectization.*

Proof. Suppose α is induced by a vector field X on some neighbourhood of N, so that $\iota_X\omega = \alpha$, and $d\alpha = \omega$, and X is is tranverse to N. Then flow along X in a neighbourhood of N to define the s-co-ordinate. □

We now use the hypersurface construction to define two fundamental examples of contact forms.

Example 2.3.5 (Standard contact structure on S^{2n-1})**.** *Let ω be the standard symplectic form on $\mathbb{R}^{2n}$. Let X be the outward-pointing vector field*

$$\sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial y^i}$$

on $\mathbb{R}^{2n}$; for $\alpha := \iota_X \omega$, we have

$$\alpha = \sum_{i=1}^{n} x^i dy^i - y^i dx^i$$

and $d\alpha = \omega$. Therefore each hypersurface of $\mathbb{R}^{2n}$ which is star-shaped about the origin is of contact type, and has a contact form induced by X, the restriction of α to that hypersurface.

However, a star-shaped-about-the-origin hypersurface of $\mathbb{R}^{2n}$ is uniquely defined by a tautological embedding of the sphere S^{2n-1} into $\mathbb{R}^{2n}$,

$$(x^1, y^1, \dots x^n, y^n) \mapsto f(x^1, y^1, \dots x^n, y^n)(x^1, y^1, \dots x^n, y^n) \in \mathbb{R}^{2n},$$

for some smooth positive function f on the sphere S^{2n-1}. Let α_f be the contact form on S^{2n-1} which is the pullback under this embedding of that induced on its image by X. Then $\alpha_{fg} = f\alpha_g$, so the contact forms α_f are precisely the representatives of a single contact structure on S^{2n-1}, which we call the standard contact structure *on S^{2n-1}.*

In particular, we call S_1 the standard contact form *on S^{2n-1}; it is the restriction of α to the unit sphere of $\mathbb{R}^{2n}$.*

Example 2.3.6 (Riemannian sphere bundles)**.** *Let W be a smooth manifold, and recall the 1-form*

$$\alpha = -\sum_{i=1}^{n} \xi^i dx^i,$$

and symplectic form

$$d\alpha = \sum_{i=1}^{n} dx^i \wedge d\xi^i$$

defined on T^*W in Example 2.2.2. Consider the vector field

$$X = \sum_{i=1}^{n} \xi^i \partial / \partial \xi^i$$

on T^*W (that is, it is a section of $T(T^*W)$. Observe that

$$\iota_X(d\alpha) = -\sum_{i=1}^{n} \xi^i dx^i = \alpha.$$

Therefore any hypersurface of T^*W transverse to X is of contact type, and α's restriction to it is a natural contact form.

In particular, the unit sphere bundle SW with respect to any Riemannian metric g on W has a natural contact form. (This is the cotangent unit sphere bundle, but the Riemannian metric gives a canonical identification with the tangent unit sphere bundle.)

The 1-form α vanishes on tangent vectors $\partial/\partial\xi^i$, so the fibres of SW are Legendrian submanifolds. The Reeb vector field is the geodesic vector field on SW.

The dynamics of a Hamiltonian vector field on a symplectic manifold are closely related to those of contact-type hypersurfaces that arise as its level sets:

Lemma 2.3.7. *Let N be a contact-type hypersurface of a symplectic manifold (M, ω), let α be some contact form on N thus induced, and let H be a smooth function on M whose level set $H^{-1}(c)$ is N.*

Then the restriction of the Hamiltonian vector field X_H to N is contained in the tangent space TN, and is a nonvanishing multiple (not necessarily constant) of the Reeb vector field of α.

Proof. Both vector fields lie in the 1-dimensional distribution $\ker(\omega|_N) = \ker(d\alpha)$ on N. □

Chapter 3

Two problems of Arnold

In this chapter we introduce the two classical [Arn86] conjectures of Arnold whose proofs will occupy the rest of the essay: the Reeb chord problem in Section 3.1, and the Lagrangian intersections problem in Section 3.2.

Parts of Section 3.1 were inspired by the survey article [Hut10b], which, though primarily devoted to a different problem, the Weinstein conjecture, containes a number of interesting examples of phenomena more generally associated with Reeb flow. Section 3.2 draws heavily on discussions of the Arnold conjecture in the texts [CdS01] and [MS98].

3.1 The chord conjecture

Let us investigate more closely the behaviour of some Reeb flows on contact manifolds. One interesting issue is the behaviour of such a flow in relation to a given Legendrian submanifold. For instance, one could ask about a Legendrian submanifold l's *Reeb chords*: those integral curves of a Reeb vector field which both start and finish on l.

Equivalently, these are intersections of a Legendrian submanifold with its image under a Reeb flow some time forward.

Example 3.1.1. *We prove that in the contact manifold* $(\mathbb{R}^3, dz + xdy)$ *of Example 2.1.3, all closed Legendrian submanifolds have Reeb chords. Indeed, that the closed Legendrian submanifolds of are uniquely determined by their* yz*-plane projections to 'cusped' loops, via*

$$x = -\frac{dz}{dy}.$$

The Reeb vector field of $(\mathbb{R}^3, dz + xdy)$ *is* $\partial/\partial z$*. Therefore the Reeb chords of a Legendrian submanifold correspond to pairs of points on its* yz*-plane projection which have the same* yz*-co-ordinate and same slope.*

In the simplest case, when this projection has only two cusps, there must be at least one such pair of points by the mean value theorem: we parametrize *the two branches of the curve to give smooth functions* $z_1, z_2 : [a, b] \to \mathbb{R}$, *and observe that since* $z_1(a) = z_2(a)$ *and* $z_1(b) = z_2(b)$ *there must be a point in* (a, b) *with* $z_1'(c) = z_2'(c)$.

A more careful treatment extends this argument to a general Legendrian submanifold; we obtain that every compact Legendrian submanifold of $(\mathbb{R}^3, dz + xdy)$ *admits a Reeb chord.*

The next example is taken from [Abb99].

Example 3.1.2. *We prove that 'overtwisted' contact form on* $\mathbb{R}^3$ *from Example 2.1.2, given in cylindrical co-ordinates by* $\alpha = \cos r dz + r \sin r d\theta$, *has Legendrian submanifolds with no Reeb chords.*

Observe that all circles centred at the origin in some plane $z = z_0$ *are Legendrian submanifolds. The Reeb vector field, though complicated at a general point in* $\mathbb{R}^3$, *is simply* $(-1)^k \partial/\partial z$ *along the cylinders* $r = \pi k$ *for* $k \in \mathbb{N}$.

It follows that the circles of radius πk *centred at the origin in the planes* $z = z_0$ *are Legendrian submanifolds of* $(\mathbb{R}^3, \alpha)$ *which admit no Reeb chords.*

For some more examples, recall from Example 2.3.6 that the sphere bundle SW of a Riemannian manifold W has a natural contact form. The fibre over each point is a Legendrian submanifold, and the geodesic vector field is the Reeb vector field. The Reeb chords of the fibre S_pW, for $p \in W$, are therefore the geodesics which self-intersect at p.

Example 3.1.3. *Every geodesic on the sphere* S^n *is periodic. Therefore the fibres of* S^n*'s sphere bundle are Legendrian submanifolds with the property:* every *integral curve starting on the Legendrian submanifold is a Reeb chord.*

Example 3.1.4. *No geodesic on* $\mathbb{R}^n$ *self-intersects. Therefore the fibres of* $\mathbb{R}^n$*'s sphere bundles are Legendrian submanifolds which admit no Reeb chords.*

We can generalize this as follows:

Example 3.1.5. *Suppose* W *is complete and of nonpositive sectional curvature. We show that each nontrivial homotopy class of* W *contains exactly one geodesic from each* $p \in W$ *to itself. Therefore the fibres of* W*'s sphere bundle each admit exactly* $|\pi_1(W)| - 1$ *Reeb chords.*

Indeed, all geodesics self-intersecting at $p \in W$ *lift to geodesics in* W*'s universal cover* $\widetilde{W}$ *which connect two preimages* p_0, p_1 *of* p. *But by the Cartan-Hadamard theorem, the exponential map from* p_0 *is a diffeomorphism onto* $\widetilde{W}$, *and thus any point in* $\widetilde{W}$ *is connected to* p_0 *by exactly one geodesic.*

And, generalizing again:

Example 3.1.6. *Suppose W is compact and $p \in W$. We give a classical variational argument (modified from one in [Cha06] IV.5.1 for a related problem) to show: in each nontrivial homotopy class of W there is a geodesic from p to p. Therefore, for a non-simply-connected compact manifold W: all fibres of W's sphere bundle admit Reeb chords.*

Indeed, take a sequence of constant-speed smooth curves from p to p in the given homotopy class whose lengths tend to the infimum. By the Arzela-Ascoli theorem some subsequence converges uniformly to a Lipschitz curve γ from p to p. Now make a piecewise smooth version γ' of γ by chopping γ into very short pieces and then replacing each very short piece by the unique minimizing constant-speed geodesic on the same domain which connects its endpoints. It is easy to check that γ' does what we want.

The existence of Reeb chords of Legendrian submanifolds was among the phenomena considered in [Arn86]. Regarding Reeb chords, Arnold's prediction was that there should be good topological lower bounds for the number admitted by a Legendrian submanifold.

What has become known as the *Arnold chord conjecture* is the following:

Conjecture. *Let α be a contact form on a compact odd-dimensional manifold N, and let l be a closed Legendrian submanifold of (N, α). Then l admits a Reeb chord.*

This conjecture is still open. However, it has been proved in numerous special cases, using a wide variety of techniques.

- In 2000, Cieliebak [Cie02] proved the chord conjecture for certain, topologically uncomplicated, Legendrian submanifolds of boundaries of subcritical Stein domains.

(We will not define subcritical Stein domains here. They are complex manifolds-with-boundary which possess a certain 'convexity' property. Their boundaries inherit natural contact forms.
A wide class of compact contact manifolds arise as boundaries of subcritical Stein domains, including all contact forms compatible with the standard contact structure on S^{2n-1}, and more generally all contact-type hypersurfaces of $\mathbb{R}^{2n}$.)

- In 2001, Mohnke [Moh01] proved the chord conjecture for (among other things) boundaries of subcritical Stein domains. Mohnke's work uses Gromov's theory of J-holomorphic curves.

- In 2010, Hutchings and Taubes [Hut10a] proved the chord conjecture for all compact 3-manifolds. Their work uses deep techniques of low-dimensional topology, including embedded contact homology and Seiberg-Witten Floer homology.

We will present Mohnke's result in Section 6.2 (restricting for simplicity to contact-type hypersurfaces of $\mathbb{R}^{2n}$), after developing in Chapters 4 and 5 the facts about J-holomorphic curves of which he makes use. The key observation is to convert the problem into a statement about discs in symplectic manifolds:

Proposition 3.1.7. *Let α be a contact form on an odd-dimensional manifold N, and let l be a closed Legendrian submanifold of (N, α). Suppose $T > 0$ such that l has no Reeb chord of length less than or equal to T. Then for each $S > 0$, the symplectic manifold $(N \times [S, 0], d(e^s\alpha))$ has a Lagrangian submanifold L, such that the symplectic areas of all discs with boundary on L are multiples of $(1 - e^S)T$.*

Proof. By compactness of N and l, in fact l has no Reeb chord of length less than or equal to $T + \epsilon$, for some $\epsilon > 0$. Therefore we have an embedding of $l \times [0, T + \epsilon]$ into N, given by

$$(p, t) \mapsto \Phi_t(p),$$

where Φ_t is the flow associated with the Reeb vector field. (It is defined for all time since N is compact.) We thence construct an embedding of $l \times [0, T + \epsilon] \times [S, 0]$ into the finite cylinder $(N \times [S, 0], d(e^s\alpha))$ of N's symplectization, via

$$(p, t, s) \mapsto (\Phi_t(p), s).$$

Since α vanishes on TN, the 1-form $e^s\alpha$ on $N \times [S, 0]$ pulls back to $e^s dt$ on $l \times [0, T + \epsilon] \times [S, 0]$.

The area with respect to the form $e^s ds \wedge dt = d(e^s t)$ of the rectangle $[0, T + \epsilon] \times [S, 0]$ is $(T + \epsilon)(1 - e^S)$. Pick a simple closed smooth curve in $[0, T + \epsilon] \times [S, 0]$,

$$(\gamma_1, \gamma_2) : S^1 \to [0, T + \epsilon] \times [S, 0]$$

which encloses an area of $T(1 - e^S)$. We claim the embedding of $l \times S^1$ into $N \times [S, 0]$ constructed from this loop,

$$(p, \theta) \mapsto (\phi_{\gamma_1(\theta)}(p), \gamma_2(\theta))$$

has image a Lagrangian submanifold of $(N \times [S, 0], d(e^s\alpha))$. This is clear since $d\alpha$ vanishes on Tl, whereas the Reeb flow and $\partial/\partial s$ are transverse to $\ker \alpha$.

By Stokes' theorem, since the symplectic form $d(e^s\alpha)$ on $N \times [S, 0]$ is exact, the symplectic areas of discs with boundary on $l \times S^1$'s image are integrals of $e^s\alpha$ along their boundaries, closed loops in $l \times S^1$'s image. Pulling back under the embedding, these become: integrals of $e^s dt$ along closed loops in $l \times [0, T + \epsilon] \times [S, 0]$ whose projection to $[0, T + \epsilon] \times [S, 0]$ is contained in the loop (γ_1, γ_2).

But by construction (γ_1, γ_2) encloses an area of $T(1 - e^S)$ with respect to $d(e^s dt)$, so all such integrals are multiples of $T(1 - e^S)$. □

3.2 Lagrangian intersections

The most famous of Arnold's conjectures in [Arn86] concerns the intersections of a Lagrangian submanifold of a symplectic manifold with its image under a *Hamiltonian isotopy*; that is, the time-1 flow of a time-dependent Hamiltonian vector field.

Such intersections are precisely the initial points, of those length-1 integral curves of the Hamiltonian flow which both start and finish on the Lagrangian submanifold. This problem is thus a natural symplectic analogue of the chord conjecture.

Let us discuss some examples.

Example 3.2.1. *Let $F : W \times [0,1] \to \mathbb{R}$ be a smooth time-dependent function on a manifold W, and consider the time-dependent Hamiltonian $H = F \circ \pi$ on the symplectic (recall Examples 2.2.2 and 2.2.5) manifold T^*W. That is, H_t is a time-dependent Hamiltonian on the cotangent bundle T^*W which is constant on T^*W's fibres.*

The Hamiltonian vector field of H_t is dF_t, where we interpret the section dF_t of T^*W tautologically as a section of $T(T^*W)$. The time-1 flow of this Hamiltonian on T^*W is the diffeomorphism

$$(p, \eta) \mapsto \left(p, \eta + \int_0^1 dF_t|_p \ dt \right).$$

Each closed section of T^*W has graph a Lagrangian submanifold of T^*W. Intersections of such a section with its image under H_t's time-1 flow, correspond to points $p \in W$ such that

$$0 = \int_0^1 dF_t|_p \ dt = d \left(\int_0^1 F_t \ dt \right) |_p;$$

that is, to critical points of the function $\int_0^1 F_t dt$ on W. If W is compact and this function is 'generic,' then Morse theory gives the lower bound

$$\dim H_*(W)$$

for the number of such critical points. In any event, if W is compact, there are at least two critical points (maximum and minimum).

Example 3.2.2. *We show the study of Lagrangian intersections subsumes the study of fixed points of Hamiltonian isotopies.*

Let H_t be a time-dependent Hamiltonian on a symplectic manifold M. Then $H_t \circ pr_2$ is a time-dependent Hamiltonian on the symplectic (recall

Examples 2.2.3 and 2.2.6) manifold $M \times M$; their time-1 flows ψ_H, $\psi_{H\circ pr_2}$ respectively satisify,

$$\psi_{H\circ pr_2}(p_1, p_2) = (p_1, \psi_H(p_2)).$$

So the image of the diagonal $\Delta \subseteq M \times M$ under $M \times M$'s Hamiltonian symplectomorphism $\psi_{H\circ pr_2}$ is the graph in $M \times M$ of M's Hamiltonian symplectomorphism ψ_H.

Thus we have a correspondence between:

- *Intersections of the Lagrangian submanifold $\Delta \subseteq M \times M$ with its image under the time-1 flow of $H_t \circ pr_2$; and,*
- *Fixed points of the time-1 flow of H_t.*

Example 3.2.3. *Let (N, α) be a manifold with contact form. Consider, for some smooth function $F : \mathbb{R} \to \mathbb{R}$, the (time-independent) Hamiltonian $H = F \circ pr_2$ on N's symplectization $(N \times \mathbb{R}, d(e^s\alpha))$. That is, H is a Hamiltonian on $N \times \mathbb{R}$ which is constant on cross-sections $N \times \{s_0\}$.*

Let Y be the Reeb vector field of α. We claim that $-F'(s)e^{-s}Y$ is the Hamiltonian vector field of H. Indeed,

$$\begin{aligned} d(e^s\alpha)(-F'(s)e^{-s}Y) &= e^s(d\alpha + ds \wedge \alpha)(-F'(s)e^{-s}Y) \\ &= -F'(s)(d\alpha(Y, \cdot) - \alpha(Y)ds) \\ &= F'(s)ds = dH. \end{aligned}$$

Each Legendrian submanifold $l \subseteq N$ gives rise to a Lagrangian submanifold $l \times \mathbb{R}$ of (N, α). Intersections of $l \times \mathbb{R}$ with its image under time-1 flow of H correspond to Reeb chords of l of length $|F'(s)|e^{-s}$ for some $s \in \mathbb{R}$.

In particular, if $s \mapsto |F'(s)|e^{-s}$ is surjective onto $\mathbb{R}^+$ (say, $F(s) = e^{2s}$), then $l \times \mathbb{R}$ intersects its image under H's time-1 flow precisely if l admits a Reeb chord.

Example 3.2.4. *This is a cautionary example: any compact Lagrangian submanifold L of $\mathbb{R}^{2n}$ can be completely displaced from itself by a compactly-supported Hamiltonian isotopy.*

For instance, enclose L in some product of open half-discs $S_1 \times \cdots \times S_n \subseteq \mathbb{R}^{2n}$, centred without loss of generality at the origin, so

$$S_i = \{(x, y) \in \mathbb{R}^2 : x > 0, x^2 + y^2 < R_i^2\}.$$

Define $H_i : \mathbb{R}^2 \to \mathbb{R}$ by

$$H_i(x, y) = \begin{cases} 0, & x^2 + y^2 \geq R_i^2 \\ \frac{1}{2}\pi(R_i^2 - x^2 - y^2), & x^2 + y^2 < R_i^2 \end{cases}$$

and $H : \mathbb{R}^{2n} \to \mathbb{R}$

$$H(x_1, y_1, \ldots x_n, y_n) = \Sigma_{i=1}^n H_i(x_i, y_i).$$

H is compactly supported and continuous, and smooth except on the boundary of $S_1 \times \cdots \times S_n$. Its Hamiltonian vector field X_H at $(x_1, y_1, \ldots x_n, y_n) \in \mathbb{R}^{2n}$ not on $S_1 \times \cdots \times S_n$'s boundary is

$$(X_{H_1}|_{(x_1,y_1)}, \ldots X_{H_n}|_{(x_n,y_n)}) \in T_{(x_1,y_1)}\mathbb{R}^2 \times \cdots \times T_{(x_n,y_n)}\mathbb{R}^2,$$

with, more explicitly,

$$X_{H_i}|(x,y) = \begin{cases} 0, & x^2 + y^2 \geq R_i^2 \\ \pi(-y, x), & x^2 + y^2 < R_i^2; \end{cases}$$

its time-1 flow is rotation by π, inside the circle $x^2 + y^2 = R_i^2$, and the identity, outside the circle $x^2 + y^2 = R_i^2$.

Smoothly approximating H gives a smooth compactly-supported Hamiltonian isotopy which displaces L.

What has become known as the *Arnold conjecture* for Lagrangian intersections, is versions of the following:

Conjecture. *Let L_0, L_1 be compact Lagrangian submanifolds of a symplectic manifold M, such that some compactly-supported Hamiltonian isotopy of M sends L_0 to L_1. Then L_0 and L_1 must have at least as many intersection points as a function on L_0 must have critical points.*

Example 3.2.4 shows we must place some restrictions on L_0 if working in complete generality. The arguments of Example 3.2.1 suggest considering, if L_0 and L_1 intersect transversely, the interesting weaker lower bound $\dim H_*(L_0)$.

The two breakthroughs on this conjecture came in the late '80s:

- In 1985, Gromov [Gro85] considered the Arnold conjecture for *tame geometrically bounded* symplectic manifolds, and Lagrangian submanifolds L_0 such that all discs with boundary on L_0 have zero symplectic area. He proved that L_0 and its image must intersect at least once.

(We will not define tame geometrical boundedness here. It is a property possessed by all compact symplectic manifolds, and also by well-behaved non-compact ones such as $\mathbb{R}^{2n}$ and cotangent bundles.)

- In 1988, Floer [Flo88], under essentially the same restrictions as Gromov, proved that if L_0 and its image intersect transversely then they intersect in at least $\dim H_*(L_0, \mathbb{Z}_2)$ points.

In Section 6.1, we will prove an sharpening, due to Chekanov [Che96], of Gromov's result (restricting for simplicity to the symplectic manifold $\mathbb{R}^{2n}$). This relates the minimal 'size' of a Hamilton isotopy which displaces a Lagrangian submanifold L to the minimal symplectic area of a disc with boundary on L.

Chapter 4

Curves in almost-complex manifolds

This chapter discusses the set of solutions to a generic *perturbed Cauchy-Riemann equation* in an *almost-complex manifold.* Section 4.1 introduces these objects. Sections 4.2-3 discuss the Sard-Smale theorem and the Riemann-Roch theorem, in preparation for the two *transversality theorems* of Section 4.4, which are the major results of the chapter.

Theorem 4.2.1's neat summary of the Morse theory paradigm is taken from the lecture notes [Hut02]. Section 4.3's discussion of Cauchy-Riemann operators, the Maslov index and the Riemann-Roch theorem is based on that of [MS04] Appendix C.

The notation and arguments of Section 4.4 are modelled on that of [MS04]'s Chapter 3 (and the lecture notes [Wen10] which amplify it). However, their discussion there has a rather different focus:

- They are interested in maps with closed domain (for instance, S^2), whereas we are most interested in the surface with boundary D^2.
- They are interested in the moduli space of J-holomorphic curves for a variable almost-complex structure J, whereas we are interested in the moduli space of solutions to a perturbed Cauchy-Riemann equation for a variable perturbation, having fixed a particular almost-complex structure.

In the second of these differences, our needs are simpler: we have a nice linear structure on the space of things we vary, and because solutions for a generic perturbation have no notrivial automorphisms our compactness proofs will be easier later on.

The statements of Theorem 4.4.2 and 4.4.3 are therefore instead modelled on [MS04]'s Section 8.3; this section is a brief discussion of a pair of theorems which matches our needs on the second point (though not on the first).

The final chapter of [AL94] contains similar arguments in concrete form and in a context very close to our intended applications, and was very useful as an introduction.

Most of this section is intended to be heuristic rather than rigorous. We are deliberately vague about the regularity demanded of functions in our 'spaces' (for instance, the 'space' of maps $u : (D^2, \partial D^2) \to (M, L)$, the 'space' of sections of $\Lambda^{0,1} D^2 \boxtimes TM$, and so on) and about the topology that we put on them. We hope that the purely formal approach of this chapter conveys some intuition despite the necessary technical gaps.

The reason that a naive argument will fail is that the natural class and topology of maps – the class of smooth maps, in the topology of uniform convergence in all derivatives – is not Banach. A rigorous argument would first carry out analogues of this chapter's arguments in the kth-order weakly differentiable category, for each $k \in \mathbb{N}$, and thence deduce the smooth-category result.

4.1 Almost-complex structures

An *almost-complex structure* on a manifold M is a section J of $T^1_1 M$, such that $J^2 = -1$. If M admits an almost-complex structure, it is forced even-dimensional: at each point $p \in M$, the endomorphism $J|_p$ of $T_p M$ can have as eigenvalues only $\pm i$, and must have an equal number of each.

An n-submanifold L of an almost-complex $2n$-manifold (M, J) is *totally real*, if for each nonzero $X \in TL$ the vector JX is not in TL.

Henceforth we fix a compact almost-complex manifold (M, J).

Let (Σ, j) a Riemann surface (possibly with boundary). From a smooth map $u : \Sigma \to M$ we define the object $\overline{\partial}_J u$, the *complex anti-linear part of u's differential*, which is a section of the complex vector bundle $\Lambda^{0,1}\Sigma \otimes_{\mathbb{C}} u^* TM$ over Σ, and is given explicitly by

$$\overline{\partial}_J u|_z = \frac{1}{2} \left(du_z + J_{u(z)} \circ du|_z \circ j_z \right).$$

A map $u : \Sigma \to M$ is *J-holomorphic*, if it satisfies the *Cauchy-Riemann equation*: for all $z \in \Sigma$,

$$\overline{\partial}_J u|_z = 0.$$

More generally, we will be interested in solutions to *perturbed Cauchy-Riemann equations*. That is, consider the complex vector bundle $\Lambda^{0,1}\Sigma \boxtimes_{\mathbb{C}} TM$ over $\Sigma \times M$, whose fibre over (z, p) is $\Lambda^{0,1}_z \Sigma \otimes_{\mathbb{C}} T_p M$, canonically isomorphic to the space of complex anti-linear homomorphisms from $T_z \Sigma$ into $T_p M$. Given a typical section g of $\Lambda^{0,1}\Sigma \boxtimes_{\mathbb{C}} TM$, we will be interested in the solutions $u : \Sigma \to M$ to the equation: for all $z \in \Sigma$,

$$\overline{\partial}_J u|_z + g|_{(z, u(z))} = 0. \tag{4.1}$$

Henceforth we will often abbreviate this equation as:

$$\overline{\partial}_J u + g|_{\widetilde{u}} = 0.$$

The rest of this section is a slight digression, which will be useful in Chapter 5. We observe that the graphs of solutions of a perturbed Cauchy-Riemann equation can be interpreted as pseudoholomorphic curves for a perturbed almost-complex structure on the product manifold.

Lemma 4.1.1. *Let (M, J) be an almost-complex manifold, (Σ, j) a Riemann surface, and g a section of $\Lambda^{0,1}\Sigma \boxtimes_{\mathbb{C}} TM$.*

(i) The $(1,1)$-tensor

$$J_g = \begin{pmatrix} j & 0 \\ 2Jg & J \end{pmatrix}$$

on $\Sigma \times M$ is an almost-complex structure.

(ii) Fix $w \in \Sigma$. Let Σ' be another Riemann surface, and let $u : \Sigma' \to M$. Then the map $(w, u) : \Sigma' \to \Sigma \times M$ into the fibre $\{w\} \times M$, defined by

$$(w, u)(z) = (w, u(z)),$$

is J_g-holomorphic if and only if u is J-holomorphic.

(iii) Let $u : \Sigma \to M$. Then u's graph $\widetilde{u} : \Sigma \to \Sigma \times M$, defined by

$$\widetilde{u}(z) = (z, u(z)),$$

is J_g-holomorphic if and only if u is a solution of (4.1).

Proof. 1. Since g is complex anti-linear, $Jgj + J^2 g = g - g = 0$. So

$$J_g^2 = \begin{pmatrix} j^2 & 0 \\ 2(Jgj + J^2 g) & J^2 \end{pmatrix}^2 = 0.$$

2. Clear.

3. Let $u : \Sigma \to M$ have graph $\widetilde{u} : \Sigma \to \Sigma \times M$, and let $z \in \Sigma$. Then

$$\begin{aligned} \overline{\partial}_{J_g}\widetilde{u}|_z &= \frac{1}{2}\left[\begin{pmatrix} I \\ du_z \end{pmatrix} + \begin{pmatrix} j_z & 0 \\ 2J_{u(z)} \circ g|_{(z,u(z))} & J_{u(z)} \end{pmatrix}\begin{pmatrix} I \\ du_z \end{pmatrix} j_z\right] \\ &= \frac{1}{2}\begin{pmatrix} I - {j_z}^2 \\ du_z + 2J_{u(z)} \circ g|_{(z,u(z))} \circ j_z + J_{u(z)} \circ du|_z \circ j_z \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \overline{\partial}_J u|_z + g|_{(z,u(z))} \end{pmatrix}. \end{aligned}$$

So $\overline{\partial}_{J_g}\widetilde{u}|_z = 0$ for all z, if and only if $\overline{\partial}_J u|_z + g|_{(z,u(z))} = 0$ for all z.

□

4.2 Banach manifolds and generic behaviour

We begin this section by summarizing, very abstractly, the analytical setup for this chapter's results on moduli spaces of solutions to perturbed Cauchy-Riemann equations. We follow Hutchings' notes [Hut02].

Definition. *Let R be a separable Banach manifold, $E \to R$ a separable Banach space bundle, and $\psi : R \to E$ a smooth section. Then the* linearization *of the section ψ at a point $r \in R$ at which ψ vanishes, is the map $D\psi|_r : T_rR \to E_r$ defined by, for $v \in T_rR$,*

$$D\psi|_r(v) = \nabla_v \psi,$$

where ∇ is an arbitrary connection on R.

Remark. *To check the linearization is well-defined, we should show that its definition is independent of the choice ∇ of connection. Indeed, if ∇ and $\overline{\nabla}$ are connections on R, then their difference is tensorial, so, since ψ vanishes at r, $(\nabla - \overline{\nabla})\psi$ does also.*

Theorem 4.2.1. *Let Q be a separable Banach manifold, R a separable Banach manifold with (possibly null) boundary, $E \to Q \times R$ a separable Banach space bundle, and $\psi : Q \times R \to E$ a smooth section. Suppose that for all $(q, r) \in \psi^{-1}(0)$, the following hold:*

1. *The linearization $D\psi|_{(q,r)} : T_{(q,r)}(Q \times R) \to E_{(q,r)}$ is surjective.*
2. *The restricted linearization $D(\psi_q)|_r : T_rR \to E_{(q,r)}$ is Fredholm of index $l \geq 0$.*

Suppose also that for all $(q, r) \in \psi^{-1}(0)$ with $r \in \partial R$,

3. *The (further-)restricted linearization $D(\psi_q)|_r : T_r(\partial R) \to E_{(q,r)}$ is surjective.*

Let Q_{reg} be the subset of Q consisting of those $q \in Q$ such that, for all $r \in R$ with $\psi(q, r) = 0$, the restricted linearization $D(\psi_q)|_r : T_rR \to E_{(q,r)}$ is surjective. Then:

1. *For each $q \in Q_{reg}$, the subset $\{r \in R : \psi(q, r) = 0\}$ of R is a a smooth manifold of dimension l, with boundary $\{r \in \partial R : \psi(q, r) = 0\}$.*
2. *The set Q_{reg} is of the second category in Q.*

Remark. *The tangent space $T_{(q,r)}(Q \times R)$ splits as $T_qQ \times T_rR$. The linearization*

$$D\psi|_{(q,r)} : T_qQ \times T_rR \to E_{(q,r)}$$

splits as $D(\psi_q)|_r + D(\psi_r)|_q$*, where*

$$D(\psi_q)|_r : T_rR \to E_{(q,r)} \quad \textit{and} \quad D(\psi_r)|_q : T_qQ \to E_{(q,r)}$$

respectively are the 'restricted linearizations' of $D\psi|_{(q,r)}$*; that is, the linearizations of the section* $\psi_q := \psi(q,\cdot)$ *of* $E_q \to R$*, and the section* $\psi_r := \psi(\cdot, r)$ *of* $E_r \to Q$*, respectively.*

Proof of Theorem 4.2.1. An application of the Sard-Smale theorem (a Banach-space generalization of Sard's theorem on regular values of smooth maps), together with the implicit function theorem. □

Roughly speaking, the plan for this chapter is to apply Theorem 4.2.1, in the context of an almost-complex manifold (M, J) and fixed totally real submanifold $L \subseteq M$, to the following two situations:

1. (a) As R, the space of maps $u : (D^2, \partial D^2) \to (M, L)$ in a fixed homology class $A \in H_2(M, L)$.
   (b) As Q, the space of sections of $\Lambda^{0,1}D^2 \boxtimes TM$.
   (c) As E, the bundle over $Q \times R$ whose fibre over (g, u) is $\Omega^{0,1}(u^*(TM))$.
   (d) As ψ, the section of E given by,

   $$\overline{\partial}_J + g|_{\cdot}.$$

   More explicitly, the section of E whose value in the fibre over (g, u) is

   $$\overline{\partial}_J u + g|_{\widetilde{u}}.$$

   (For this notation, see the previous section.)

   Theorem 4.2.1 will tell us that for generic sections g of $\Lambda^{0,1}D^2 \boxtimes TM$, the space

   $$\mathcal{M}(A; g) := \{u \in R : \psi(g, u) = 0\},$$

   consisting of the solutions to the perturbed Cauchy-Riemann equation (4.1) in the homotopy class A, is a smooth oriented manifold.

2. Fix sections g_0 and g_1 of $\Lambda^{0,1}D^2 \boxtimes TM$.

   (a) As R, the product of $[0, 1]$ with the space of maps $u : (D^2, \partial D^2) \to (M, L)$ in a fixed homology class $A \in H_2(M, L)$.
   (b) As Q, the space of *homotopies* $(g_\lambda)_\lambda$ from g_0 to g_1 through sections of $\Lambda^{0,1}D^2 \boxtimes TM$.
   (c) As E, the bundle over $Q \times R$ whose fibre over $((g_\lambda)_\lambda, \lambda, u)$ is $\Omega^{0,1}(u^*(TM))$.

(d) As ψ, the section of E given by,

$$\overline{\partial}_J + g\models.$$

More explicitly, the section of E whose value in the fibre over $((g_\lambda)_\lambda, \lambda, u)$ is

$$\overline{\partial}_J u + g_\lambda|_{\widetilde{u}}.$$

Theorem 4.2.1 will tell us that for generic homotopies $(g_\lambda)_\lambda$ from g_0 to g_1, the space

$$\mathcal{W}(A; (g_\lambda)_\lambda) = \{(\lambda, u) : \lambda \in [0,1], u \in \mathcal{M}(A; g_\lambda)\}$$

consisting of the disjoint union of the solutions in the homology class A to the perturbed Cauchy-Riemann equations along the homotopy $(g_\lambda)_\lambda$, is a smooth manifold with boundary

$$\mathcal{M}(A; g_0) \cup \mathcal{M}(A; g_1).$$

It is clear from our intended applications that the linearization at a solution $u \in \mathcal{M}(A; g)$ of the section $\overline{\partial}_J + g\models$ will be of crucial interest. We conclude this section by describing an operator which, again roughly speaking, is this linearization. The objects of Section 4.3 will be to show this operator is Fredholm and to calculate its index.

Lemma 4.2.2. *Let g be a section of $\Lambda^{0,1}D^2 \boxtimes_{\mathbb{C}} TM$, let $u \in \mathcal{M}(A; g)$, and let $\xi \in \Omega^0(u^*(TM))$. Pick a symmetric connection ∇ on M, and consider the element of $\Lambda^1 D^2 \otimes u^*(TM)$ given by*

$$(\nabla\xi)^{0,1} + \frac{1}{2}(\nabla_\xi J) \circ du \circ j + \nabla_\xi g;$$

or, more explicitly, as the bundle map from TD^2 to $u^*(TM)$ given by,

$$v \mapsto \frac{1}{2}\left[(u^*\nabla)_v \xi + J(u^*\nabla)_{jv}\xi + (\nabla_{\xi(z)}J)(du(jv))\right] + \left[\nabla_{\xi(z)}(g|_z)\right](v).$$

Then this element of $\Lambda^1 D^2 \otimes u^*(TM)$ is independent of the choice of ∇, and is complex anti-linear in v.

Proof. Straightforward calculations applying the perturbed Cauchy-Riemann equation (4.1). □

We write

$$D_{g,u} : \Omega^0(u^*(TM)) \to \Omega^{0,1}(u^*(TM))$$

for the thus-defined operator

$$\xi \mapsto (\nabla\xi)^{0,1} + \frac{1}{2}(\nabla_\xi J) \circ du \circ j + \nabla_\xi g.$$

Roughly speaking, the tangent space at u of the manifold

$$\left\{\text{maps from } (D^2, \partial D^2) \text{ to } (M, L) \text{ in the homology class } A\right\}.$$

is the space of sections of the pullback complex vector bundle $u^*(TM) \to D^2$ which, on the disc's boundary ∂D^2, lie in the totally real sub-bundle $u|_{\partial D^2}{}^*(TL)$, which we write as

$$\Omega^0(u^*(TM), u|_{\partial D^2}{}^*(TL)).$$

Again roughly speaking, the restriction of $D_{g,u}$ to

$$\Omega^0(u^*(TM), u|_{\partial D^2}{}^*(TL)),$$

is the linearization at u of the section $\overline{\partial}_J + g\dagger$.

4.3 The Riemann-Roch theorem

Definition. *Let E be a complex vector bundle over a Riemann surface (possibly with boundary) Σ. We write J for the complex structure on E.*

1. *A* real linear Cauchy-Riemann operator *on E is a linear operator*

$$D : \Omega^0(E) \to \Omega^{0,1}(E)$$

   which satisfies the Leibnitz rule: for $\xi \in \Omega^0(E)$ and smooth real functions $f \in \mathcal{C}^\infty(\Sigma)$,

$$D(f\xi) = fD\xi + \frac{1}{2}\left[df\xi + (df \circ j)J\xi\right].$$

2. *A* (complex linear) Cauchy-Riemann operator *on E is a real linear Cauchy-Riemann operator which commutes with J: for $\xi \in \Omega^0(E)$,*

$$D(J\xi) = JD\xi.$$

The idea of the preceding definition is as follows: A (complex linear) Cauchy-Riemann operator is a very natural geometric object. (In fact it can be shown that a Cauchy-Riemann operator is essentially the same data as a choice of holomorphic structure on a bundle pair.) A real linear Cauchy-Riemann operator is a 'zeroth-order perturbation' of a complex linear Cauchy-Riemann operator; that is, a real linear Cauchy-Riemann operator differs by a tensor from some complex linear one.

The following example is our intended use of the concept.

Example 4.3.1. *Let (M, J) be an almost-complex manifold, let L be a totally real submanifold of M, let g be a section of $\Lambda^{0,1}D^2 \boxtimes_{\mathbb{C}} TM$, let $A \in H_2(M, L)$, and let $u \in \mathcal{M}(A; g)$. Then the operator $D_{g,u}$ is a real linear Cauchy-Riemann operator on the complex bundle $u^*(TM)$ (with complex structure J) over the Riemann surface D^2.*

Proof. Let $\xi \in \Omega^0(u^*(TM))$ and $f \in \mathcal{C}^\infty(D^2)$, and pick a symmetric connection ∇ on M. Then

$$\begin{aligned}
D_{g,u}(f\xi) &= \frac{1}{2}\left[\nabla f\xi + J(\nabla f\xi) \circ j + (\nabla_{f\xi} J) \circ du \circ j\right] + \nabla_{f\xi} g \\
&= \frac{1}{2}\left[f\nabla\xi + (df)\xi + fJ(\nabla\xi) \circ j + (df \circ j)J\xi + f(\nabla_\xi J) \circ du \circ j\right] + f\nabla_\xi g \\
&= fD_{g,u}\xi + \frac{1}{2}\left[df\xi + (df \circ j)J\xi\right].
\end{aligned}$$

□

The classical Riemann-Roch theorem can be interpreted as a formula for the index of a complex-linear Cauchy-Riemann operator. Since the index of a Fredholm operator is invariant under perturbation, it is therefore plausible that some variant of the Riemann-Roch theorem should give a formula more generally for the indices of real linear Cauchy-Riemann operators. We spend the rest of the section describing such a variant on the disc D^2.

First we specify some boundary conditions. Following [MS04], we define a *bundle pair* over a Riemann surface Σ to be a pair (E, F), where E is a complex vector bundle over Σ and F is a totally real sub-bundle of $E|_{\partial\Sigma}$.

Example 4.3.2. *Let (M, J) be an almost-complex manifold, let L be a totally real submanifold of M, and let $u : (D^2, \partial D^2) \to (M, L)$. Then $(u^*(TM), u|_{\partial D^2}{}^*(TL))$ is a bundle pair over the disc D^2.*

A useful invariant of bundle pairs is the *boundary Maslov index*, defined by the following lemma.

Lemma 4.3.3. *There is a unique operation, assigning to each bundle pair (E, F) over the complex disc an integer $\mu(E, F)$, for which the following conditions are satisfied:*

1. *Let (E_1, F_1) and (E_2, F_2) be isomorphic bundle pairs over D^2. Then*

$$\mu(E_1, F_1) = \mu(E_2, F_2).$$

2. *Let (E_1, F_1) and (E_2, F_2) be bundle pairs over D^2. Then*

$$\mu(E_1 \oplus E_2, F_1 \oplus F_2) = \mu(E_1, F_1) + \mu(E_2, F_2).$$

3. *Let* $E = D \times \mathbb{C}$ *be the trivial line bundle over* D^2*, and let* F_k *be the totally real sub-bundle of* $E|_{\partial D^2}$ *whose fibre over* $e^{i\theta} \in \partial D^2$ *is* $\mathbb{R}e^{ik\theta/2}$*. Then*
$$\mu(E, F_k) = k.$$

Lemma 4.3.4 (Homology invariance for pullback bundles)**.** *Let* (M, J) *be an almost-complex manifold, let* L *be a totally real submanifold of* M*, and let* $u : (D^2, \partial D^2) \to (M, L)$*. Then the Maslov index of the bundle pair* $(u^*(TM), u|_{\partial D^2}{}^*(TL))$ *over* D^2 *depends only on* u*'s homology class in* M *relative to* L*.*

We therefore introduce the notation $\mu(A)$, for $A \in H_2(M, L)$. The integer $\mu(A)$ is the Maslov index of the bundle pair $(u^*(TM), u|_{\partial D^2}{}^*(TL))$, for any map $u : (D^2, \partial D^2) \to (M, L)$ in the class A.

Example 4.3.5. *If* u *is nullhomologous relative to* L*, then the bundle pair* $(u^*(TM), u|_{\partial D^2}{}^*(TL))$ *is isomorphic to the trivial bundle pair* $(D^2 \times \mathbb{C}^n, D^2 \times \mathbb{R}^n)$*, and the Maslov index of* $(u^*(TM), u|_{\partial D^2}{}^*(TL))$ *is 0.*

Using the Maslov index we can state the appropriate version of Riemann-Roch:

Theorem 4.3.6 (Riemann-Roch for the disc)**.** *Let* (E, F) *be a bundle pair over* D^2*. Let* D *be a real linear Cauchy-Riemann operator on* E*, and consider its restriction*
$$D : \Omega^0(E, F) \to \Omega^{0,1}(E)$$
to the space of smooth sections of E *which, on* ∂D^2*, lie in* F*. Then this restriction is Fredholm, with index* $n + \mu(E, F)$*.*

Example 4.3.7. *Consider the standard complex anti-linear differential*
$$\overline{\partial} : \Omega^0(\Sigma \times \mathbb{C}, \partial\Sigma \times \mathbb{R}) \to \Omega^{0,1}(\Sigma \times \mathbb{C});$$
it is (in fact the prototypical example of) a Cauchy-Riemann operator.

The kernel of $\overline{\partial}$ *with the given boundary conditions is the set of holomorphic functions on the disc which are real on the disc's boundary. By the mean value property of harmonic functions, the complex parts of such functions are uniformly 0, so such functions must be constant. The real dimension of* $\overline{\partial}$*'s kernel is therefore 1.*

Since the bundle pair $(\Sigma \times \mathbb{C}, \partial\Sigma \times \mathbb{R})$ *is trivial, it has zero Maslov index. Therefore by the Riemann-Roch theorem* $\overline{\partial}$*'s restriction to* $\Omega^0(\Sigma \times \mathbb{C}, \partial\Sigma \times \mathbb{R})$ *is Fredholm of index* 1*.*

We conclude that with these boundary conditions $\overline{\partial}$ *has cokernel dimension* $1 - 1 = 0$*, so is surjective.*

Corollary 4.3.8. *Let (M, J) be an almost-complex manifold, let L be a totally real submanifold of M, let g be a section of $\Lambda^{0,1}D^2 \boxtimes_{\mathbb{C}} TM$, let $A \in H_2(M, L)$, and let $u \in \mathcal{M}(A; g)$. Then the restriction to*

$$\Omega^0(u^*(TM), u|_{\partial D^2}{}^*(TL)),$$

of the operator $D_{g,u}$ is Fredholm, with index $n + \mu(A)$.

Proof. The homology invariance of the Maslov index for pullback bundles, Lemma 4.3.4. □

4.4 Generic perturbed Cauchy-Riemann equations

As in previous sections, we let (M, J) be a compact almost-complex manifold and L a compact totally real submanifold of M, and denote by $(D^2, \partial D^2)$ the unit disc with its standard Riemann-surface-with-boundary structure.

We continue to write $\mathcal{M}(A; g)$ for the set of solutions in a homology class A to the perturbed Cauchy-Riemann equation (4.1).

Definition. *Let $\mathcal{G}$ be a Banach subspace of the space of sections of $\Lambda^{0,1}D^2 \boxtimes TM$. We will say $\mathcal{G}$ is* sufficiently full, *if for each map $u : (D^2, \partial D^2) \to (M, L)$ and each $v \in \Omega^{0,1}(u^*(TM))$, there is an element $g \in \mathcal{G}$ which agrees with v along the graph of u; that is, such that for all $z \in \Sigma$,*

$$g|_{(z,u(z))} = v_z \in \Lambda_z^{0,1}\Sigma \otimes T_{u(z)}M.$$

We now fix some more notation for the rest of the section. Let $\mathcal{B}$ be a Banach space, and $\widehat{}$ (more explicitly,

$$B \mapsto \widehat{B})$$

a bounded linear map from $\mathcal{B}$ into the sections of $\Lambda^{0,1}D^2 \boxtimes TM$. We impose the following hypothesis on $(\mathcal{B}, \widehat{})$: we demand its image $\widehat{\mathcal{B}}$, a subspace of the space of sections of $\Lambda^{0,1}D^2 \boxtimes TM$, be sufficiently full.

We also fix a homology class $A \in H_2(M, L)$, and an open subset V of $\mathcal{B}$.

Definition. *The* regular subset *of V, denoted $V_{reg}(A)$, consists of those $B \in V$ such that, for all $u : (D^2, \partial D^2) \to (M, L)$ in the class A solving*

$$\overline{\partial}_J u + \widehat{B}|_{\tilde{u}} = 0,$$

the linearization $D_{\widehat{B},u}$ is surjective.

Example 4.4.1. *Let us show that if the homology class A is zero, then $0 \in V_{reg}$. We need to prove: for all J-holomorphic $u : (D^2, \partial D^2) \to (M, L)$ which are nullhomologous relative to L, the the associated linearization*

$$D_{0,u} : \Omega^0(u^*(TM), u|_{\partial D^2}^*(TL)) \to \Omega^{0,1}(u^*(TM))$$

is surjective.

The key observation is that a J-holomorphic curve which is zero in relative homology is constant. (Indeed, J-holomorphic maps preserve orientation, so nonconstant ones have strictly positive symplectic area and hence nonzero homology class.)

So the bundle pair $(u^*(TM), u|^*_{\partial D^2}(TL))$ is isomorphic to the trivial bundle pair

$$(\Sigma \times \mathbb{C}^n, \partial\Sigma \times \mathbb{R}^n),$$

and the operator $D_{0,u}$ must correspond under some such isomorphism to the trivial operator

$$D^n : \Omega^0(\Sigma \times \mathbb{C}^n, \partial\Sigma \times \mathbb{R}^n) \to \Omega^{0,1}(\Sigma \times \mathbb{C}^n)$$

constructed from the complex anti-linear differential $\overline{\partial}$ with trivial boundary conditions,

$$\overline{\partial} = D : \Omega^0(\Sigma \times \mathbb{C}, \partial\Sigma \times \mathbb{R}) \to \Omega^{0,1}(\Sigma \times \mathbb{C}),$$

as discussed in Example 4.3.7. We proved in that example that $\overline{\partial}$ is surjective. Therefore its n-fold product is surjective too.

Theorem 4.4.2. *(i) For each $B \in V_{reg}(A)$, the space $\mathcal{M}(A; \widehat{B})$ is a smooth oriented manifold of dimension $n + \mu(A)$.*

(ii) The set $V_{reg}(A)$ is of the second category in V.

Proof. We apply the setup outlined (in a special case) previously: the idea is to apply Theorem 4.2.1 with:

1. As R, the space of maps $u : (D^2, \partial D^2) \to (M, L)$ in the homology class A.
2. As Q, the open subset V of $\mathcal{B}$.
3. As E, the bundle over $Q \times R$ whose fibre over (B, u) is $\Omega^{0,1}(u^*(TM))$.
4. As ψ, the section $\overline{\partial}_J + \widehat{B}|$- of E.

For a solution $u \in \mathcal{M}(A; \widehat{B})$, the restricted linearizations at (B, u) are:

- In the R-direction, the map $D(\psi_B)|_u = D_{\widehat{B},u} : \Omega^0(u^*(TM)) \to \Omega^{0,1}(u^*(TM))$ described in Sections 4.2-3: for $\xi \in \Omega^0(u^*(TM))$, we have

$$D_{\widehat{B},u}\xi = (\nabla\xi)^{0,1} + \frac{1}{2}(\nabla_\xi J) \circ du \circ j + \nabla_\xi \widehat{B}.$$

- In the Q-direction, the map $D(\psi_u)|_B : \mathcal{B} \to \Omega^{0,1}(u^*(TM))$ given by, for $B' \in \mathcal{B}$,

$$D(\psi_u)|_B(B') = \widehat{B}'|_{\widetilde{u}};$$

  that is, $D(\psi_u)|_B(B')$ is the section of $\Lambda^{0,1}\Sigma \otimes u^*(TM)$ which at $z \in \Sigma$ gives $\widehat{B}'|_{(z,u(z))} \in \Lambda^{0,1}_z\Sigma \otimes T_{u(z)}M$.

We need only check that Theorem 4.2.1's hypotheses hold. The condition that $\mathcal{B}$'s image be sufficiently full means that, for each $B \in V$ and each $u \in \mathcal{M}(A; \widehat{B})$, the Q-direction restricted linearization $D(\psi_u)|_B$ is surjective. Therefore the full linearization $D\psi|_{(B,u)}$ is also surjective, which is the first of Theorem 4.2.1's hypotheses.

By the results of Section 4.3, for each $B \in V$ and each $u \in \mathcal{M}(A; \widehat{B})$, the R-direction restricted linearization $D_{\widehat{B},u}$ is Fredholm of index $n + \mu(A)$. This establishes the the second of Theorem 4.2.1's hypotheses. □

Let $B_0, B_1 \in V_{reg}$. By a *homotopy from* B_0 *to* B_1 *through* V we mean a map from $[0,1]$ into V, such that 0 maps to B_0 and 1 to B_1. Following [MS04], our standard notation for a homotopy from B_0 to B_1 will be $(B_\lambda)_\lambda$. Note the distinction between the 'variable' λ in this notation and the use of a fixed $\lambda \in [0,1]$, which, confusingly, may sometimes occur in the same formula.

We denote by $V(B_0, B_1)$ the set of homotopies $(B_\lambda)_\lambda$ through V from B_0 to B_1.

For a homotopy $(B_\lambda)_\lambda$ through V, we continue to write

$$\mathcal{W}(A; (\widehat{B}_\lambda)_\lambda) = \{(\lambda, u) : \lambda \in [0,1], u \in \mathcal{M}(A; \widehat{B}_\lambda)\}$$

for the disjoint union of the sets of solutions in the homology class A to any of the equations along the homotopy.

Definition. *The* regular subset *of* $V(B_0, B_1)$*, denoted* $V_{reg}(A; B_0, B_1)$*, consists of those homotopies* $(B_\lambda)_\lambda$ *through* V *from* B_0 *to* B_1 *such that, for all* $\lambda \in [0,1]$ *and all* $u : (D^2, \partial D^2) \to (M, L)$ *in the class* A *solving*

$$\overline{\partial}_J u + \widehat{B}_\lambda|_{\widetilde{u}} = 0,$$

the linearization

$$D_{\widehat{B}_\lambda, u} + \frac{d\widehat{B}_\lambda}{dt}|_{\widetilde{u}}$$

(see more detailed description in the proof below) is surjective.

Theorem 4.4.3. *(i) For each homotopy* $(B_\lambda)_\lambda \in \mathcal{G}_{reg}(A; B_0, B_1)$*, the space* $\mathcal{W}(A; (\widehat{B}_\lambda)_\lambda)$ *is a smooth oriented manifold-with-boundary of dimension* $n+1$*, with boundary*

$$\mathcal{M}(A; \widehat{B}_1) - \mathcal{M}(A; \widehat{B}_0).$$

(ii) The set $V_{reg}(A; B_0, B_1)$ *is of the second category in* $V(B_0, B_1)$*.*

Proof. We apply the setup outlined (in a special case) previously: the idea is to apply Theorem 4.2.1 with:

1. As R, the product of $[0,1]$ with the space of maps $u : (D^2, \partial D^2) \to (M, L)$ in the homology class $A \in H_2(M, L)$. Its boundary is the product of $\{0, 1\}$ with the space of maps.
2. As Q, the space $V(B_0, B_1)$.
3. As E, the bundle over $Q \times R$ whose fibre over $((B_\lambda)_\lambda, \lambda, u)$ is $\Omega^{0,1}(u^*(TM))$.
4. As ψ, the section of E given by,

$$((B_\lambda)_\lambda, \lambda, u) \mapsto \overline{\partial}_J u + \widehat{B}_\lambda|_{\widetilde{u}}.$$

For a solution $(\lambda, u) \in \mathcal{W}(A; (B_\lambda)_\lambda)$, the restricted linearizations at the point $((B_\lambda)_\lambda, \lambda, u)$ in $Q \times R$ are:

- In the R-direction, since R is itself a product manifold, let us consider the further restrictions of the linearization to each of the factors:
  - In the $\{\text{maps } u : (D^2, \partial D^2) \to (M, L)\}$-direction, the restricted linearization is the Cauchy-Riemann operator

    $$D(\psi_{((B_\lambda)_\lambda, \lambda)})|_u = D_{\widehat{B}_\lambda, u} : \Omega^0(u^*(TM)) \to \Omega^{0,1}(u^*(TM))$$

    described in previous sections: for $\xi \in \Omega^0(u^*(TM))$, we have

    $$D_{\widehat{B}_\lambda, u}\xi = (\nabla \xi)^{0,1} + \frac{1}{2}(\nabla_\xi J) \circ du \circ j + \nabla_\xi \widehat{B}_\lambda.$$

  - In the $[0,1]$-direction, the restricted linearization is the map

    $$D(\psi_{((B_\lambda)_\lambda, u)})|_\lambda : \mathbb{R} \to \Omega^{0,1}(u^*(TM))$$

    defined by multiplication by

    $$\frac{dB_\lambda}{dt}|_{\widetilde{u}} \in \Omega^{0,1}(u^*(TM)).$$

- In the Q-direction, the tangent space at the point $(B_\lambda)_\lambda$ is $\mathcal{B}(0,0)$, the vector space of homotopies from 0 to 0 through $\mathcal{B}$. The restricted linearization is the map

  $$D(\psi_{(\lambda, u)})|_{(B_\lambda)_\lambda} : \mathcal{B}(0,0) \to \Omega^{0,1}(u^*(TM))$$

  given by, for $(B'_\lambda)_\lambda \in \mathcal{B}(0,0)$,

  $$D(\psi_{(\lambda,u)})|_{(B_\lambda)_\lambda}((B'_\lambda)_\lambda) = \widehat{B}'_\lambda|_{\widetilde{u}};$$

  that is, $D(\psi_{(\lambda,u)})|_{(B_\lambda)_\lambda}((B'_\lambda)_\lambda)$ is the section of $\Lambda^{0,1}\Sigma \otimes u^*(TM)$ which at $z \in \Sigma$ gives $\widehat{B}'_\lambda|_{(z,u(z))} \in \Lambda^{0,1}_z\Sigma \otimes T_{u(z)}M$.

We need only check that Theorem 4.2.1's hypotheses hold. The condition that $\mathcal{B}$'s image be sufficiently full means that, for each $(B_\lambda)_\lambda \in V_{reg}(A; B_0, B_1)$ and each $(\lambda, u) \in \mathcal{W}(A; (B_\lambda)_\lambda)$, the Q-direction restricted linearization $D(\psi_{(\lambda,u)})|_{(B_\lambda)_\lambda}$ is surjective. Therefore the full linearization $D\psi|_{((B_\lambda)_\lambda,\lambda,u)}$ is also surjective, which is the first of Theorem 4.2.1's hypotheses.

By the results of Section 4.3, for each $(B_\lambda)_\lambda \in V_{reg}(A; B_0, B_1)$ and each $(\lambda, u) \in \mathcal{W}(A; (B_\lambda)_\lambda)$, the R-direction restricted linearization $D_{\widehat{B}_\lambda,u} + \frac{d\widehat{B}_\lambda}{dt}|_{\widetilde{u}}$ restricts on a codimension-1 subspace to an operator which is Fredholm of index $n + \mu(A)$. Therefore it itself is Fredholm of index $n + \mu(A) + 1$. This establishes the the second of Theorem 4.2.1's hypotheses.

Finally, since B_0 and B_1 are in $V_{reg}(A)$, for each $(B_\lambda)_\lambda \in V_{reg}(A; B_0, B_1)$ and each $(i, u) \in \mathcal{W}(A; (B_\lambda)_\lambda)$ (i zero or 1), the restriction

$$D_{\widehat{B}_i,u} = D(\psi_{((B_\lambda)_\lambda,i)})|_u : \Omega^0(u^*(TM)) \to \Omega^{0,1}(u^*(TM))$$

of the linearization to the boundary tangent space at $((B_\lambda)_\lambda, i, u)$ is surjective. □

Chapter 5

Curves in tame almost-complex manifolds

The object of this chapter is Proposition 5.3.1, which gives a lower bound for the "radius of solubility" of certain families of perturbed Cauchy-Riemann equations in a symplectic manifold in terms of the symplectic areas of discs in the manifold. Its proof has two key technical inputs: the transversality theorems of the previous chapter, and Gromov's compactness theorem which we describe in Section 5.1.

Section 5.2 introduces the PDE of interest: *Hamiltonian perturbations* of the Cauchy-Riemann equations of a symplectic manifold's *tame* almost-complex structures. Section 5.3 is devoted to the proof of Proposition 5.3.1.

Our major source for this chapter is [MS04]. Specifically, our statement the compactness theorem is essentially [MS04] Theorem 4.6.1, the example is theirs, and the proof of the compactness theorem which we outline is their proof. The concept of Hamiltonian perturbations and the development of their properties in Section 5.2 is distilled from [MS04] Section 8.1. Our Proposition 5.3.1 and its proof are modelled on the (similar though slightly weaker) [MS04] Proposition 9.2.16.

Corollary 5.2.4, the explicit re-working of Gromov's compactness theorem for sequences of solutions to Hamiltonian perturbed Cauchy-Riemann equation, does not have an analogue in [MS04], since the sharp estimates using curvature are not required for the applications there. The set-up of this step in the argument owes much to Polterovich's presentation ([Pol93] Proposition 3.1) of the analogous technical result for his work.

The final chapter of [AL94] sketches an argument similar in spirit to that with which we, following [MS04], prove Proposition 5.3.1. This sketch was invaluable as an introduction to the subject.

5.1 Gromov's compactness theorem

This section is devoted to a fundamental theorem of Gromov. The theorem concerns compactness properties for families of pseudoholomorphic curves; that is, the question of whether a suitably bounded sequence of pseudoholomorphic curves must have an (in some sense) convergent subsequence.

We will see that, for a symplectic manifold's *tame* almost-complex structures, "suitably bounded" includes any sequence of pseudoholomorphic curves in a fixed homology class, and that we can ensure the existence of a subsequence "convergent" in a very strong way: uniformly in all derivatives on all compact subsets which avoid a finite number of singular points. We can also obtain a detailed understanding of what happens near these singular points: pseudoholomorphic "bubbles" form, absorbing all the energy and area and topology otherwise lost in the limit.

We state Gromov's theorem, and describe some aspects of the proof.

Definition. *An almost-complex structure J on a symplectic manifold (M, ω) is* tame*, or ω-*tame*, if for all nonzero $Y \in TM$, $\omega(Y, JY) > 0$.*

Theorem 5.1.1. *Let (M, ω) be a compact symplectic manifold, L a compact Lagrangian submanifold, and Σ a Riemann surface (possibly with boundary). Let J be an ω-tame almost complex structure, and J_n a sequence of almost-complex structures, such that J_n converges to J in the $\mathcal{C}^\infty$ topology. Let $u_n : (\Sigma, \partial\Sigma) \to (M, L)$ be a sequence of maps, J_n-holomorphic respectively, whose symplectic area satisfies*

$$\sup_n \Omega(u_n) < \infty.$$

Then there exists a subsequence (still denoted by u_n), a J-holomorphic curve $u : (\Sigma, \partial\Sigma) \to (M, L)$, and a finite subset $Z = \{z_1, \dots z_l\}$ of Σ, such that the following holds:

(i) u_n converges to u uniformly with all derivatives on compact subsets of $\Sigma \setminus Z$.

(ii) For every j, "bubbling occurs at z_j": For every $\epsilon > 0$ such that $B_\epsilon(z_j) \cap Z = \{z_j\}$, the limit

$$m_\epsilon(z_j) := \lim_{n\to\infty} \Omega(u_n; B_\epsilon(z_j))$$

exists and is a continuous function of ϵ. Moreover there exists either a non-constant J-holomorphic sphere $v : S^2 \to M$ or a non-constant J-holomorphic disc $v : (D^2, \partial D^2) \to (M, L)$, whose image can be approximated for each $\epsilon > 0$ and $N > 0$ by points in $\{u_n(z) : n \geq N, z \in B_\epsilon(z_j)\}$, and whose symplectic area $\Omega(v)$ is at most

$$m(z_j) := \lim_{\epsilon\to 0} m_\epsilon(z_j).$$

(iii) For every compact subset $K \subseteq \Sigma$ *with* $Z \subseteq int(K)$,

$$\Omega(u; K) + \sum_{j=1}^{l} m(z_j) = \lim_{n\to\infty} \Omega(u_n; K).$$

Example 5.1.2. *As* Σ*, the unit disc with boundary; as* (M, ω, J)*, the Riemann sphere, identified with* $\mathbb{C} \cup \infty$*; as* L*, the unit circle in* $\mathbb{C} \subset M$*. Pick a subsequence* (z_n) *of the open unit disc which converges to* 1*, and define* $u_n : \Sigma \to \mathbb{C} \subset M$ *by,*

$$u_n(z) = \frac{z - z_n}{1 - zz_n}.$$

This has one bubble, a disc, at $1 \in \Sigma$*: On compact subsets* $K \subset \Sigma \setminus \{1\}$*, the sequence* (u_n) *converges uniformly in all derivatives to the constant map onto* -1*. However, for arbitrarily large* N *and arbitrarily small* ϵ*, the union*

$$\bigcup_{n\geq N} u_n(B_\epsilon(1)) \subset \mathbb{C} \subset M$$

covers all of the unit disc in M *except the point* -1*.*

We now briefly discuss the compactness theorem's proof. The following claim is key:

Proposition 5.1.3. *Under the hypotheses of Theorem 5.1.1, there exists a subsequence (still denoted* (u_n)*), and and a finite subset* $Z = \{z_1, \dots z_l\}$ *of* Σ*, such that on each compact subset* K *of* $\Sigma \setminus Z$*, the sequence* $(du_n|_K)$ *is uniformly bounded.*

Note that for a family of maps, such as $(u_n|_K)$, between fixed compact manifolds, it is meaningful to speak of $(du_n|_K)$ as being uniformly bounded, even when, as here, we have no metrics in place with which to measure the differentials du; this is because all metrics on a compact manifold are equivalent.

Sketch proof. Typically established by some geometric argument. For instance, [MS04] shows: points with arbitrarily large du_n nearby must be the sites of pseudoholomorphic bubbles, and pseudoholomorphic bubbles absorb at least a certain fixed 'quantum' of symplectic area; hence there is a global bound on how many singular points there can be. Away from the singular points, passing to a subsequence if necessary, we have uniform boundedness.

Alternatively, Gromov's original [Gro85] approach, also presented in [AL94] and in [Hum97], gives an argument in hyperbolic geometry.

□

Having established Proposition 5.1.3, one can immediately conclude, for instance by the Arzela-Ascoli theorem, that u_n has a subsequence converging uniformly on compact subsets of $\Sigma \setminus F$. But in fact much more is true: elliptic regularity theory makes pseudoholomorphic curves very rigid in their convergence, so that a sequence which converges uniformly in fact turns out to converge uniformly in all derivatives. This establishes the existence of a subsequence with the property Theorem 5.1.1(i). The other parts of the theorem follow from a more detailed analysis of the points of singularity.

5.2 Hamiltonian perturbations

Let (M, ω, J) be a tame almost-complex manifold, and Σ a Riemann surface (possibly with boundary).

Definition. *Let L be a compact Lagrangian submanifold of M. A* Hamiltonian form on $\Sigma \times (M, L)$ *(or, if L is irrelevant or clear from context,* on $\Sigma \times M$*) is a section of $\Lambda^1\Sigma \boxtimes \mathbb{R}$ which vanishes on $T(\partial\Sigma) \times L$.*

Let H be a Hamiltonian form on $\Sigma \times M$. Notice that the bundle of 1-forms on $\Lambda^1(\Sigma \times M)$ splits as

$$\Lambda^1(\Sigma \times M) \cong (\Lambda^1\Sigma \boxtimes \mathbb{R}) \oplus (\mathbb{R} \oplus \Lambda^1 M).$$

In particular, H is naturally a 1-form on $\Sigma \times M$.

We thus have three natural exterior derivative operations on H:

- $d_1 H$ is a section of $\Lambda^2\Sigma \boxtimes \mathbb{R}$.
- $d_2 H$ is a section of $\Lambda^1\Sigma \boxtimes \Lambda^1 M$.
- dH is a section of $\Lambda^2(\Sigma \times M)$.

They are related by the formula $dH = d_1 H - 2\mathrm{Alt}(d_2 H)$.

Denote by X_H the section of $\Lambda^1\Sigma \boxtimes TM$ which satisfies,

$$pr_2{}^*\omega(X_H, \cdot) = d_2 H.$$

As previously, we can identify a fibre $\Lambda^1_z\Sigma \otimes T_pM$ of this bundle with the space of real homomorphisms from $T_z\Sigma$ into T_pM. Thus we can extract the complex antilinear part $X_H^{0,1}$ of X_H, defined by

$$X_H^{0,1}|_{(z,p)} = \frac{1}{2}\left[X_H|_{(z,p)} + J_p \circ X_H|_{(z,p)} \circ j_z\right],$$

a section of $(\Lambda^1\Sigma \boxtimes_{\mathbb{R}} TM)^{0,1} \cong \Lambda^{0,1}\Sigma \boxtimes_{\mathbb{C}} TM$.

Definition. *Let L be a compact Lagrangian submanifold of M. A* Hamiltonian perturbation on $\Sigma \times (M, L)$ *(or,* on $\Sigma \times M$*) is a section of $\Lambda^{0,1}\Sigma \boxtimes_{\mathbb{C}} TM$ which arises as $X_H^{0,1}$, for some Hamiltonian form H on $\Sigma \times (M, L)$.*

Solutions to Hamiltonian-perturbed Cauchy-Riemann equations – that is, maps $u : (\Sigma, \partial\Sigma) \to (M, L)$, such that for all $z \in \Sigma$,

$$\overline{\partial}_J u|_z + X_H^{0,1}|_{(z,u(z))} = 0. \tag{5.1}$$

will be our focus for the rest of the essay. Let us first prove that the space of Hamiltonian perturbations is sufficiently full, so that the results of Section 4.4 apply. This gives us a guarantee that the moduli space of solutions to a generic Hamiltonianian-perturbed Cauchy-Riemann equation is well-behaved.

Lemma 5.2.1. *The space of Hamiltonian perturbations is sufficiently full.*

Proof. Let $u : (\Sigma, \partial\Sigma) \to (M, L)$, and pick a section v of $\Lambda^{0,1} \otimes u^*(TM)$. We need to show there is a Hamiltonian form H such that $X_H^{0,1}$'s restriction to u's graph agrees with v. To construct such a Hamiltonian form, do so locally, then patch together local solutions using a partition of unity. □

Next we will use Gromov's compactness theorem to deduce some compactness properties of solutions to Hamiltonian-perturbed Cauchy-Riemann equations. We can do this because of Lemma 4.1.1, which interprets graphs of solutions to perturbed Cauchy-Riemann equations as J-holomorphic curves in a product manifold.

More precisely, we recall: $X_H^{0,1}$ will induce an almost-complex structure

$$J_H := J_{X_H^{0,1}}$$

on $\Sigma \times M$. This almost-complex structure has the property that a curve $u : \Sigma \to M$ has its graph $\widetilde{u} := (\cdot, u(\cdot))$ be J_H-holomorphic precisely if u solves (5.1).

The special property of *Hamiltonian* perturbations which will let us deduce useful compactness results, is that we can give a good definition for such a perturbation's 'size'.

Definition. *The* curvature *of a Hamiltonian form H is the section R_H of $\Lambda^2\Sigma \boxtimes \mathbb{R}$ defined by,*

$$R_H = d_1 H + (pr_2{}^*\omega)(X_H, X_H).$$

Remark. *The term 'curvature' comes from an interpretation of this quantity as the curvature of a connection on the trivial principal G-bundle over Σ, where G is the group of Hamiltonian symplectomorphisms of M, see eg McDuff-Salamon [REF].*

In Chapter 6 we will make use the following bound provided by the curvature:

Lemma 5.2.2. *Let (M, ω, J) be a tame almost-complex manifold, L a compact Lagrangian submanifold of M, $\Sigma \subset \mathbb{C}$ a Riemann surface (possibly with boundary) contained in the plane, and H a Hamiltonian form on $\Sigma \times (M, L)$. Write the natural metric on M as*

$$|X|^2 = \omega(X, JX).$$

Then for each solution $u : (\Sigma, \partial\Sigma) \to (M, L)$ to

$$\overline{\partial}_J u|_z + X_H^{0,1}|_{(z,u(z))} = 0,$$

with respect to the natural conformal co-ordinates (s, t) on $\mathbb{C}$,

$$\int_\Sigma |\partial_t u + X_H(\partial_t)|_{\widetilde{u}}|^2 ds \wedge dt = \int_\Sigma u^*\omega + \int_\Sigma R_H|_{\widetilde{u}}.$$

Proof. It is easy to check that

$$|\partial_t u + X_H(\partial_t)|_{\widetilde{u}}|^2 ds \wedge dt = u^*\omega + R_H|_{\widetilde{u}} - \widetilde{u}^* dH.$$

The final term is exact, and vanishes on $T(\partial\Sigma)$; therefore by Stokes' theorem its integral vanishes. □

Let us say that a section κ of $\Lambda^2\Sigma$ *dominates* a section R of $\Lambda^2\Sigma \boxtimes \mathbb{R}$, if for each $(z, p) \in \Sigma \times M$, we have, with respect to the natural orientation on $\Lambda^2_z\Sigma$, that $\kappa(z) > R(z, p)$.

Proposition 5.2.3. *Let (M, ω, J) be a tame almost-complex manifold, L a compact Lagrangian submanifold of M, Σ a Riemann surface (possibly with boundary), H a Hamiltonian form on $\Sigma \times (M, L)$, and κ a section of $\Lambda^2\Sigma$ which dominates the curvature R_H. Then:*

(i) The section of $\Lambda^2(\Sigma \times M)$ defined by

$$\omega_{H,\kappa} := pr_1{}^*\kappa + pr_2{}^*\omega - dH$$

is a symplectic form on $\Sigma \times M$.

(ii) The symplectic form $\omega_{H,\kappa}$ tames the almost-complex structure J_H.

(iii) Fix $w \in \Sigma$. Let Σ' be another Riemann surface, and let $u : \Sigma' \to M$. Then the $\omega_{H,\kappa}$-symplectic area of $(w, u) : \Sigma' \to \Sigma \times M$ is the same as the ω-symplectic area of $u : \Sigma' \to M$.

(iv) The submanifold $\partial\Sigma \times L$ of $(\Sigma \times M, \omega_{H,\kappa})$ is Lagrangian.

(v) Let $u : (\Sigma, \partial\Sigma) \to (M, L)$. Then the $\omega_{H,\kappa}$-symplectic area of its graph $\widetilde{u} : (\Sigma, \partial\Sigma) \to (\Sigma \times M, \partial\Sigma \times L)$ is

$$\int_\Sigma \kappa + \int_\Sigma u^*\omega.$$

Proof. (i) The 2-form $\omega_{H,\kappa}$ is a sum of closed terms, so closed. To check that it is nondegenerate, we will show that the $(2n+2)$-form

$$\omega_{H,\kappa}{}^{n+1} = (pr_1{}^*\kappa + pr_2{}^*\omega - dH)^{n+1}$$

is nonvanishing.

First, it's clear that $\kappa \wedge \kappa$ and $\kappa \wedge dH$ and ω^{n+1} and $dH \wedge dH \wedge dH$ vanish, so most of the terms in consideration disappear. Moreover, straightforward computation shows that

$$\begin{aligned} dH \wedge dH \wedge (pr_2{}^*\omega)^{n-1} &= -\frac{2}{n}(pr_2{}^*\omega)(X_H, X_H) \wedge (pr_2{}^*\omega)^n, \\ dH \wedge (pr_2{}^*\omega)^n &= d_1 H \wedge (pr_2{}^*\omega)^n. \end{aligned}$$

So we can evaluate the whole expression at once:

$$\begin{aligned} & (pr_1{}^*\kappa + pr_2{}^*\omega - dH)^{n+1} \\ = \; & (n+1) pr_1{}^*\kappa \wedge (pr_2{}^*\omega)^n - (n+1) dH \wedge (pr_2{}^*\omega)^n + \frac{n(n+1)}{2} dH \wedge dH \wedge (pr_2{}^*\omega)^{n-1} \\ = \; & (n+1)\left[pr_1{}^*\kappa \wedge (pr_2{}^*\omega)^n - d_1 H \wedge (pr_2{}^*\omega)^n - (pr_2{}^*\omega)(X_H, X_H) \wedge (pr_2{}^*\omega)^n\right] \\ = \; & (n+1)\left[pr_1{}^*\kappa - R_H\right](pr_2{}^*\omega)^n. \end{aligned}$$

Since κ is chosen to dominate the curvature R_H, this expression vanishes nowhere.

(ii) We will use the following identity: for $(z,p) \in \Sigma \times M$, and $v, v' \in T_z\Sigma$, $Y, Y' \in T_pM$,

$$(pr_2{}^*\omega - dH)(v+Y, v'+Y') = \omega(X_H(v)+Y, X_H(v')+Y') - K_H(v,v').$$

To see this, observe that both sides reduce to

$$\omega(Y,Y) + d_2H(v,Y') - d_2H(v',Y) - d_1H(v,v').$$

Using this identity, evaluating the expression $\omega_{H,\kappa}(\cdot, J_H(\cdot))$ on a typical element $v + Y \in T_z\Sigma \oplus T_pM$ of $T(\Sigma \times M)$ gives,

$$\begin{aligned} & (pr_1{}^*\kappa + pr_2{}^*\omega - dH)(v+Y, J_H(v+Y)) \\ = \; & \kappa(v, jv) + (pr_2{}^*\omega - dH)(v+Y, jv + 2JX_H^{0,1}(v) + JY) \\ = \; & (\kappa - R_H)(v, jv) + \omega(X_H(v)+Y, X_H(jv) + 2JX_H^{0,1}(v) + JY) \\ = \; & (\kappa - R_H)(v, jv) + \omega(X_H(v)+Y, J(X_H(v)+Y)). \end{aligned}$$

Since $\kappa - R_H$ tames j (since κ dominates R_H, so their difference is positively-oriented) and ω tames J, we conclude that this expression is positive except when $v = Y = 0$. Therefore $\omega_{H,\kappa}$ tames J_H.

(iii) Clear.

(iv) The submanifold $\partial\Sigma \times L$ of $\Sigma \times M$ is of the correct dimension. We need to check that for each $(z,p) \in \partial\Sigma \times L$, the subspace $T_z(\partial\Sigma) \times T_pL$ of $T_{(z,p)}(\Sigma \times L)$ is isotropic.

Indeed, let $v, v' \in T_z(\partial\Sigma)$, $Y, Y' \in T_pL$. Then

$$\omega_{H,\kappa}(v+Y, v'+Y') = \kappa(v,v') + \omega(Y,Y') - d_1H(v,v') + d_2H(v,Y') - d_2H(v',Y).$$

The first and third terms vanish since κ and d_1H are alternating and $T_z(\partial\Sigma)$ is one-dimensional. The second term vanishes since L is a Lagrangian submanifold of (M,ω). The fourth and fifth terms vanish since the assumption that H vanishes on $T(\partial\Sigma) \times L$ implies that d_2H vanishes on $T(\partial\Sigma) \times TL$.

(v) Clear.

□

Corollary 5.2.4. *Let (M,ω,J) be a tame compact symplectic manifold, L a compact Lagrangian submanifold, H a Hamiltonian form on $\Sigma \times (M,L)$ whose curvature satisfies*

$$\int_\Sigma \sup_{p\in M} R_H|_{(\cdot,p)} < \infty,$$

and (H_n) a sequence of Hamiltonian forms on $\Sigma \times (M,L)$, such that (H_n) converges to H in the $\mathcal{C}^\infty$ topology. Let $u_n : (\Sigma, \partial\Sigma) \to (M,L)$ be a corresponding sequence of maps, which are nullhomologous relative to L, and for which, for all n and all $z \in \Sigma$,

$$\overline{\partial}_J u|_z + X^{0,1}_{H_n}|_{(z,u(z))} = 0.$$

Then one of the following holds:

1. *There exists a subsequence (still denoted by u_n), and solution $u : (\Sigma, \partial\Sigma) \to (M,L)$ (nullhomologous relative to L) to*

$$\overline{\partial}_J u|_z + X^{0,1}_{H}|_{(z,u(z))} = 0, \tag{5.2}$$

such that u_n converges to u uniformly in all derivatives on compact subsets of Σ.

2. *For each $\epsilon > 0$, there exists either a J-holomorphic sphere $u : S^2 \to M$ or a J-holomorphic disc $u : (D^2, \partial D^2) \to (M,L)$, of symplectic area at most*

$$\int_\Sigma \sup_{p\in M} R_H|_{(\cdot,p)} + \epsilon.$$

Proof. Let $\epsilon > 0$. Pick a section κ of $\Lambda^2\Sigma$, which dominates K_H and satisfies

$$\int_\Sigma \kappa < \int_\Sigma \sup_{p\in M} R_H|_{(\cdot,p)} + \epsilon.$$

We will produce either a subsequence of (u_n) which converges as required in (1), or a J-holomorphic curve in M with symplectic area constrained as in (2). We will do this by applying Gromov's compactness theorem to the manifold $\Sigma \times M$ and the graphs of the u_n's. Let us therefore check that its hypotheses hold. Indeed:

- By Proposition 5.2.3(i), $(\Sigma \times M, \omega_{H,\kappa})$ is a compact symplectic manifold (with boundary).
- By Proposition 5.2.3(iv), $\partial\Sigma \times L$ is a compact Lagrangian submanifold of $(\Sigma \times M, \omega_{H,\kappa})$.
- Using Lemma 4.1.1(i), form an almost-complex structure J_H on $\Sigma \times M$. By Proposition 5.2.3(ii), J_H is $\omega_{H,\kappa}$-tame.
- Using Lemma 4.1.1(i), form for each n an almost-complex structure J_{H_n} on $\Sigma \times M$. The sequence (J_{H_n}) of almost-complex structures converges uniformly in all derivatives to J_H.
- By Lemma 4.1.1(iii), for each n, the map u_n satisfies, for all $z \in \Sigma$,

  $$\overline{\partial}_J u_n|_z = h_n|_{(z,u_n(z))},$$

  so its graph $\widetilde{u}_n : (\Sigma, \partial\Sigma) \to (\Sigma \times M, \partial\Sigma \times L)$ is J_{H_n}-holomorphic.
- By Proposition 5.2.3(v), the maps u_n are all nullhomologous relative to L,, and therefore their graphs $\widetilde{u}_n$ all have $\omega_{H,\kappa}$-symplectic area

  $$\int_\Sigma \kappa.$$

  In particular their symplectic areas are uniformly bounded.

Therefore, applying the compactness theorem, we obtain a subsequence (still denoted by $\widetilde{u}_n$), a J_H-holomorphic curve $\widetilde{u} : (\Sigma, \partial\Sigma) \to (\Sigma\times M, \partial\Sigma\times L)$, and a finite subset $Z = \{z_1, \dots z_l\}$ of Σ, such that the following holds:

1. $(\widetilde{u}_n)$ converges to $\widetilde{u}$ uniformly with all derivatives on compact subsets of $\Sigma \setminus Z$.
2. For every j, "bubbling occurs at z_j": For every $\epsilon > 0$ such that $B_\epsilon(z_j)\cap Z = \{z_j\}$, the limit

   $$m_\epsilon(z_j) := \lim_{n\to\infty} \Omega_{H,\kappa}(u_n; B_\epsilon(z_j))$$

exists and is a continuous function of ϵ. Moreover there exists either a non-constant J_H-holomorphic sphere $\widetilde{v}_j : S^2 \to \Sigma \times M$ or a non-constant J_H-holomorphic disc $\widetilde{v}_j : (D^2, \partial D^2) \to (\Sigma \times M, \partial\Sigma \times L)$, whose image can be approximated for each $\epsilon > 0$ and $N > 0$ by points in $\{(z, u_n(z)) : n \geq N, z \in B_\epsilon(z_j)\}$, and whose symplectic area $\Omega_{H,\kappa}(\widetilde{v}_j)$ is at most

$$m(z_j) := \lim_{\epsilon \to 0} m_\epsilon(z_j).$$

3. $\Omega_{H,\kappa}(\widetilde{u}) + \sum_{j=1}^{l} m(z_j) = \lim_{n\to\infty} \Omega_{H,\kappa}(\widetilde{u}_n)$.

Case 1: Z is empty.
Then $(\widetilde{u}_n)$ tends to $\widetilde{u}$ uniformly in all derivatives on compact subsets of Σ. So (u_n) converges uniformly in all derivatives on compact subsets to the M-projection

$$u : (\Sigma, \partial) \to (M, L)$$

defined by, $u = pr_2 \circ \widetilde{u}$. Moreover this limit u must also be nullhomologous relative to L, and a solution of (5.1).

Case 2: Z is nonempty.
Then $l \geq 1$; let us study the singularity z_1. The image of the J_H-holomorphic sphere or disc $\widetilde{v}_1$ can be approximated, for each $\epsilon > 0$ and $N > 0$, by points in

$$\{(z, u_n(z)) : n \geq N, z \in B_\epsilon(z_1)\} \subseteq B_\epsilon(z_1) \times M.$$

Therefore the image of $\widetilde{v}_1$ is contained in the fibre

$$\bigcap_{\epsilon > 0} B_\epsilon(z_1) \times M = \{z_1\} \times M.$$

Let v_1 be the projection $pr_2 \circ \widetilde{v}_1$ of the v_1 into M, so that for each $z \in \Sigma$, $\widetilde{v}_1(z) = (z, v_1(z))$. We obtain a sphere or disc with boundary on L. Then by Lemma 4.1.1(ii), v_1 is J-holomorphic, and by Proposition 5.2.3(iii), it has ω-symplectic area at most the $\omega_{H,\kappa}$-symplectic area of $\widetilde{v}_1$,

$$m(z_1) \leq \lim_{n\to\infty} \Omega_{H,\kappa}(\widetilde{u}_n) - \Omega_{H,\kappa}(\widetilde{u}) \leq \int_\Sigma \kappa \leq \int_\Sigma \sup_{p\in M} R_H|_{(\cdot,p)} + \epsilon.$$

□

5.3 A J-holomorphic Fredholm alternative

Let (M, ω, J) be a tame compact symplectic manifold, and L a compact Lagrangian submanifold of M. Suppose that no sphere in M or disc in M with boundary on L has positive symplectic area less than σ.

Proposition 5.3.1. *For each Hamiltonian form H on $D^2 \times (M, L)$ whose curvature satisfies*

$$\int_\Sigma \sup_{p \in M} R_H|_{(\cdot,p)} < \sigma,$$

each $w \in \partial D^2$ and each $p \in L$, there is a solution $u : (D^2, \partial D^2, w) \to (M, L, p)$ to equation (5.1).

For convenience we re-state equation (5.1) here : it says, for all $z \in D^2$,

$$\overline{\partial}_J u|_z + X_H^{0,1}|_{(z,u(z))} = 0.$$

Our proof will deduce the existence of a solution nullhomologous relative to L, by showing the moduli space of L-nullhomologous solutions is sufficiently similar to the moduli space of L-nullhomologous J-holomorphic discs.

Proof. Let $\mathcal{H}$ be the Banach space of Hamiltonian forms $D^2 \times (M, L)$, and V the open subset of $\mathcal{H}$ consisting of forms H whose curvature satisfies

$$\int_\Sigma \sup_{p \in M} R_H|_{(\cdot,p)} < \sigma,$$

We have a natural map

$$H \mapsto X_H^{0,1}$$

from $\mathcal{H}$ into the space of sections of $\Lambda^{0,1} D^2 \boxtimes_{\mathbb{C}} TM$. By Lemma 5.2.1, the image of this map is sufficiently full.

Recall the notation of Chapter 4:

- by $\mathcal{M}(0; X_H^{0,1})$, (for $H \in V$) the set of solutions to (5.1) in the homology class $0 \in H_2(M, L)$.
- by $\mathcal{W}(0; (X_{H_\lambda}^{0,1})_\lambda)$, (for a homotopy $(H_\lambda)_\lambda$ through forms in V), the disjoint union

  $$\{(\lambda, u) : \lambda \in [0, 1], u \in \mathcal{M}(0; X_{H_\lambda}^{0,1})\},$$

  of the sets of zero-homology solutions to any of the equations along the homotopy.

By Theorems 4.4.2 and 4.4.3, together with the fact that V is path-connected, and the index calculation Example 4.3.5, there exists a second-category subset V_{reg} of V, such that:

- For each Hamiltonian form $H \in V_{reg}$, the space $\mathcal{M}(0; X_H^{0,1})$ is a smooth oriented manifold of dimension $n + \mu(0) = n$, and carries a natural orientation.

- For each pair (H_0, H_1) of Hamiltonian forms in V_{reg}, there exists a homotopy $(H_\lambda)_\lambda$ from H_0 to H_1 through V, such that the space

$$\mathcal{W}(H_0, H_1) := \mathcal{W}(0; (X_{H_\lambda}^{0,1})_\lambda)$$

  is a smooth oriented manifold-with-boundary of dimension $n+1$, with (oriented) boundary $\mathcal{M}(0; X_{H_1}^{0,1}) - \mathcal{M}(0; X_{H_0}^{0,1})$.

Remark. *Of course, Theorem 4.4.3 in fact tells us that the space of such homotopies is of second category in $V_{reg}(H_0, H_1)$. However, we just need the existence of one!*

Since all J-holomorphic spheres or discs in (M, L) have symplectic area at least σ, greater than any

$$\int_\Sigma \sup_{p \in M} R_H|_{(\cdot,p)},$$

for $H \in V$, we conclude by the bubble-bounding Corollary 5.2.4 that:

- For each Hamiltonian form $H \in V_{reg}$, the oriented n-manifold $\mathcal{M}(0; X_H^{0,1})$ is compact.
- For each pair (H_0, H_1) of Hamiltonian forms in V_{reg}, the oriented $(n+1)$-manifold-with-boundary $\mathcal{W}(H_0, H_1)$ is compact.

Thus the manifolds $\{\mathcal{M}(0; X_H^{0,1}) : H \in V_{reg}\}$ are all mutually oriented-cobordant.

Henceforth fix $w \in \partial D^2$. For each $H \in V$, we have a *w-evaluation map* $\mathrm{ev}_H : \mathcal{M}(0; X_H^{0,1}) \to L$ defined by,

$$\mathrm{ev}_H(u) = u(w).$$

We need to show that for each $H \in V$, ev_H is surjective.

First we will show this for V_{reg}. For each pair (H_0, H_1) of Hamiltonian forms in V_{reg}, the w-evaluation maps ev_{H_0} (on $\mathcal{M}(0; X_{H_0}^{0,1})$) and ev_{H_1} (on $\mathcal{M}(0; X_{H_1}^{0,1})$) extend to a map $\mathrm{ev}_{H_0,H_1} : \mathcal{W}(H_0, H_1) \to L$ on the cobordism between $\mathcal{M}(0; X_{H_0}^{0,1})$ and $\mathcal{M}(0; X_{H_1}^{0,1})$, defined by,

$$\mathrm{ev}_{H_0,H_1}(\lambda, u) = u(w).$$

Therefore the parity of the degrees of ev_{H_0} and ev_{H_1} is the same. Thus the degrees of the w-evaluation maps $\{\mathrm{ev}_H : H \in V_{reg}\}$ all have the same parity.

Let's determine what this parity is by checking it in the easiest case. Recall from Example 4.4.1 that $0 \in V_{reg}$. The elements of $\mathcal{M}(0; 0)$ are precisely those J-holomorphic discs in M with boundary in L which are nullhomologous relative to L. But the only such discs are the constant ones, so $\mathcal{M}(0;0)$ is just L, and $\mathrm{ev}_0 : \mathcal{M}(0;0) \to L$ is just the identity. Thus the parity of the degree of ev_0, and hence of ev_H for any $H \in V_{reg}$, is odd.

It follows that for each $H \in V_{reg}$, the degree of ev_H is nonzero, hence the map ev_H is surjective. Can we extend this to $V \setminus V_{reg}$? Yes: Corollary 5.2.4 implies that surjectivity of ev_H for a dense subset of V implies surjectivity for all of it.

More explicitly: let $H \in V \setminus V_{reg}$ and $p \in L$. Since V_{reg} is of second category in V, there is a sequence (H_n) in V_{reg} which tends $\mathcal{C}^\infty$ to H. Our result on V_{reg} shows that for each n, there is a solution $u_n : (D^2, \partial D^2, w) \to (M, L, p)$ to

$$\overline{\partial}_J u_n|_z + X^{0,1}_{H_n}|_{(z,u_n(z))} = 0.$$

So, by Corollary 5.2.4 again, a limit of some subsequence of the u_n's provides a solution $u : (D^2, \partial D^2, w) \to (M, L, p)$ to equation (5.1). □

Corollary 5.3.2. *For each compactly supported Hamiltonian form H on $\mathbb{R} \times [0,1] \times (M, L)$ whose curvature satisfies*

$$\int_\Sigma \sup_{p \in M} R_H|_{(\cdot,p)} < \sigma,$$

each $w \in \partial D^2$ and each $p \in L$, there is a solution $u : \mathbb{R} \times [0,1], \mathbb{R} \times \{0,1\}, w) \to (M, L, p)$ to equation (5.1).

Proof. Identify the Riemann surfaces $\mathbb{R} \times [0,1]$ and $D^2 \setminus \{-1, 1\}$. Since H is compactly supported, the resulting Hamiltonian form on $D^2 \setminus \{-1,1\} \times (M, L)$ extends to a Hamiltonian form on $D^2 \times (M, L)$. Now apply Proposition 5.3.1.

We obtain a solution $u : (D^2, \partial D^2, w) \to (M, L, p)$ to equation (5.1). Restricting to $D^2 \setminus \{-1, 1\} \cong \mathbb{R} \times [0, 1]$ gives what we want. □

Chapter 6

Symplectic corollaries

We are now in a position to prove the two theorems described in the introduction and in Chapter 3. We prove the Lagrangian intersections theorem in Section 6.1, following the argument [MS04] Theorem 9.2.14 (with appropriate modifications). In Section 6.1, following [Moh01], we prove the chord conjecture.

6.1 Lagrangian intersections and displacement energy

Recall from Chapter 3 that a Hamiltonian isotopy is a symplectomorphism obtained as the time-1 flow of a (possibly time-dependent) Hamiltonian vector field. In this section we will define a notion of 'size' for Hamiltonian isotopies, and relate the size of Hamiltonian isotopies which displace a Lagrangian submanifold to the symplectic area of discs the submanifold bounds.

Definition. *Let $F : M \times [0,1] \to \mathbb{R}$ be a compactly supported time-dependent Hamiltonian on a manifold M. Then the* Hofer norm *of F is*

$$||F|| := \int_0^1 \sup_p F_t(p) - \inf_p F_t(p) dt.$$

The displacement energy *(possibly infinite) of a subset X of M is the infimum of the Hofer norms of compactly supported Hamiltonians whose time-1 flow displaces X.*

Example 6.1.1. *Recall, from Example 3.2.4, the time-independent, compactly supported Hamiltonians $H : \mathbb{R}^2 \to \mathbb{R}$ which we construct as smooth approximations of*

$$H(x,y) = \begin{cases} 0, & x^2 + y^2 \geq R^2 \\ \frac{1}{2}\pi(R^2 - x^2 - y^2), & x^2 + y^2 < R^2; \end{cases}$$

they have Hofer norm just over $\frac{1}{2}\pi R^2$ and their flow displaces the open half-disc

$$S = \{(x, y) \in \mathbb{R}^2 : x > 0, x^2 + y^2 < R^2\},$$

which has area $\frac{1}{2}\pi R^2$, from itself.

Therefore a connected simply connected open subset of $\mathbb{R}^2$, of area A, has displacement energy at most A. The same holds for a simple loop in $\mathbb{R}^2$ which encloses the area A.

We will use the results of Chapter 5 to prove the following result, due to Chekanov [Che96].

Theorem 6.1.2. *Let L be a compact Lagrangian submanifold of*

1. *$\mathbb{R}^{2n}$ such that no disc with boundary on L has positive symplectic area less than σ; or,*
2. *a compact manifold M such that no sphere in M or disc in M with boundary on L has positive symplectic area less than σ.*

Then the displacement energy of L is at least σ.

The strength of this theorem is clear already from Example 6.1.1, which shows that this bound is tight.

Proof that 6.1.2(2) implies 6.1.2(1). Let L be a compact Lagrangian submanifold of $\mathbb{R}^{2n}$, and $F : \mathbb{R}^{2n} \times [0, 1] \to \mathbb{R}$ a compactly-supported time-dependent Hamiltonian function on $\mathbb{R}^{2n}$ whose flow displaces L. Take a torus $\mathbb{R}^{2n}/K\mathbb{Z}^{2n}$ such that L and $\mathrm{supp}(F)$ are contained (translating if necessary) in the fundamental domain $(0, K)^{2n} \subseteq \mathbb{R}^{2n}$.

Applying Theorem 6.1.2(2) to this torus, we obtain in in $\mathbb{R}^{2n}/K\mathbb{Z}^{2n}$ either a disc with boundary on L or a sphere which has symplectic area positive but less than σ. Since the disc and sphere are simply connected, this lifts to a disc or sphere with this property in $\mathbb{R}^{2n}$. In fact it will have to be a disc, since by Stokes' theorem and the contractibility of $\mathbb{R}^{2n}$ all spheres in it have vanishing symplectic area. □

Proof of 6.1.2(2). Let L be a compact Lagrangian submanifold of M, and let $\sigma > 0$ such that no disc with boundary on L has positive symplectic area less than σ. Suppose that $F : M \times [0, 1] \to \mathbb{R}$ is a compactly-supported time-dependent Hamiltonian function on M with $||F|| < \sigma$. We will produce an integral curve of X_{F_t} which both starts and finishes on L.

First, fix $N \in \mathbb{N}$. Choose a smooth bump function $\beta_N : \mathbb{R} \to \mathbb{R}$, such that

$$\beta_N(s) = \begin{cases} 1, & |s| \leq N \\ \text{monotone}, & N \leq |s| \leq N + 1 \\ 0, & |s| \geq N + 1. \end{cases}$$

Observe that $H := ((s,t),p) \mapsto \beta_N(s)F(p,t)dt$ is a compactly supported Hamiltonian form on $(\mathbb{R}\times[0,1])\times(M,L)$, whose curvature satisfies

$$\begin{aligned}\int_{\mathbb{R}\times[0,1]} \sup_{p\in M} R_H|_{((s,t),p)} &= \int_{-\infty}^{\infty}\int_0^1 \sup_{p\in M} \beta_N'(s)F(p,t)dtds \\ &= \int_{-\infty}^{0}\beta_N'(s)ds\int_0^1 \sup_{p\in M} F(p,t)dt + \int_0^{\infty}\beta_N'(s)ds\int_0^1 \inf_{p\in M} F(p,t)dt \\ &= ||F||.\end{aligned}$$

Since $||F|| < \sigma$, we may therefore apply Corollary 5.3.2 to H. We obtain a map $u_N : (\mathbb{R}\times[0,1], \mathbb{R}\times\{0,1\}) \to (M,L)$ whose restriction to $[-N,N]\times[0,1], [-N,N]\times\{0,1\})$ satisifies: for all $(s,t)\in[-N,N]\times[0,1]$,

$$\overline{\partial}u_N|_{(s,t)} - J_{u(s,t)}X_{F_t}|_{u(s,t)} = 0.$$

By Lemma 5.2.2, together with the curvature estimate just computed,

$$\begin{aligned}\int_{-N}^{N}\int_0^1 |\partial_t u_N + X_{F_t}|_{\widetilde{u_N}}|^2 dtds &\leq \int_{\mathbb{R}\times[0,1]} |\partial_t u_N + \beta_N(s)X_{F_t}|_{\widetilde{u_N}}|^2 ds\wedge dt \\ &= \int_{\Sigma} R_H|_{\tilde{u}} \leq ||F||.\end{aligned}$$

Therefore some strand $x_N = u_N(s_0,\cdot) : ([0,1],\{0,1\}) \to (M,L)$ of this map satisifes

$$\int_0^1 |\dot{x}_N(t) - X_{F_t}|_{x_N(t)}|^2 dt \leq ||F||/2N.$$

Now, consider the sequence (x_N) of maps : $([0,1],\{0,1\}) \to (M,L)$ thus produced. Some subsequence converges uniformly to some map $x : ([0,1],\{0,1\}) \to (M,L)$, which must have

$$\int_0^1 |\dot{x}(t) - X_{F_t}|_{x_N(t)}|^2 dt = 0,$$

hence $\dot{x}(t) - X_{F_t} = 0$ for all t. Thus x is an integral curve of the Hamiltonian F which both starts and finishes on L. □

6.2 Proof of the chord conjecture

Theorem 6.2.1. *Let (N,α) be a compact simply-connected contact-type hypersurface of $\mathbb{R}^{2n}$. Then each compact Legendrian submanifold of (N,α) admits a Reeb chord.*

This follows from the following more quantitative theorem:

Theorem 6.2.2. *Suppose that (N, α) is a compact simply-connected manifold-with-contact-form, and that the finite cylinder $(N \times [S, 0], d(e^s\alpha))$ of N's symplectization admits a symplectic embedding into $\mathbb{R}^{2n}$, with displacement energy σ. Then each compact Legendrian submanifold of (N, α) admits a Reeb chord of length at most $\sigma/(1 - e^S)$.*

by making the following two observations:

- By Lemma 2.3.4, every compact contact-type hypersurface of a symplectic manifold can be extended to an embedding into the symplectic manifold of a finite cylinder of the contact manifold's symplectization.
- By Example 3.2.4, every compact subset of $\mathbb{R}^{2n}$ can be displaced from itself by a compactly-supported Hamiltonian flow: proved

Proof of Theorem 6.2.2. We prove a contrapositive. Suppose the finite cylinder $(N \times [S, 0], d(e^s\alpha))$ of N's symplectization admits a symplectic embedding into $\mathbb{R}^{2n}$. Let l be a compact Legendrian submanifold of (N, α). Suppose that l admits no Reeb chord of length less than or equal to T.

By Proposition 3.1.7, there is a smooth $(1 - e^S)T$-rational Lagrangian embedding of $l \times S^1$ into $(N \times [S, 0], d(e^s\alpha))$. Let L denote its image. We claim that L is still $(1 - e^S)T$-rational as a Lagrangian submanifold of $\mathbb{R}^{2n}$. We will prove this by showing that for each disc in $\mathbb{R}^{2n}$ with boundary in L, there is a disc in $N \times [S, 0]$ with boundary in L and with the same symplectic area.

Indeed, since N and hence $N \times [S, 0]$ are simply-connected, any loop in $N \times [S, 0]$ is contractible and therefore bounds a disc in $N \times [S, 0]$. So if A is a disc in $\mathbb{R}^{2n}$ with boundary ∂A contained in $L \subseteq N \times [S, 0]$, then ∂A also bounds a disc in $N \times [S, 0]$. Since the standard symplectic form on $\mathbb{R}^{2n}$ is exact, by Stokes' Theorem the symplectic area of this disc is the same as that of A.

So by Theorem 6.1.2, the displacement energy of L in $\mathbb{R}^{2n}$, and hence also of $N \times [S, 0]$ in $\mathbb{R}^{2n}$, is at least $(1 - e^S)T$. It follows that if $N \times [S, 0]$ has displacement energy σ, then $T \leq \sigma/(1 - e^S)$. $\square$

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