1) RECAP: SPLITTING THEOREM, IDEA OF PROOF
- morally a “convex” metric space can’t contain an entire straight line unless that line splits off as an isometric ℝ-factor (but “concave” spaces often contain plenty of entire lines that don’t split off); Cheeger-Gromoll: this is true in particular for complete Riemannian mfd’s with Ric ≥ 0 if “line” is taken to mean “bi-infinite geodesic that minimizes distance between any two of its points”
- important application: π₁ of a compact manifold with Ric ≥ 0 is virtually abelian; I completely messed up the proof of this last time; I just refer to the original paper in JDG 6 (1971)
- recall Busemann function bᵦ(x) = limᵦ→∞(d(x, γ(ᵦ)) − s) associated with a ray γ, i.e. a geodesic of the form γ : [0, ∞) → M that minimizes distance between any two of its points
- recall idea of the proof pretending that Busemann functions are smooth: ∆bᵦ ≤ 0 by comparison theory; if σ is a line, divide it into two rays γ±; b⁺ᵦ + b⁻ᵦ ≥ 0 everywhere by the triangle inequality, with equality precisely along σ; strong maximum principle ⇒ b⁺ᵦ + b⁻ᵦ ≡ 0; in particular, b±ᵦ are harmonic: |∇²bᵦ| + Ric(∇b, ∇b) = 0 by the Bochner formula; hence ∇b is parallel
- problem: even if bᵦ was smooth a.e. with ∆bᵦ ≤ 0 at all smooth points, this would not be enough for the conclusion of the maximum principle to hold (recall standard example x ↦ x on (−1, 1), which is concave on (−1, 0) and (0, 1), yet attains an interior minimum); need to make sure that bᵦ is superharmonic “everywhere” in a suitable sense (bᵦ is Lipschitz, thus C¹ almost everywhere by Rademacher’s theorem, but conceivably nowhere C² without any further information)

2) SPLITTING THEOREM: DETAILS OF PROOF
- will now prove that (a) Busemann functions are superharmonic in the sense of barriers (hence subharmonic in the viscosity sense, using PDE jargon), (b) the strong maximum principle still holds for C⁰ functions that are subharmonic in this generalized sense; whole idea ⊗ Calabi; application to proving the splitting theorem ⊗ Eschenburg & Heintze in Ann Glob Anal Geom 2 (1984)
- recall that the Laplacian comparison inequalities also hold in the sense of distributions; while this does not a priori help much with the maximum principle (though the original Cheeger-Gromoll proof did in fact proceed along these lines), it will play a role in its own right later on
- define barrier superharmonic: u ∈ C⁰ is called barrier superharmonic if for all p ∈ M and δ > 0 there exists a smooth function uᵦ,δ defined in some open neighborhood of p (an “upper barrier” for u near p) such that uᵦ,δ ≥ u, with equality at p, and moreover ∆uᵦ,δ ≤ δ

• Busemann functions are barrier superharmonic under Ric ≥ 0
- consider a ray γ : [0, ∞) → M and a point p ∈ M at which we want to verify the above definition for bᵦ; our barriers will take the form bᵦ,σ + bᵦ(γ(p)) with s ≥ 1, where σ is a ray in M with σ(0) = p which is “parallel” to γ (construction: take a sequence tᵦ → ∞, choose minimal geodesics σᵦ from p to γ(tᵦ), and limit out σᵦ as i → ∞) and bᵦ,σᵦ(σᵦ) := d(x, σᵦ(s)) − s → bᵦ(γ(p)) as s → ∞

- bᵦ,σᵦ is certainly smooth in some small open neighborhood of p, and satisfies ∆bᵦ,σᵦ ≤ C(n)σᵦ there by the comparison inequalities, because up to adding a constant bᵦ,σᵦ is simply the distance function from the fixed point σ(s), which must be smooth near p because σ is still minimizing between p and σ(s + ε) for some (in fact, every) ε > 0, so that σ(s) /∈ Cut(p) and hence p /∈ Cut(σ(s))
- it remains to check that bᵦ,σᵦ + b⁺ᵦ(γ(p)) ≥ b⁺ᵦ with equality at p; equality at p is clear; the inequality is slightly subtle, or at least surprising in the way it works out: we would like to show that

\[ \lim_{t → ∞} |d(x, γ(t)) − t| ≤ d(x, σ(γ)) − s + \lim_{t → ∞} |d(p, γ(t)) − t|; \]

now consider that σ(γ(tᵦ)) = limᵦ→∞ σᵦ(γ(tᵦ)), where σᵦ is the chosen minimal geodesic from p to γ(tᵦ); the triangle inequality in the triangle x, σᵦ(γ(tᵦ)), γ(tᵦ) tells us that

\[ d(x, γ(tᵦ)) ≤ d(x, σᵦ(γ(tᵦ))) + d(σᵦ(γ(tᵦ)), γ(tᵦ)) = d(x, σᵦ(γ(tᵦ))) + [d(p, γ(tᵦ)) − s] \]

because p, σᵦ(γ(tᵦ)) line up along σᵦ; subtract tᵦ from both sides and pass to the limit

• barrier superharmonics still satisfy the strong maximum principle
- proof of the strong MP ⊗ Evans from Lecture 3 isn’t flexible enough; follow Cheeger/Calabi
- enough: u ≥ 0 continuous and barrier superharmonic on B₁ ⊂ ℝⁿ with u(0) = 0 ⇒ u ≡ 0
- if not, then wlog $u(1,0,...,0) > 0$, hence still $> 0$ in some definite neighborhood of $(1,0,...,0)$
- idea is to replace $u$ by an upper barrier $u_{\delta}$ and so contradict the strong MP for $u_{\delta}$; but we only have $\Delta u_{\delta} \leq \delta$, not $\leq 0$; hence we first need to subtract a small $C^\infty$ function with positive Laplacian from $u$ in such a way that the new function $\hat{u}$ still has an interior global minimum in $B_1$
- first candidate for a function to subtract from $u$: $\phi(x) \equiv x_1 - A(x_2^2 + ... + x_n^2)$; if $A \gg 1$ then along $\partial B_1$ this is positive only in a disk of radius $\approx \frac{1}{\sqrt{A}}$ around $(1,0,...,0)$, negative elsewhere
- unfortunately $\Delta \phi \approx -A$ everywhere; the trick is to replace $\phi$ by $\psi := e^{B\phi} - 1$ for $B \gg 1$; then $\psi$ still has the same sign as $\phi$ at each point but $\Delta \psi = (B\Delta \phi + B^2|\nabla \phi|^2)e^{B\phi}$ and $|\nabla \phi| \geq |\frac{\partial \phi}{\partial x_1}| = 1$; thus in particular $\Delta \psi \geq (B^2 - AB)e^{-CnAB} > 0$ if we make $B \gg A$, on the whole of $B_1$
- fix $A$ such that $\{x \in \partial B_1 : \phi(x) > 0\} \subset \{x \in \partial B_1 : u(x) \geq \frac{1}{2}u(1,0,...,0) > 0\}$, then $B$ such that $\Delta \psi > 0$ on the whole of $B_1$, then $\varepsilon > 0$ such that if $x \in \partial B_1$ satisfies $\phi(x) > 0$ then $(u - \varepsilon \psi)(x) > 0$; since $\hat{u} := u - \varepsilon \psi$ satisfies $\hat{u}(0) = 0$, but on the other hand $\hat{u} \geq 0$ and even $\hat{u} > 0$ near $(1,0,...,0)$ on $\partial B_1$, we see that $\hat{u}$ still attains a global minimum in the interior of $B_1$ (though not necessarily at $x = 0$ anymore); plus, $\Delta \hat{u} \leq -\delta$ in the barrier sense on all of $B_1$, where $\delta \approx \varepsilon e^{-B} > 0$
- to get a contradiction we now take a point $p \in B_1$ where $\hat{u}$ attains an interior global minimum, and an upper barrier $\hat{u}_{p,\delta}$ for $\hat{u}$ near $p$ such that $\hat{u}_{p,\delta} \geq \hat{u}$ in some small open neighborhood $U \subset B_1$ of $p$, with equality at $p$, and $\Delta \hat{u}_{p,\delta} \leq -\frac{\delta}{2} < 0$ on $U$; then $\hat{u}_{p,\delta}$ clearly still attains a global minimum on $U$ at $p$, but is smooth with strictly negative Laplacian; contradiction

(3) CHAPTER 4: GROMOV-HAUSDORFF LIMITS

- Intuition about convergence of metric spaces
  - $\exists$ many different notions of convergence of mfs and/or metric spaces at various levels of smoothness, but all can be viewed as variations on the same basic theme that we try to explain here
  - phenomena captured by the GH topology: (i) manifolds converging to a manifold but in a rough fashion [surface in $\mathbb{R}^3$ acquiring small creases that get sharper and sharper, but also smaller in size, so that the limit is smooth]; (ii) $n$-manifolds converging to an $n$-dimensional space with singularities [cone in $\mathbb{R}^3$ with the cone tip rounded off, but more and more sharply]; (iii) $n$-manifolds converging to an $m$-manifold with $m < n$ [flat 2-tori $\mathbb{R}^2/(\mathbb{Z} + \varepsilon \mathbb{Z})$ converge to $S^1$ as $\varepsilon \rightarrow 0$, “collapsing”]

  - how to make a good definition? intuitively clear what we mean by a sequence of subsets of $\mathbb{R}^N$ converging to a limit subset; the KEY is that all sets are jointly realized inside the same $\mathbb{R}^N$
  - there is an important “gauge fixing” issue here: we often picture Riemannian mfs as submfs of $\mathbb{R}^N$, however the formal definition makes no reference to any such embedding; Riemannian mfs may look completely different if we compare their local matrices $(g_{ij}(x))$ on a standard coordinate patch $B_1 \subset \mathbb{R}^n$, yet be isometric to almost the same submf of $\mathbb{R}^N$; useful exercise on 2nd problem sheet: why has the Schwarzschild metric two Euclidean ends for $\alpha > \frac{2}{n-2}$ and not just one?

- Definition of Gromov-Hausdorff distance
  - definition: given compact metric spaces $X, Y$ and $\varepsilon > 0$, then we say that $d_{GH}(X,Y) < \varepsilon$ if there exist maps $F : X \rightarrow Y$ and $G : Y \rightarrow X$ (typically highly discontinuous in at least one direction, i.e. the interesting cases are when $X$ and $Y$ are topologically different), then often referred to as $\varepsilon$-GH approximations, such that $\forall x_1, x_2 \in X$: $|d_Y(F(x_1), F(x_2)) - d_X(x_1, x_2)| < \varepsilon$, and $\forall y \in Y \exists x \in X$: $d_Y(F(x), y) < \varepsilon$, and likewise for $G$; i.e. $X$ and $Y$ are isometric up to absolute errors $< \varepsilon$
  - main example: the grid $\varepsilon \mathbb{Z}^n \rightarrow \mathbb{R}^n$ with subspace metric from $\mathbb{R}^n$ has GH distance $\sim \varepsilon$ to $\mathbb{R}^n$; the map $G : \mathbb{R}^n \rightarrow \varepsilon \mathbb{Z}^n$ can be any “binning” map: for $y \in \mathbb{R}^n$ pick $x \in \varepsilon \mathbb{Z}^n$ of minimal distance to $y$; if the GH distance was $= \delta < \varepsilon$ then any two points in $\varepsilon \mathbb{Z}^n$ would be either $C\delta$ close or $\frac{\varepsilon}{\delta} - C\delta$ apart
  - remarks: (i) GH distance only depends on the isometry classes of $X$ and $Y$; (ii) it’s not important to compute GH distance precisely, but to understand what it means that GH distance $\rightarrow 0$; (iii) idea is to take the standard Hausdorff distance of $X$ and $Y$ after isometric embedding into some common metric space $Z$, then inf over all possible common isometric embeddings $X, Y \rightarrow Z$; (iv) for unbounded metric spaces it is better to say that $(X_i, p_i) \rightarrow (X, p)$ in the pointed GH topology if $B_{X_i}(p_i, R) \rightarrow B_X(p, R)$ in the usual GH topology for all $R > 0$ (choice of basepoints matters)
- **Gromov’s compactness theorem**
  - statement: GH limits almost always exist; more precisely, given a sequence \((X_i, d_i)\) of compact metric spaces, every subsequence contains a GH-convergent subsequence \(\Leftrightarrow \forall \varepsilon > 0: \exists N(\varepsilon) < \infty: \forall i: (X_i, d_i)\) contains a finite \(\varepsilon\)-dense set consisting of at most \(N(\varepsilon)\) distinct points
  - the way this is most often applied in Riemannian geometry: given \(C > 0\), the set of all compact Riemannian manifolds with dimension \(\leq C\), diameter \(\leq C\), and Ricci curvature \(\geq -C\) is sequentially precompact in the GH topology (the possible limit spaces are geodesic, i.e. \(d(x, y)\) is equal to the inf over all curves \(\gamma\) from \(x\) to \(y\) of the sup over all partitions \(\{t_k\}_{k=1}^N\) of \(\sum_{k=1}^{N-1} d(\gamma(t_k), \gamma(t_{k+1}))\)
  - proof of the application: given \(M^n\) satisfying these bounds, and \(\varepsilon > 0\), construct a finite subset \(S_\varepsilon \subset M\) which is maximal subject to the condition that its points are pairwise at least \(\varepsilon\) apart; then \(S_\varepsilon\) is indeed \(\varepsilon\)-dense in \(M\) but the \(\frac{\varepsilon}{2}\)-balls are pairwise disjoint, so \(#S_\varepsilon \leq |M|/\min_{x \in S_\varepsilon} |B(x, \frac{\varepsilon}{2})|\)
  - proof of the abstract compactness theorem: exercise in basic analysis, see Cheeger’s grey book; \(\Rightarrow\): obvious by contradiction; \(\Leftarrow\): for each \(i\) construct an infinite sequence \(p_1^i, ..., p_{N_i}^i, ..., p_{N_2}^i, ...\) of points in \(X_i\) such that for every \(k\) the first \(N_{k,i}\) points from this sequence are \(\frac{1}{k}\)-dense in \((X_i, d_i)\); it is then clear by hypothesis that we can assume \(N_{k,i} = N_k\) to be independent of \(i\); the desired limit space \(X_\infty\) can then be constructed as the completion of the countable set indexed by \(j \in \mathbb{N}\) of all infinite sequences \(\langle p_j^i \rangle = \langle p^i_{j1}, p^i_{j2}, p^i_{j3}, ... \rangle\) under the pseudometric \(d_\infty(\langle p^i_{j1}\rangle, \langle p^i_{j2}\rangle) = \lim_{j \to \infty} d_i(p^i_{j1}, p^i_{j2})\)
  - to be slightly more precise, for any fixed \(k\) and \(j_1, j_2 < k\) we have \(d_i(p^i_{j1}, p^i_{j2}) \leq N_k/k\) uniformly in \(i\), so that we get a limit \(\in [0, N_k/k]\) after passing to a subsequence in \(i\); now diagonalize over \(k\)

- **Applications and relevance of GH convergence**
  - will very quickly go over three examples to illustrate the relevance of GH limits: (1) groups of polynomial growth, (2) existence of Einstein metrics, and (3) Cheeger-Colding theory; beware that I’m not really competent to discuss (1) and at best partially so for (2) and (3)
  - GH limits are easy to get because the definition is weak; yet the limit spaces have reasonably good properties to serve as a basis for further analysis; also GH convergence is often quite adequate, e.g. because in many interesting examples the topology does tend to change in the limit
  - however, “further analysis” may involve different regularity theory, e.g. solution to Hilbert’s 5th problem in (1), Uhlenbeck’s \(\varepsilon\)-regularity and removable singularities theorems in (2) in dimension 4 (the appropriate regularity theorems for (2) in higher dimensions are far from understood)
  - (1) Gromov’s theorem on groups of polynomial growth (Publ Math IHES 53 (1981)): this is what GH limits were invented for; given \(\Gamma\) a finitely generated group of polynomial growth, Gromov shows that \(\Gamma\) contains a nilpotent subgroup of finite index (compare Lecture 5)
  - known before if \(\Gamma\) sits inside a Lie group \(G\); Gromov constructs \(G\) as the isometry group of the pointed GH limit \(X_\infty\) of the rescalings \(\varepsilon X\) of the Cayley graph \(X\) of \(\Gamma\) as \(\varepsilon \to 0\) (polynomial growth allows him to apply Gromov compactness & prove that \(X_\infty\) is finite-dimensional & locally connected; the isometry group \(G\) of \(X_\infty\) is then a Lie group by the solution to Hilbert’s 5th problem, and \(\Gamma\) maps to \(G\) with sufficiently large image in order for previously known results to apply)
  - for \(\Gamma = \mathbb{Z}^n\) with its standard generators, \(X_\infty\) is \(\mathbb{R}^n\) with the translation invariant Finsler metric gotten from the \(\|\cdot\|_\infty\)-norm; for general \(\Gamma\) the metric on \(X_\infty\) is not even Finsler but sub-Finsler (or Finsler-Carnot), i.e. we have a Lie group and a (sufficiently nonintegrable) subbundle of its tangent bundle, and we only inf the length functional over paths that are tangent to this subbundle everywhere; e.g. the integer \(3 \times 3\) Heisenberg group gives a Finsler-Carnot structure of Hausdorff dimension 4 on \(\mathbb{R}^3\), invariant under left multiplication in the real \(3 \times 3\) Heisenberg group
  - see Enrico Le Donne, Lecture notes on sub-Riemannian geometry (online)