(1) RECALL VOLUME COMPARISON

- recall idea of comparing a manifold $M$ with a warped product type model $dr^2 + (f(r)/f(a))^2g_X$ on $[a,b] \times \mathbb{X}$ (with prescribed warping factor $f$ but arbitrary cross-section $(X,g_X)$)

- assuming lower Ricci bounds, we get three types of comparison results for a distance function $r$ i.e. $|\nabla r| \equiv 1$ in $M$: (1) upper bounds on $\Delta r$; (2) monotonicity of the area of the level sets of $r$ in relation to the area in the model space; (3) monotonicity of the volume ratio of the sublevels

- student question: do we need completeness to prove the comparison inequalities? correct answer: formally the proof given last time requires that the level sets of $r$ are compact; if not, the change of area involves boundary terms; for example, $M$ formally the proof given last time requires that the level sets of $r$ are compact; if not, the change of area involves boundary terms; for example, $M$ formally the proof given last time requires that the level sets of $r$ are compact; if not, the change of area involves boundary terms; for example, $M$ formally the proof given last time requires that the level sets of $r$ are compact; if not, the change of area involves boundary terms; for example, $M$ formally the proof given last time requires that the level sets of $r$ are compact; if not, the change of area involves boundary terms; for example, $M$

- meaning of Laplacian comparison [e.g. $\Delta r \leq \frac{2\pi^2}{r}$ for $r = \text{dist}(p,-)$ and $Ric \geq 0$ across the cut locus, i.e. where $r$ isn’t smooth: for smooth $u$, recall that $\Delta u \leq 0$ implies a weak concavity property: $u$ attains its minimum at the boundary of any given domain; this fails miserably if $u$ is only continuous and almost everywhere smooth with $\Delta u < 0$ where $u \in C^2$, see $u(x) = \sqrt{x}$ on $\mathbb{R}$; “not a problem” for $r = \text{dist}(p,-)$ because this has upward pointing singularities at the cut locus

- however, making this precise is a major technical issue; last time: the comparison inequalities hold globally in the sense of distributions (see current problem sheet); however, to make sure that the concavity/minimum principle still holds (needed for the splitting theorem), we must talk about nonsmooth $C^0$ functions satisfying differential inequalities in the viscosity sense; next time

- some basic examples and intuition

- if $\text{Ric} \geq 0$, then $|B(p,r)|/|\mathbb{R}^n(r)|$ is nonincreasing; constant $\Leftrightarrow$ balls are isometric; here we use $f(r) = r$ and $a = 0$, which means singular initial conditions, but the ODE comparison argument is still OK; trivial consequence: the only complete noncompact manifold with $Ric \geq 0$ all of whose ends are asymptotic to $\mathbb{R}^n$ is $\mathbb{R}^n$ itself (later: complete with $\text{Ric} \geq 0$ and more than one end implies isometric to a cylinder $\mathbb{R} \times X$ with metric $dt^2 + g_X$; consequence of the splitting theorem)

- consider Schwarzschild metric: two Euclidean ends, identically vanishing scalar curvature, $g(x) = (1 + m/|x|^2)^2g_{\mathbb{R}^4}$ on $\mathbb{R}^4 \setminus 0 = S^3 \times \mathbb{R}$

- if $\text{Ric} \geq -(n-1)$, then the appropriate model warping factor is $f(r) = \sinh(r)$, $a = 0$; model space is $\mathbb{R}^n$ with metric $dr^2 + \sinh(r)^2g_{S^{n-1}}$; this is hyperbolic space written in normal or geodesic polar coordinates, constant sectional curvature $-1$; volume of a ball of radius $R$ is $\sim R^ne^{(n-1)R}$; some resemblance with $\mathbb{R}^n$ if $R$ is bounded by a definite constant, but not at all as $R \to \infty$

- general principle: can hope to obtain definite estimates on the geometry of a manifold with scale invariant lower Ricci bounds, i.e. compare a ball $B(p,R)$ with $\text{Ric} \geq -(n-1)R^{-2}$ to $\mathbb{R}^n(R)$

- e.g. classical Margulis lemma: global geometry of a compact hyperbolic manifold $M = \mathbb{H}^n/\Gamma$, which has $\text{Ric} \equiv -(n-1)$, is very intricate, e.g. $\Gamma$ must be huge, imagine a free group or $\text{SL}(2,\mathbb{Z})$; but there exist $\varepsilon < \varepsilon$ depending only on $n$ such that either $|B(x,1)| > \varepsilon^n$, in which case $B(x,\varepsilon')$ is diffeo to $\mathbb{R}^n(\varepsilon')$ with good estimates comparing the two metrics; or else $|B(x,1)| < \varepsilon^n$, then $B(x,\varepsilon')$ is diffeo to $(0,\varepsilon') \times F$ for a flat mfd $F = \mathbb{R}^n/\Lambda$ of small volume, again with estimates; for general symmetric spaces, $(0,\varepsilon')$ gets replaced by a ball in $\mathbb{R}^k$ and $F$ gets replaced by a nilquotient

(2) DIGRESSION: $\frac{1}{2}$-HÖLDER BOUND FOR JACOBI FIELDS

- instructive application of the Laplacian comparison theorem, not unrelated to the proof of the splitting theorem we will be looking at later; taken from the intro of Colding & Naber (whose main theorem is a tremendous strengthening of this), however seems to go back to Calabi

- Thm: $\text{Ric} \geq -(n-1)$, $\gamma : [0,1] \to M$ minimal geodesic, $J$ Jacobi field along $\gamma$ with $J(0) = 0$, then for all $s,t \in [\delta,1-\delta]$ we have $|J(s) - J(t)|^2 \leq C(n)\delta^{-1}|t-s|$, i.e. a uniform $\frac{1}{2}$-Hölder bound (of course Jacobi fields are smooth and not just $\frac{1}{2}$-Hölder; what matters is the uniformity of the bound)
- easy to see that this could be upgraded to Lipschitz in the case of a lower sectional curvature bound; in a precise sense, see Colding & Naber, the Hölder behavior is optimal for lower Ricci

- proof: proceed from basic identity $\frac{d}{dt} |J|^2 = (\nabla^2 d_p)(J, J)$ along $\gamma$, where $d_p$ is the distance from $p$ (smooth in the interior of $\gamma$ because $\gamma$ is minimizing); to control the Hessian, we need the Bochner formula: $\frac{d}{dt}(\Delta d_p) + |\nabla^2 d_p|^2 \leq C(n)$ (a lower sectional bound would give us an upper Hessian bound straight away; also notice that we previously used Bochner to derive an upper estimate on $\Delta d_p$ by essentially dropping the Hessian term from this inequality; now we go back and turn things around); best estimate we can get from this is an $L^2$ type bound along $\gamma$:

$$\int_\delta^{1-\delta} |\nabla^2 d_p|_{\gamma(t)}^2 \, dt \leq C(n) + (\Delta d_p)(\gamma(\delta)) - (\Delta d_p)(\gamma(1-\delta)) \leq \frac{C(n)}{\delta};$$

here we use the Laplacian comparison for $\Delta d_p$ at $\gamma(\delta)$ (upper bound), and the following very useful argument to reverse this at $\gamma(1-\delta)$, not unlike the reverse monotonicity arguments last time that also depended on having a long minimal geodesic: $d_p + d_q$ is smooth in the vicinity of $\gamma([\delta, 1-\delta])$, is $\equiv$ 1 along $\gamma$, and $\geq$ 1 everywhere else by the triangle inequality, so that $\Delta (d_p + d_q) \geq 0$ along $\gamma$, and hence $-(\Delta d_p)(\gamma(1-\delta)) \leq (\Delta d_q)(\gamma(1-\delta)) \leq \frac{C(n)}{8}$ by Laplacian comparison again; now use

$$\log \frac{|J(t)|}{|J(s)|} \leq \frac{1}{2} \int_s^t |\nabla^2 d_p|_{\gamma(t)} \, d\tau \leq \sqrt{t-s} \frac{C(n)}{\sqrt{\delta}}$$

(3) RICCI CURVATURE AND THE FUNDAMENTAL GROUP

- finiteness of topology, Milnor conjecture

- complete noncompact mfdς of nonnegative sectional curvature: Cheeger-Gromoll soul theorem [3] compact totally geodesic submanifold $S$ such that $M$ is diffeomorphic to the vector bundle $N_S/M$; moreover [Perelman] sec $> 0$ at one point implies $S =$ point, $M$ diffeo to $\mathbb{R}^n$! shows that the topology of $M$ is finite; Toponogov splitting theorem applied to $S \Rightarrow \pi_1(M)$ has an abelian subgroup $\mathbb{Z}^k$ with $k \in \{0, ..., n\}$ of finite index; eventually reduce to the study of compact mfdς with strictly positive sectional curvature; then Gromov $\Rightarrow \exists$ bound depending only on $n$ for all Betti numbers of $M$ with coefficients in any field; all this based on subtle use of the triangle comparison theorems

- nothing anywhere nearly as strong can hold for nonnegative Ricci, though some things can still be said, especially about $b_1$ and $\pi_1$ (these are still “controlled” by the Bochner formula on 1-forms); can write down explicit complete Ricci-flat 4-manifolds with trivial $\pi_1$ but infinite $b_2$ (Anderson & Kronheimer & LeBrun, Comm Math Phys 125 (1989)); see also Sha & Yang, J Diff Geom 33 (1991), for compact mfdς of any dimension with Ric $> 0$ and arbitrarily large higher Betti numbers

- fundamental problem, actively studied: Milnor conjecture: the $\pi_1$ of a complete noncompact mfd with Ric $\geq 0$ is finitely generated; only known instances are in “highly extremal” cases (Sormani, J Diff Geom 54 (2000); Kapovitch & Kleiner & Wilking, in prep); Milnor, J Diff Geom 2 (1968)

- polynomial growth and consequences, Margulis lemma

- IF $\pi_1$ is finitely generated, then not so hard to show (© Milnor) that it has polynomial growth: the number $w(m)$ of distinct group elements representable as words of length $\leq m$ in the generators is at worst polynomial in $m$ (for instance, $O(m^b)$ in a rank $b$ lattice, $\mathbb{Z}^b$, as opposed to $O(b^m)$ in a free group on $b$ generators, $F_b$, which looks more like the global $\pi_1$ of a hyperbolic manifold)

- proof: same as proving that $w(m) = O(m^n)$ in $\mathbb{Z}^n$; view $\pi_1$ as deck group of universal cover; choose generators $g_1, ..., g_k$ and basepoint $x$; let $D = \max_{i} d(x, g_i x)$ (distance taken in the universal cover) and pick $\varepsilon < \min_{i} d(x, g_i x)$; then for every $m \in \mathbb{N}$ the ball $B(x, mD)$ contains at least $w(m)$ distinct balls of radius $\varepsilon$ (all of them isometric copies of the fixed ball $B(x, \varepsilon)$), so that the volume comparison theorem yields $w(m)|B(x, \varepsilon)| \leq |B(x, mD)| \leq \text{const} \cdot (mD)^n$

- Gromov’s theorem (will briefly discuss this in a later lecture, nice new proof by Kleiner in JAMS 23 (2010)): finitely generated groups of polynomial growth are virtually nilpotent

- example: $\mathbb{Z}^3$ vs nonabelian nilpotent group $H_3(\mathbb{Z}) = \{ \left( \begin{array}{ccc} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{array} \right) : x, y, z \in \mathbb{Z} \}$ with $w(m) \sim m^4$

- Wei/Wilking: every finitely generated virtually nilpotent group does appear as $\pi_1(\text{Ric} > 0)$; the case proved by Guofang Wei (torsion free nilpotent) is well explained in her thesis (online)
- will see as a consequence of the Cheeger-Gromoll splitting theorem that $\pi_1$ of a compact mfd with $\text{Ric} \geq 0$ is virtually abelian rather than merely virtually nilpotent

- optimal result in any of these directions short of the Milnor conjecture itself has been proved very recently by Kapovitch and Wilking (arXiv 2011): let $M^n$ be complete such that $\text{Ric} \geq -(n-1)$ on some ball $B(p, 1)$, then the image of $\pi_1(B(p, \varepsilon_n))$ in $\pi_1(B(p, 1))$ contains a nilpotent subgroup of index at most $C_n$ with a triangular basis of length at most $n$; refer to introduction of that paper for details, e.g. the very interesting diameter ratio theorem (Thm 8)

(4) CHEEGER-GROMOLL SPLITTING THEOREM

- idea: a convex set in $\mathbb{R}^n$ that contains an entire line must be a cylinder, i.e. split as the product of $\mathbb{R}$ and a convex set in $\mathbb{R}^{n-1}$; Toponogov: also holds for mfds with nonnegative sectional curvature (line = bi-infinite geodesic that minimizes distance between any two of its points); fails for concave spaces, e.g. interior of one-sheeted hyperboloid or, on the level of mfds, hyperbolic space

- notice: on every complete, noncompact, convex surface in $\mathbb{R}^3$ which is not a cylinder, such as a paraboloid, every geodesic must eventually cease to minimize distance between its endpoints

- Cheeger-Gromoll: Toponogov splitting still holds under $\text{Ric} \geq 0$; proof is PDE based and is in some ways more natural than T’s proof once some basic understanding of PDE’s is in place

- easy application: if a complete noncompact manifold with $\text{Ric} \geq 0$ has more than one end, then it must already split isometrically as $\mathbb{R} \times X$, where $X$ is compact (i.e. exactly two ends)

- more serious application: $\pi_1$ of a compact manifold with $\text{Ric} \geq 0$ is virtually abelian

- proof: absolutely not what I was trying to do in class; maybe next time

- proof of the splitting theorem: need to introduce Busemann functions first

- geodesic of the form $\gamma : [0, \infty) \to M$ is called a ray if it minimizes distance between any two of its points; the associated Busemann function $b_\gamma(x) = \lim_{s \to \infty}(d(x, \gamma(s)) - s)$ [expression in parentheses is nonincreasing in $s$ with $x$ fixed, and bounded below by $-d(x, \gamma(0))$, from the triangle inequality]; ex: $\gamma(s) = se_n$ in $\mathbb{R}^n$, then $b_\gamma(x) = -x_n$; intuition is that geodesics emanating from $p$ should look almost parallel at a point $x$ far away from $p$, now rather keep $x$ fixed and move $p$ to infinity

- level sets of Busemann functions are called horospheres, e.g. in the unit ball model of hyperbolic space the horospheres associated with a given radial ray through the origin are spheres touching the boundary of the unit ball from within at the point where the ray hits the boundary

- proof of the splitting theorem is beautifully simple if we pretend that Busemann functions are smooth: $\Delta b_\gamma \leq 0$ from Laplacian comparison for every ray $\gamma$, i.e. superharmonic/weakly concave; if $\sigma$ is any line, we can split it into two rays $\gamma^\pm$, and the triangle inequality tells us that $b_{\gamma^+} + b_{\gamma^-} \geq 0$ everywhere with equality precisely along $\sigma$; strong maximum principle $\Rightarrow b_{\gamma^+} + b_{\gamma^-} \equiv 0$; in particular, both are harmonic (hence smooth by elliptic regularity even if we hadn’t been assuming them to be smooth to begin with, granted that everything else up to this point would have still worked then); now the Bochner formula tells us that $|\nabla^2 b|^2 + \text{Ric}(\nabla b, \nabla b) = 0$, hence $\nabla b$ is parallel and so the flow lines of $\nabla b$ split off as isometric factors (de Rham), so $M = \mathbb{R} \times \text{horosphere}$

- great technical difficulty: a priori the Busemann functions are only Lipschitz, i.e. have a gradient almost everywhere (of unit length wherever it exists), but no second derivative in any sense