Investigating $\mathfrak{sl}(2)$ conformal blocks

Zachery Wills
Advisor: Dr. David Swinarski

Submitted in partial fulfillment of the requirements of the
Honors Program

Fordham College at Lincoln Center

April 19, 2013

1. Introduction

We studied the behavior of $\mathfrak{sl}(2)$ conformal blocks related to $\overline{M}_{0,6}$. We began our research by investigating Fakhruddin's conjecture, then moved towards studying when the first Chern class vectors of these conformal blocks are ample in the nef cone. Computationally, we proved Fakhruddin's conjecture for levels $\ell = 1, 2, ..., 10$, and noticed some interesting patterns in the weights of ample conformal blocks interior to the Fakhruddin cone. We also determined which $\mathfrak{sl}(2)$ conformal blocks with symmetric weights are ample and computed their first Chern class vectors.

2. $\overline{M}_{0,n}$ and conformal blocks

Definition 2.1. *$M_{0,n}$ is the moduli space of the ordered configurations of n distinct points on a sphere. Sets of points $(p_1, \ldots, p_n)$ and $(q_1, \ldots, q_n)$ are equivalent if there exists a 2×2 invertible matrix M such that $Mp_i = q_i$ for $i = 1, \ldots, n$.*

We can see an example of points on a sphere in Figure 1.

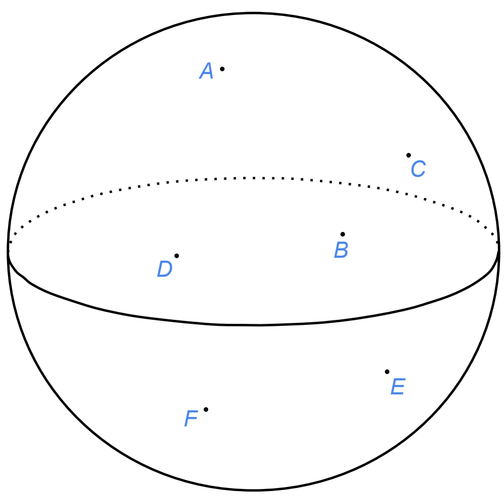

Figure 1. Example of $M_{0,6}$

In $M_{0,n}$, the points $p_1, \ldots, p_n$ must be distinct. However, $M_{0,n}$ does not have a way to describe when two or more points in a configuration approach a common point.

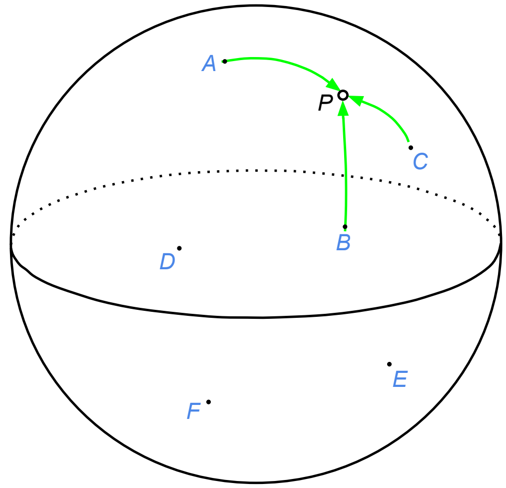

FIGURE 2. Points A, B, and C approaching a common point P

We use a larger space $\overline{M}_{0,n}$ to describe the limiting behavior. In $\overline{M}_{0,n}$ colliding points (A, B, and C) move through the point P they mutually approach onto another sphere attached at P, as shown in Figure 3.

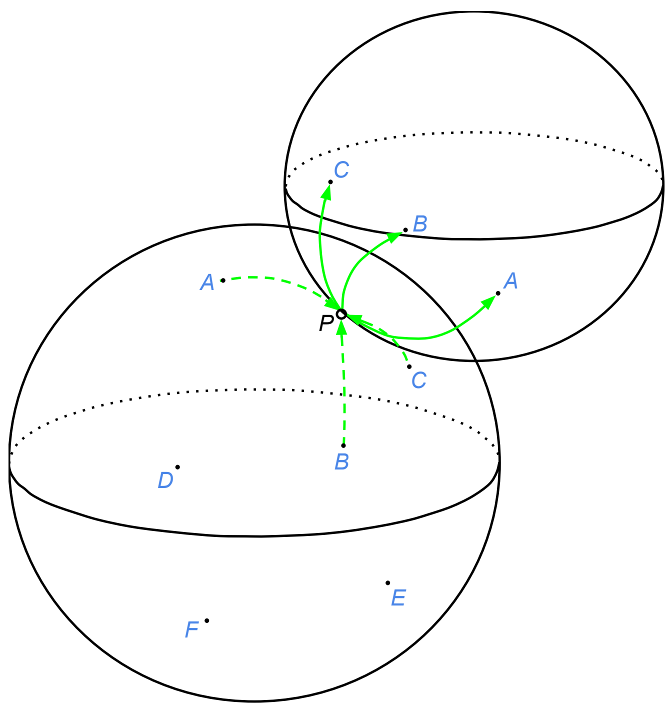

FIGURE 3. A, B, and C move onto a second sphere

The study of $\overline{M}_{0,n}$ is important in algebraic geometry. Conformal blocks are one useful tool with which we can study $\overline{M}_{0,n}$.

The work of Deligne, Mumford, Mayer, and Knudsen demonstrates that $\overline{M}_{0,n}$ can be described using polynomial equations [1, 6–8]. To each configuration of points in $\overline{M}_{0,n}$ we may assign a vector space called a *conformal block*. Furthermore, these conformal blocks form an algebraic vector bundle, which means that if the point configuration varies continuously, the vector space varies continuously as well. The rank of the conformal block tells us the dimension of this vector space. Ueno and Fakhruddin (see [3, Lemma 2.5]) proved that vector bundles of conformal blocks are globally generated, which implies that vector bundles of conformal blocks can be used to define polynomial maps $\overline{M}_{0,n} \to X$, where X is another space defined by polynomials, typically simpler than $\overline{M}_{0,n}$. The first Chern class vector of a conformal block encodes information about these maps. Specifically, the first Chern class vector can be useful to compute which parts of $\overline{M}_{0,n}$ are contracted by the map $\overline{M}_{0,n} \to X$. For a more thorough treatment of conformal blocks, see [11, Chapter 4].

Conformal blocks are denoted $\mathbb{V}(\mathfrak{g}, \ell, \vec{\lambda})$, where $\mathfrak{g}$ is a Lie algebra, ℓ is a positive integer, and $\vec{\lambda}$ is a six coordinate weight vector whose coordinates $\lambda_i \leq \ell$. The Lie algebra $\mathfrak{sl}(2)$ is the set of 2×2 matrices with trace zero.

3. Picard Space

The first Chern class of a conformal block is a vector in the Picard space. The Picard space is a vector space over the field of rational numbers $\mathbb{Q}$ and is denoted P_n. For our purposes we have fixed $n = 6$, and so our first Chern class vectors are in P_6.

Definition 3.1. *The Picard space is spanned by vectors denoted D_{I,I^C}. Here I and I^C partition $\{1, 2, \dots, n\}$ and are also required to satisfy the conditions:*

i. $\#I, \#I^C \geq 2$

ii. *If $\#I = \frac{n}{2}$ then $1 \in I$*

Since we have set $n = 6$, $I \subset \{1, 2, 3, 4, 5, 6\}$ and $I^C = \{1, 2, 3, 4, 5, 6\} \backslash I$ following Definition 3.1. P_n is spanned by these D_{I,I^C} vectors (which are not linearly independent).

4. F-functionals

Definition 4.1. *An* F-functional *is a linear transformation* $F_{I_1,I_2,I_3,I_4} : P_n \to \mathbb{Q}$ *such that*

$$F_{I_1,I_2,I_3,I_4}(D_{I,I^C}) = \begin{cases} -1 & \text{if } I = I_j \text{ or } I^C = I_j \text{ for } j \in \{1,2,3,4\} \\ 1 & \text{if } I = I_j \cup I_k \text{ or } I^C = I_j \cup I_k \text{ for } j,k \in \{1,2,3,4\},\ j \neq k \\ 0 & \text{otherwise.} \end{cases}$$

Here I_1, I_2, I_3, I_4 *partition* $\{1, 2, \ldots, n\}$ *such that no* I_j *for* $j \in \{1, 2, 3, 4\}$ *is empty and* $I_1 \cup I_2 \cup I_3 \cup I_4 = \{1, 2, \ldots, n\}$.

Definition 4.2. *When we fix* $n = 6$*, we can express a vector* D *in* P_6 *in terms of a certain basis, called the* non-adjacent basis. *The non-adjacent basis is*

$$\left\{ \begin{array}{lllll} D_{\{1,3\},\{2,4,5,6\}}, & D_{\{1,4\},\{2,3,5,6\}}, & D_{\{1,5\},\{2,3,4,6\}}, & D_{\{2,4\},\{1,3,5,6\}}, & D_{\{2,5\},\{1,3,4,6\}}, \\ D_{\{2,6\},\{1,3,4,5\}}, & D_{\{3,5\},\{1,2,4,6\}}, & D_{\{3,6\},\{1,2,4,5\}}, & D_{\{4,6\},\{1,2,3,5\}}, & D_{\{1,2,4\},\{3,5,6\}}, \\ D_{\{1,2,5\},\{3,4,6\}}, & D_{\{1,3,4\},\{2,5,6\}}, & D_{\{1,3,5\},\{2,4,6\}}, & D_{\{1,3,6\},\{2,4,5\}}, & D_{\{1,4,5\},\{2,3,6\}}, \\ D_{\{1,4,6\},\{2,3,5\}} & & & & \end{array} \right\}$$

The F-functionals can be used to determine the coefficients of vectors in this basis. We describe this formula now. If we fix $n = 6$, we can write $\{1, 2, \ldots, 6\}$ in a circle as

$$\begin{array}{ccc} & 1 & \\ 6 & & 2 \\ 5 & & 3 \\ & 4 & . \end{array}$$

We then partition $\{1, 2, \ldots, 6\}$ into four subsets, named block 1, gap 1, block 2, and gap 2 such that $\{1, 2, \ldots, 6\}$ is divided into blocks and gaps. Additionally, blocks are only next to gaps and gaps are only next to blocks. For example, if block 1 is $\{1\}$, gap 1 is $\{2\}$, block 2 is $\{3\}$, and gap 2 is $\{4, 5, 6\}$, we would have

$$\begin{array}{ccc} & \boxed{1} & \\ 6 & & 2 \\ 5 & & \boxed{3} \\ & 4 & . \end{array}$$

We can use this division into blocks and gaps to find which F-functional gives the coefficient for D_{I,I^C}. Specifically, the F-functional $F_{\text{block 1,gap 1,block 2,gap 2}}$ will give the coefficient for $D_{\{\text{block 1} \bigcup \text{block 2}\},\{\text{gap 1} \bigcup \text{gap 2}\}}$. This gives us a rule with which we can find the coefficient of any element of the non-adjacent basis.

Proposition 4.3. *A vector D in P_6 can be expressed in the non-adjacent basis with coefficients given by:*

$$\begin{aligned}
D = (F_{\{1\},\{2\},\{3\},\{4,5,6\}}(D))D_{\{1,3\},\{2,4,5,6\}} \quad & +(F_{\{1\},\{2,3\},\{4\},\{5,6\}}(D))D_{\{1,4\},\{2,3,5,6\}} \\
+(F_{\{1\},\{6\},\{5\},\{2,3,4\}}(D))D_{\{1,5\},\{2,3,4,6\}} \quad & +(F_{\{2\},\{3\},\{4\},\{1,5,6\}}(D))D_{\{2,4\},\{1,3,5,6\}} \\
+(F_{\{2\},\{3,4\},\{5\},\{1,6\}}(D))D_{\{2,5\},\{1,3,4,6\}} \quad & +(F_{\{2\},\{1\},\{6\},\{3,4,5\}}(D))D_{\{2,6\},\{1,3,4,5\}} \\
+(F_{\{3\},\{4\},\{5\},\{1,2,6\}}(D))D_{\{3,5\},\{1,2,4,6\}} \quad & +(F_{\{3\},\{1,2\},\{6\},\{4,5\}}(D))D_{\{3,6\},\{1,2,4,5\}} \\
+(F_{\{4\},\{5\},\{6\},\{1,2,3\}}(D))D_{\{4,6\},\{1,2,3,5\}} \quad & +(F_{\{1,2\},\{3\},\{4\},\{5,6\}}(D))D_{\{1,2,4\},\{3,5,6\}} \\
+(F_{\{1,2\},\{3,4\},\{5\},\{6\}}(D))D_{\{1,2,5\},\{3,4,6\}} \quad & +(F_{\{1\},\{2\},\{3,4\},\{5,6\}}(D))D_{\{1,3,4\},\{2,5,6\}} \\
+[F_{\{1,3\},\{2,4\},\{5\},\{6\}}(D) \quad & +F_{\{1\},\{2\},\{3\},\{4,5,6\}}(D) \\
+F_{\{2\},\{3\},\{4\},\{1,5,6\}}(D) \quad & +F_{\{1,6\},\{2\},\{3\},\{45\}}(D)]D_{\{1,3,5\},\{2,4,6\}} \\
+(F_{\{1,6\},\{2\},\{3\},\{4,5\}}(D))D_{\{1,3,6\},\{2,4,5\}} \quad & +(F_{\{1\},\{2,3\},\{4,5\},\{6\}}(D))D_{\{1,4,5\},\{2,3,6\}} \\
+(F_{\{1,6\},\{2,3\},\{4\},\{5\}}(D))D_{\{1,4,6\},\{2,3,5\}}. \quad &
\end{aligned}$$

5. Calculating $c_1\mathbb{V}(\mathfrak{sl}(2), \ell, \vec{\lambda})$

The first Chern class of $\mathbb{V}(\mathfrak{sl}(2), \ell, \vec{\lambda})$ is a 16-coordinate vector in the Picard space and therefore can be expressed in terms of the non-adjacent basis. We denote the first Chern class of $\mathbb{V}(\mathfrak{sl}(2), \ell, \vec{\lambda})$ as $c_1\mathbb{V}(\mathfrak{sl}(2), \ell, \vec{\lambda})$.

Lemma 5.1 ([3, Proposition 2.5])**.** *For $D = c_1\mathbb{V}(\mathfrak{sl}(2), \ell, \vec{\lambda})$, we can calculate $F_{I_1,I_2,I_3,I_4}(D)$ as follows:*

$$F_{I_1,I_2,I_3,I_4}(D)$$
$$= \sum_{(\mu_1,\mu_2,\mu_3,\mu_4)\in P_\ell^4} \deg(\mathbb{V}(\mathfrak{sl}(2), \ell, \{\mu_1, \mu_2, \mu_3, \mu_4\})) \cdot r(\lambda_{I_1}, \mu_1) \cdot r(\lambda_{I_2}, \mu_2) \cdot r(\lambda_{I_3}, \mu_3) \cdot r(\lambda_{I_4}, \mu_4)$$

with the conditions

- I_1, I_2, I_3, I_4 *are fixed,*
- $0 < \mu_1, \mu_2, \mu_3, \mu_4 \leq \ell$,
- $\lambda_{I_J} = \{\lambda_k \mid k \in I_j\}$.

Definition 5.2. *The* rank *of a conformal block is the integer assigned to that set of weights by the formulas given in Lemma 5.3 and Lemma 5.4 below. The rank is denoted $r_\ell(\vec{\lambda})$.*

Lemma 5.3 ([10, 3.1])**.**

$$r_\ell(a,b) = \begin{cases} 1 & \text{if} \quad a = b \\ 0 & \text{otherwise} \end{cases}$$

$$r_\ell(a,b,c) = \begin{cases} 1 & \text{if} \quad \begin{array}{c} a+b+c \text{ even} \\ a \le b+c \\ b \le a+c \\ c \le a+b \\ a+b+c \le 2\ell \end{array} \\ 0 & \text{otherwise} \end{cases}$$

Lemma 5.4 ([10, 3.2])**.**

$$r_\ell(a,b,c,d) = \sum_{i=0}^{\ell} r_\ell(a,b,i) r_\ell(c,d,i)$$

Lemma 5.5 ([10, 3.5])**.**

$$\deg(\mathbb{V}(\mathfrak{sl}(2), \ell, \{a,b,c,d\})) = r_\ell(a,b,c,d) \cdot \max\left\{0, \frac{a+b+c+d-2\ell}{2}\right\}$$

Lemmas 5.1, 5.3, 5.4, and 5.5 allow us to calculate $F_{I_1,I_2,I_3,I_4}(c_1\mathbb{V}(\mathfrak{sl}(2), \ell, \vec{\lambda}))$.

Example Calculation

Let $\ell = 6$ and $\vec{\lambda} = \{2,3,3,4,4,6\}$. Let $D = c_1\mathbb{V}(\mathfrak{sl}(2), 6, \{2,3,3,4,4,6\})$. If we pick an F-functional F_{I_1,I_2,I_3,I_4}, we can calculate $F_{I_1,I_2,I_3,I_4}(D)$. Pick $F_{\{1,6\},\{2,3\},\{4\},\{5\}}$, which is used to compute the last coefficient in the non-adjacent basis. Specifically, we are calculating:

$$F_{\{1,6\},\{2,3\},\{4\},\{5\}}(c_1\mathbb{V}(\mathfrak{sl}(2), 6, (2,3,3,4,4,6))).$$

Here $I_1 = \{1,6\}$ so $\lambda_{I_1} = \{\lambda_1, \lambda_6\} = \{2,6\}$ because 2 is the first element of $\vec{\lambda}$ and is thus selected by λ_1. Likewise 6 is the sixth element of $\vec{\lambda}$ and is thus selected by λ_6. Similarly $\lambda_{I_2} = \{\lambda_2, \lambda_3\} = \{3,3\}$, $\lambda_{I_3} = \{\lambda_4\} = \{4\}$, and $\lambda_{I_4} = \{\lambda_5\} = \{4\}$. Then, by Lemma 5.1 we have

$$\sum_{\{\mu_1,\mu_2,\mu_3,\mu_4\} \in P_\ell^4} \deg(\mathbb{V}(\mathfrak{sl}(2), 6, \{\mu_1,\mu_2,\mu_3,\mu_4\})) \cdot r(2,6,\mu_1) \cdot r(3,3,\mu_2) \cdot r(4,\mu_3) \cdot r(4,\mu_4)$$

We can use the conditions from Lemma 5.3 to find μ_i such that the product of each term is non-zero. In this way we see that $r(4,\mu_3) \neq 0$ if and only if $\mu_3 = 4$ and $r(4,\mu_4) \neq 0$ if and only if $\mu_3 = 4$. By Lemma 5.3 $r_6(2,6,\mu_1) = 1$ if and only if $\mu_1 = 4$. By Lemma 5.3 $r_6(3,3,\mu_2) = 1$ if and only if $\mu_2 \in \{2,4,6\}$. These conditions give the only positive terms for the summation in Lemma 5.1. Additionally, under these conditions, $r_6(\lambda_{I_1}, \mu_1) = r_6(\lambda_{I_2}, \mu_2) = r(\lambda_{I_3}, \mu_3) = r(\lambda_{I_4}, \mu_4) = 1$, so the summation given by Lemma 5.1 becomes

$$\begin{aligned} & r_6(4,2,4,4) \max\left\{0, \frac{14-12}{2}\right\} \\ & + r_6(4,4,4,4) \max\left\{0, \frac{16-12}{2}\right\} + r_6(4,6,4,4) \max\left\{0, \frac{18-12}{2}\right\}. \end{aligned}$$

By Lemmas 5.4 and 5.3 , $r_6(4,2,4,4) = 2$, $r_6(4,4,4,4) = 3$, and $r_6(4,6,4,4) = 1$. Therefore the summation simplifies to:

$$2 \cdot \max\{0, 1\} + 3 \cdot \max\{0, 2\} + 1 \cdot \max\{0, 3\} = 2 + 4 + 3 = 11.$$

We can check this with the `ConformalBlocks` [9] package for `Macaulay2` [4].

```
Macaulay2, version 1.4
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
               PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : loadPackage("ConformalBlocks")

o1 = ConformalBlocks

o1 : Package

i2 : FCurveDotConformalBlockDivisor({{1,6},{2,3},{4},{5}},
          {"A",1,6},{{2},{3},{3},{4},{4},{6}})

o2 = 11

o2 : QQ
```

6. Polyhedral Cones

Definition 6.1. *For a set of n vectors, X, the* cone over X *is the region in $\mathbb{R}^n$ given by $\sum_{i=1}^{n} c_i x_i$ for all $c_i \geq 0$, where $c \in \mathbb{R}$. A cone can also be described in terms of linear inequalities, which serve as the walls of the cones. These inequalities are called the* facets *of the cone* [12, Theorem 1.3].

Consider the cone in Figure 4. We can see that this cone can be expressed in either of the manners described in 6.1.

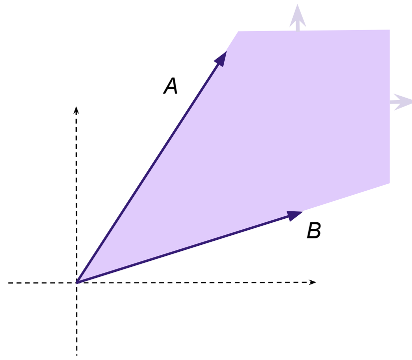

FIGURE 4. Cone in $\mathbb{R}^2$

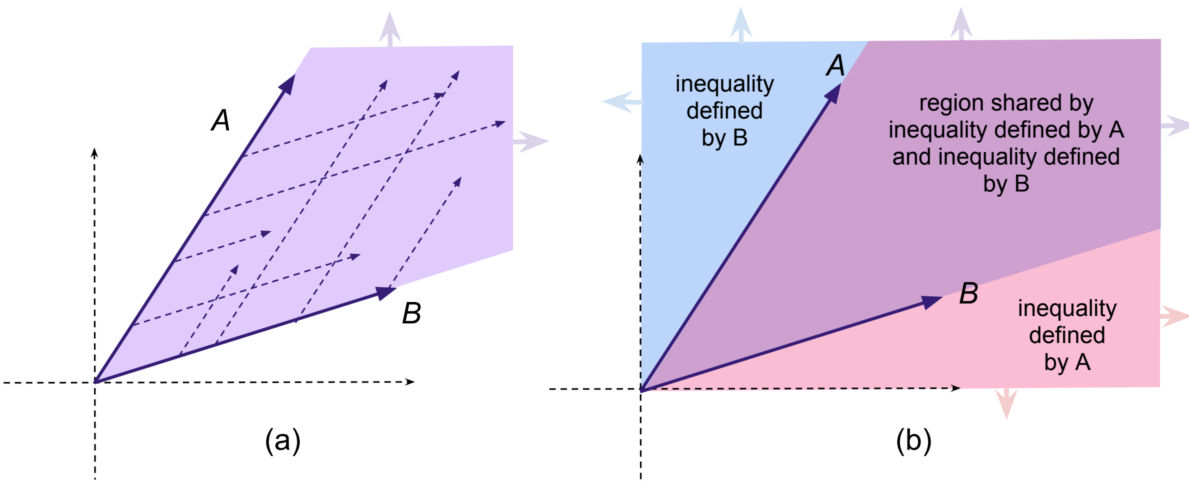

FIGURE 5. (a) Cone from Fig. 4 as linear combinations of A and B. (b) Cone from Fig. 4 as system of inequalities given by A and B.

Definition 6.2. *Consider a d-dimensional cone. Each of the facets defines a $(d-1)$-dimensional plane. The 1-dimensional ray obtained by intersecting $d-1$ linearly independent facets is called an* extremal ray*.*

The example above is a two-dimensional cone, but cones can have n dimensions for any integer $n \geq 1$.

It is important to note that our usage of the word "cone" differs from its use in common English. Our cones are not round like a traffic cone, but instead have finitely many flat facets. Also, our cones are infinitely long.

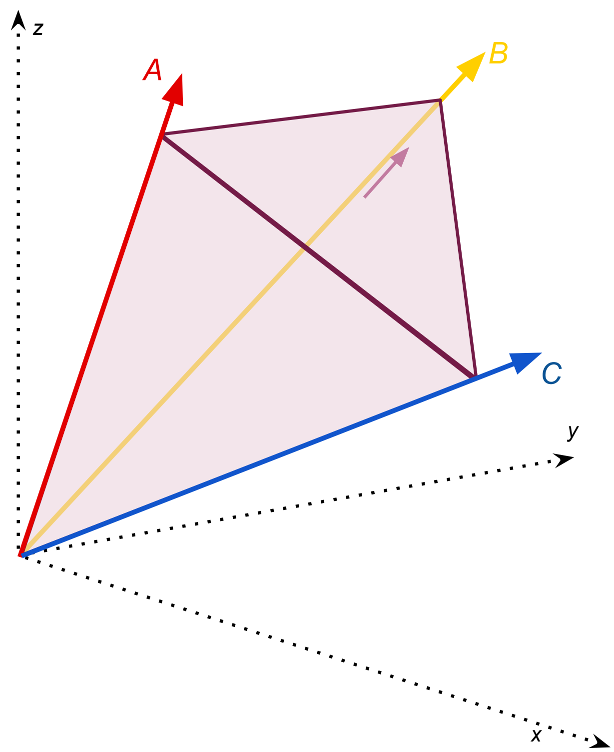

FIGURE 6. A three dimensional cone

7. Nef Cone and Fakhruddin Cone

We will omit a general definition of the nef cone of a projective algebraic variety, and instead describe it only for the spaces $\overline{M}_{0,4}$, $\overline{M}_{0,5}$, and $\overline{M}_{0,6}$.

Theorem 7.1. *Let $n = 4, 5, 6$ and P_n be the Picard space of $\overline{M}_{0,n}$.*

A vector $\vec{v}$ in P_n is nef *if $F_{I_1,I_2,I_3,I_4}(\vec{v}) \geq 0$ for every F-functional. For fixed n, the inequalities of the form $F_{I_1,I_2,I_3,I_4}(\vec{v}) \geq 0$ define a cone, which we call the* nef cone. [2]

Definition 7.2. *If $F_{I_1,I_2,I_3,I_4}(\vec{v}) = 0$ for some F-functional, we call $\vec{v}$* strictly nef. *If $F_{I_1,I_2,I_3,I_4}(\vec{v}) > 0$ for all F-functionals, then we call $\vec{v}$* ample.

Corollary 7.3 ([3, Lemma 2.5])**.** *Let $D = c_1\mathbb{V}(\mathfrak{g}, \ell, \vec{\lambda})$ be the first Chern class of a conformal block on $\overline{M}_{0,n}$. Then D is nef.*

7.1. **The Fakhruddin Cone.** Najmuddin Fakhruddin proposed that for $\overline{M}_{0,6}$ there is a subcone of the nef cone which contains all $\mathfrak{sl}(2)$ conformal blocks [3, Section 6].

Definition 7.4. *The* Fakhruddin cone [10] *is the cone generated by the set of vectors*

$D_1 = c_1\mathbb{V}(\mathfrak{sl}(2), 1, (1, 1, 1, 1, 1, 1)),$

$D_2 = c_1\mathbb{V}(\mathfrak{sl}(2), 2, (1, 1, 1, 1, 1, 1)),$

$D_3 = c_1\mathbb{V}(\mathfrak{sl}(2), 3, (3, 1, 1, 1, 1, 1)),$

$D_4 = c_1\mathbb{V}(\mathfrak{sl}(2), 1, (1, 1, 1, 1, 0, 0)),$

$D_5 = c_1\mathbb{V}(\mathfrak{sl}(2), 3, (2, 2, 1, 1, 1, 1)),$

$D_6 = c_1\mathbb{V}(\mathfrak{sl}(2), 2, (2, 1, 1, 1, 1, 0)),$

$D_7 = c_1\mathbb{V}(\mathfrak{sl}(2), 4, (3, 2, 2, 1, 1, 1)),$

and all permutations of the weights of these conformal blocks.

For example, $c_1\mathbb{V}(\mathfrak{sl}(2), 3, (1, 3, 1, 1, 1, 1))$ and $c_1\mathbb{V}(\mathfrak{sl}(2), 3, (1, 1, 3, 1, 1, 1))$ (as well as the other permutations of the weights) are included because we include all permutations of the weights of $D_1 \dots D_7$ [10, Definition 4.2].

We constructed the Fakhruddin cone using `polymake` [5].

We can visualize the Fakhruddin cone as in Figure 7, although we know that in fact this cone is 16-dimensional.

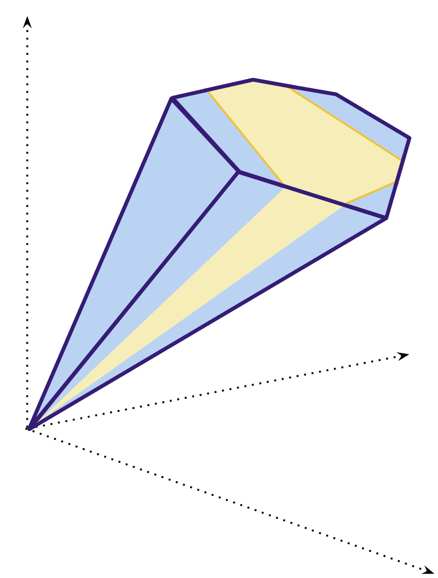

FIGURE 7. The Fakhruddin cone (lighter) as contained in the nef cone (darker)

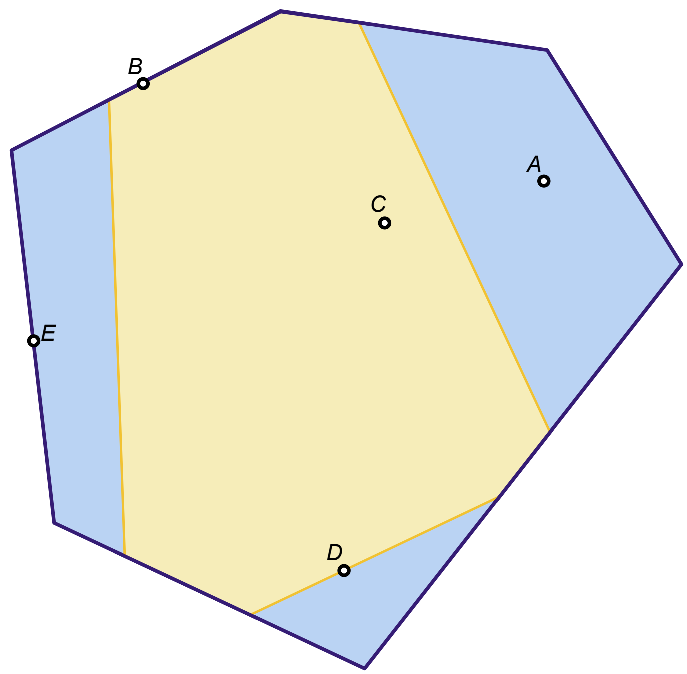

FIGURE 8. A cross section of Figure 7 with certain points labelled

Figure 8 illustrates possible relations between the Fakhruddin cone and the nef cone. We see that A is ample, but not included in the Fakhruddin cone. B is on a facet of the Fakhruddin cone shared with the nef cone, and is therefore strictly nef. Notice that the facets that the Fakhruddin cone shares with the nef cone are strictly nef, and therefore represented as darker in Figure 8. C is a point in the interior of the Fakhruddin cone, and D is a point on the boundary of the Fakhruddin cone. Note that D lies on a facet not shared with the nef cone. E is a strictly nef point that is not in the Fakhruddin cone.

The Fakhruddin cone has 128 extremal rays and 20,516 facets; however, we expect that many of these are generated from permutations of the same set of weights. We are currently working on sorting these facets into a smaller collection that does not include redundant facets generated by permutations of the weights of the same conformal block.

Fakhruddin conjectured that all first Chern class vectors of $\mathfrak{sl}(2)$ conformal blocks land on the boundary or in the interior of the Fakhruddin cone [10, Conjecture 4.2]. This conjecture guided our research as we worked toward proving it.

8. Symmetric weights $\ell - j$

Recall Definition 7.2, in which we define ampleness.

Consider the case when all weights are equal, that is $\vec{\lambda} = (\lambda, \lambda, \lambda, \lambda, \lambda, \lambda)$. In this case, we see that I_1, I_2, I_3, I_4 will select the same elements from $\vec{\lambda}$ by 4. Therefore I_1, I_2, I_3, I_4 are only distinguishable by their size. The possible cardinalities are 1,2, or 3. Additionally, without loss of generality we may reorder I_1, I_2, I_3, and I_4 in order of decreasing size. Therefore all F-functionals will be of the form F_{I_1,I_2,I_3,I_4} with $\#I_1 = 3$ and $\#I_2 = \#I_3 = \#I_4 = 1$, or F_{I_1,I_2,I_3,I_4} with $\#I_1 = \#I_2 = 2$ and $\#I_3 = \#I_4 = 1$. In this way, we see that $F_{\{4,5,6\},\{1\},\{2\},\{3\}}$ will act on $c_1\mathbb{V}(\mathfrak{sl}(2), \ell, (\lambda, \lambda, \lambda, \lambda, \lambda, \lambda))$ in the same way as any F-functional with $\#I_1 = 3, \#I_2 = \#I_3 = \#I_4 = 1$. Likewise $F_{\{1,2\},\{3,4\},\{5\},\{6\}}$ will act the same on this conformal block as any F-functional with $\#I_1 = \#I_2 = 2, \#I_3 = \#I_4 = 1$.

Lemma 8.1 ([10, Lemma 4.4])**.** *Let* $D = c_1\mathbb{V}(\mathfrak{sl}(2), \ell, (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6))$, $\#I_1 = 3$, $\#I_2 = \#I_3 = \#I_4 = 1$, *and* $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \geq 1$.

Then $F_{\{4,5,6\},\{1\},\{2\},\{3\}}(D) > 0$ *if all of the following inequalities are satisfied:*

i. $2 + \lambda_5 + \lambda_6 \leq \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$,

ii. $2 + \lambda_4 + \lambda_6 \leq \lambda_1 + \lambda_2 + \lambda_3 + \lambda_5$,

iii. $2 + \lambda_4 + \lambda_5 \leq \lambda_1 + \lambda_2 + \lambda_3 + \lambda_6$,

iv. $2\ell + 2 \leq \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6$,

v. $2 + \ell \leq \lambda_1 + \lambda_2 + \lambda_3$.

Lemma 8.2. *Fix* F_{I_1,I_2,I_3,I_4} *such that* $\#I_1 = 3$ *and* $\#I_2 = \#I_3 = \#I_4 = 1$.

If $0 < j \leq \frac{2}{3}(\ell - 1)$ *and* $j \in \mathbb{Z}$, *then* $F_{I_1,I_2,I_3,I_4}(c_1\mathbb{V}(\mathfrak{sl}(2), \ell, (\ell - j, \ell - j, \ell - j, \ell - j, \ell - j, \ell - j))) > 0$.

Proof. From 8.1i. we see

$$2 + \ell - j + \ell - j \leq \ell - j + \ell - j + \ell - j + \ell - j,$$

so

$$j \leq \ell - 1,$$

and 8.1ii. and 8.1iii. give the same result. From 8.1iv.

$$2\ell + 2 \leq \ell - j + \ell - j + \ell - j + \ell - j + \ell - j + \ell - j,$$

so

$$j \leq \frac{2}{3}\ell - \frac{1}{3}.$$

From 8.1v.

$$2 + \ell \leq \ell - j + \ell - j + \ell - j$$

so

$$j \leq \frac{2}{3}(\ell - 1).$$

Therefore $0 < j \leq \frac{2}{3}(\ell - 1)$ implies $F_{I_1,I_2,I_3,I_4}(c_1\mathbb{V}(\mathfrak{sl}(2), \ell, (\ell-j, \ell-j, \ell-j, \ell-j, \ell-j, \ell-j))) > 0$ when $\#I_1 = 3$ and $\#I_2 = \#I_3 = \#I_4 = 1$. $\square$

Lemma 8.3 ([10, Lemma 4.5])**.** *Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \geq 1$, $\ell \geq 3$, and $D = c_1\mathbb{V}(\mathfrak{sl}(2), \ell, (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6))$.*

$F_{\{1,2\},\{3,4\},\{5\},\{6\}}(D) > 0$ *if and only if:*

i. $2\ell + 2 \leq \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6$,

ii. $\lambda_1 + \lambda_2 \leq 2\ell - 1$,

iii. $\lambda_3 + \lambda_4 \leq 2\ell - 1$,

iv. $2 + \lambda_1 + \lambda_2 \leq \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6$,

v. $2 + \lambda_3 + \lambda_4 \leq \lambda_1 + \lambda_2 + \lambda_5 + \lambda_6$,

vi. $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \leq 2\ell - 2 + \lambda_5 + \lambda_6$,

vii. $\lambda_1 + \lambda_2 + \lambda_5 + \lambda_6 \geq \ell + 2$,

viii. $\lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 \geq \ell + 2$,

ix. $\lambda_1 + \lambda_2 \leq \ell - 2 + \lambda_5 + \lambda_6$,

x. $\lambda_3 + \lambda_4 \leq \ell - 2 + \lambda_5 + \lambda_6$.

Lemma 8.4. *Let $\#I_1 = \#I_2 = 2$ and $\#I_3 = \#I_4 = 1$.*

If $0 < j \leq \frac{3}{4}(\ell - \frac{2}{3})$ and $j \in \mathbb{Z}$ then $F_{I_1,I_2,I_3,I_4}(c_1\mathbb{V}(\mathfrak{sl}(2), \ell, (\ell-j, \ell-j, \ell-j, \ell-j, \ell-j, \ell-j))) >$ 0.

Proof. For $\vec{\lambda} = \{\ell - j, \ell - j, \ell - j, \ell - j, \ell - j, \ell - j\}$, by 8.3i.

$$j \leq \frac{1}{3}(2\ell - 1).$$

By 8.3ii. and iii.

$$j \geq \frac{1}{2}.$$

By 8.3iv. and v.

$$j \leq \ell - 1.$$

By 8.3vi.

$$j \geq 1.$$

By 8.3vii. and viii.

$$j \leq \frac{3}{4}(\ell - \frac{2}{3}).$$

By 8.3ix. and x.

$$\ell \geq 2.$$

Therefore $1 \leq j \leq \frac{3}{4}(\ell - \frac{2}{3})$. ∎

Theorem 8.5. *Suppose $1 \leq j \leq \frac{2}{3}(\ell - 1)$ where $j \in \mathbb{Z}$.*

$D = c_1\mathbb{V}(\mathfrak{sl}(2), \ell, (\ell - j, \ell - j, \ell - j, \ell - j, \ell - j, \ell - j))$ is ample.

Proof. Use 8.2 and 8.4. ∎

Additionally, we know that all conformal blocks with symmetric weights will be between the conformal blocks with symmetric weights that generate the Fakhruddin cone. This means that, for all levels, conformal blocks with weights of the form 8.5 will be interior to the Fakhruddin cone.

9. An Example with symmetric weights

Let

$$z = F_{I_1,I_2,I_3,I_4}(c_1\mathbb{V}(\mathfrak{sl}(2),\ell,(\ell-1,\ell-1,\ell-1,\ell-1,\ell-1,\ell-1))).$$

Since all of the weights are equal, the cardinalities of I_1, I_2, I_3, and I_4 determine z. That is, if $I_1 = \{2,3\}$ then $\lambda_{I_1} = \{\ell-1,\ell-1\}$. Likewise if $I_1 = \{4,6\}$, then $\lambda_{I_1}$is still equal to $\{\ell-1,\ell-1\}$, because all of the weights are equal. Therefore, when computing the first Chern class of a conformal block with symmetric weights, it does not matter which weights are specifically selected by I_1, I_2, I_3, I_4; only the number of weights each selects is important. We see that two cases emerge, the case where $\#I_1 = 3$ and $\#I_2 = \#I_3 = \#I_4 = 1$, and the case where $\#I_1 = \#I_2 = 2$ and $\#I_3 = \#I_4 = 1$.

Proposition 9.1. *If $\#I_1 = 3$ and $\#I_2 = \#I_3 = \#I_4 = 1$, then $z = 5\ell - 11$.*

If $\#I_1 = \#I_2 = 2$ and $\#I_3 = \#I_4 = 1$, then $z = 2$.

Proof. By Lemma 5.1,

(9.2)

$$z = \sum_{(\mu_1,\mu_2,\mu_3,\mu_4)\in P_\ell^4} \deg(\mathbb{V}(\mathfrak{sl}(2),\ell,\{\mu_1,\mu_2,\mu_3,\mu_4\}))\cdot r(\lambda_{I_1},\mu_1)\cdot r(\lambda_{I_2},\mu_2)\cdot r(\lambda_{I_3},\mu_3)\cdot r(\lambda_{I_4},\mu_4).$$

Case 1: $\#I_1 = 3$ and $\#I_2 = \#I_3 = \#I_4 = 1$. Therefore $\lambda_{I_1} = \{\ell-1,\ell-1,\ell-1\}$ and $\lambda_{I_2} = \lambda_{I_3} = \lambda_{I_4} = \{\ell-1\}$. We see immediately, using Lemma 5.3, that the summand is zero unless $\mu_2 = \mu_3 = \mu_4 = \ell - 1$. Lemma 5.4 gives us

$$r_\ell(\lambda_{I_1},\mu_1) = r_\ell(\mu_1,\ell-1,\ell-1,\ell-1) = \sum_{i=0}^{\ell} r_\ell(\mu_1,\ell-1,i)\cdot r_\ell(\ell-1,\ell-1,i). \tag{9.3}$$

The only non-zero terms in the summation (9.3) occur when $i \in \{0,2\}$, and so Lemma 5.3 dictates that $r_\ell(\mu_1,\ell-1,i) = 1$ when $\mu_1 \in \{\ell-1,\ell-3\}$. In summary, we see that the only nonzero terms in (9.2) occur when $\mu_1 = \ell-1$, $r_\ell(\ell-1,\ell-1,\ell-1,\ell-1) = 2$ and for

$\mu_1 = \ell - 3$, $r_\ell(\ell-3, \ell-1, \ell-1, \ell-1) = 1$. When $\mu_1 = \ell - 1$, Lemma 5.5 gives

$$\begin{aligned}
&\deg(\mathbb{V}(\mathfrak{sl}(2), \ell, \{\ell-1, \ell-1, \ell-1, \ell-1\})) \\
&= r_\ell(\ell-1, \ell-1, \ell-1, \ell-1) \cdot \max\left\{0, \frac{\ell-1+\ell-1+\ell-1+\ell-1-2\ell}{2}\right\} \\
&= 2 \cdot \max\{0, \ell-2\} \\
&= 2\ell - 4.
\end{aligned}$$

When $\mu_1 = \ell - 3$, Lemma 5.5 gives

$$\begin{aligned}
&\deg(\mathbb{V}(\mathfrak{sl}(2), \ell, \{\ell-3, \ell-1, \ell-1, \ell-1\})) = \\
&= r_\ell(\ell-3, \ell-1, \ell-1, \ell-1) \cdot \max\left\{0, \frac{\ell-3+\ell-1+\ell-1+\ell-1-2\ell}{2}\right\} = \\
&= 1 \cdot \max\{0, \ell-3\} = \ell - 3.
\end{aligned}$$

Therefore, by Lemma 5.1 z simplifies to

$$z = (2\ell - 4) \cdot 2 \cdot 1 \cdot 1 \cdot 1 + (\ell - 3) \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 5\ell - 11.$$

This concludes Case 1.

Case 2: $\#I_1 = \#I_2 = 2$ and $\#I_3 = \#I_4 = 1$. Therefore $\lambda_1 = \lambda_2 = \{\ell-1, \ell-1\}$ and $\lambda_3 = \lambda_4 = \{\ell - 1\}$. We see immediately, using Lemma 5.3, that the summand is zero unless $\mu_3 = \mu_4 = \ell - 1$. By Lemma 5.3 $r_\ell(\mu_1, \ell-1, \ell-1) = 1$ when $\mu_1 = 2$ and likewise from $r_\ell(\mu_2, \ell-1, \ell-1) = 1$ we see that $\mu_2 = 2$. When $\mu_1 = \mu_2 = 2$, Lemma 5.5 gives

$$\begin{aligned}
&\deg(\mathbb{V}(\mathfrak{sl}(2), \ell, \{2, 2, \ell-1, \ell-1\})) = \\
&= r_\ell(2, 2, \ell-1, \ell-1) \cdot \max\left\{0, \frac{2+2+\ell-1+\ell-1-2\ell}{2}\right\} = \\
&= r_\ell(2, 2, \ell-1, \ell-1) \cdot \max\{0, 1\}
\end{aligned}$$

Considering $r_\ell(2, 2, \ell-1, \ell-1)$, by Lemma 5.4 this becomes

$$r_\ell(2, 2, \ell-1, \ell-1) = \sum_{i=0}^{\ell} r_\ell(2, 2, i) r_\ell(\ell-1, \ell-1, i)$$

which, by Lemma 5.3, has non-zero terms in its summation when $i \in \{0, 2\}$. $r_\ell(2, 2, 0) = 1$ and $r_\ell(2, 2, 2) = 1$, so $r_\ell(2, 2, \ell-1, \ell-1) = 2$. Therefore the expression for z becomes

$$z = r(2, 2, \ell-1, \ell-1) \cdot \max\{0, 1\} = 2.$$

This concludes Case 2. □

10. Computational Results

10.1. **Testing Fakhruddin's Conjecture.** We tested which first Chern class vectors are interior to the Fakhruddin cone for levels $\ell = 1, 2, \ldots, 10$. We did this by comparing each first Chern class vector for a given weight against the inequalities of each facet of the Fakhruddin cone. For these levels, all first Chern class vectors either landed on the boundary of the Fakhruddin cone, or in the interior. This confirms Fakhruddin's predictions for these levels. Specifically, first Chern class vectors for a given level landed in the interior or on the boundary with frequencies:

ℓ:	1	2	3	4	5	6	7	8	9	10
# interior:	0	0	1	4	8	17	35	62	109	185
# boundary:	2	8	25	65	151	310	587	1050	1774	2871.

We carried out these calculations using the `ConformalBlocks` [9] package in `Macaulay2` [4].

10.2. **Weight sums of conformal blocks interior to the Fakhruddin cone.** We further investigated the first Chern class vectors interior to the Fakhruddin cone by studying the weights of these conformal blocks. We noticed a pattern in the allowable sum of the weights for a given level, and also a pattern in the frequencies of these sums.

ℓ	Possible weight sums (with occurence frequencies) for ample conformal blocks
3	12(×1)
4	12(×1) 14(×2) 18(×1)
5	16(×2) 18(×2) 20(×2) 24(×1)
6	18(×2) 20(×5) 22(×5) 24(×2) 26(×2) 30(×1)
7	18(×1) 20(×2) 22(×8) 24(×9) 26(×5) 28(×5) 30(×2) 32(×2) 36(×1)
8	22(×2) 24(×6) 26(×16) 28(×12) 30(×11) 32(×5) 34(×5) 36(×2) 38(×2) 42(×1)
9	24(×2) 26(×8) 28(×19) 30(×22) 32(×20) 34(×12) 36(×11) 38(×5) 40(×5) 42(×2) 44(×2) 48(×1)
10	24(×1) 26(×2) 28(×8) 30(×22) 32(×34) 34(×35) 36(×24) 38(×22) 40(×12) 42(×11) 44(×5) 46(×5) 48(×2) 50(×2) 54(×1)

These calculations were also carried out using the `ConformalBlocks` package in `Macaulay2`. We noticed from this data that the weight sums increase by two's, and the last one is an increase by four. This last weight sum comes from the weight vector $\vec{\lambda} = (\ell-1, \ell-1, \ell-1, \ell-1, \ell-1, \ell-1)$. We proved above that $c_1\mathbb{V}(\mathfrak{sl}(2), \ell, (\ell-1, \ell-1, \ell-1, \ell-1, \ell-1, \ell-1)$ is ample, and in fact this first Chern class vector is additionally in the interior of the Fakhruddin cone. In fact, we see that conformal blocks with weight vector $\vec{\lambda} = (\ell - j, \ell - j, \ell - j, \ell - j, \ell - j, \ell - j)$ as in Theorem 8.5 are all contained in the interior of the Fakhruddin cone for $\ell = 3, 4, \ldots, 10$. There is also a noticeable pattern in the frequencies of the weight sums. We can see that the largest weight sum will occur once, then the second and third largest each occur two times, the fourth and fifth largest five times, the sixth eleven times, and the seventh twelve times. It seems that we can expect this pattern to continue and grow for larger levels.

Expressing the weight vectors in terms of the level, we can see that the list of interior weight vectors at each level is included in the same list for highter levels. In other words, we can write the list of all weight vectors interior to the Fakhruddin cone for $\ell = 6$ and show that this includes the lists of interior weight vectors for all $\ell < 6$.

Interior for level	Weight vector
$\ell = 6$	$\{\ell-4, \ell-3, \ell-3, \ell-3, \ell-3, \ell-2\}$
$\ell = 6$	$\{\ell-3, \ell-3, \ell-3, \ell-3, \ell-3, \ell-3\}$
$\ell = 6$	$\{\ell-4, \ell-3, \ell-3, \ell-3, \ell-2, \ell-1\}$
$\ell = 6$	$\{\ell-4, \ell-3, \ell-3, \ell-2, \ell-2, \ell-2\}$
$\ell = 6$	$\{\ell-4, \ell-3, \ell-3, \ell-3, \ell-3, \ell\}$
$\ell = 6$	$\{\ell-3, \ell-3, \ell-3, \ell-3, \ell-3, \ell-1\}$
$\ell = 6$	$\{\ell-3, \ell-3, \ell-3, \ell-3, \ell-2, \ell-2\}$
$\ell = 6$	$\{\ell-3, \ell-3, \ell-3, \ell-3, \ell-2, \ell\}$
$\ell = 6$	$\{\ell-3, \ell-3, \ell-3, \ell-3, \ell-1, \ell-1\}$
$\ell = 6$	$\{\ell-3, \ell-3, \ell-3, \ell-2, \ell-2, \ell-1\}$
$\ell = 5, 6$	$\{\ell-4, \ell-2, \ell-2, \ell-2, \ell-2, \ell-2\}$
$\ell = 5, 6$	$\{\ell-3, \ell-3, \ell-2, \ell-2, \ell-2, \ell-2\}$
$\ell = 5, 6$	$\{\ell-3, \ell-2, \ell-2, \ell-2, \ell-2, \ell-1\}$
$\ell = 4, 5, 6$	$\{\ell-2, \ell-2, \ell-2, \ell-2, \ell-2, \ell-2\}$
$\ell = 4, 5, 6$	$\{\ell-2, \ell-2, \ell-2, \ell-2, \ell-2, \ell\}$
$\ell = 4, 5, 6$	$\{\ell-2, \ell-2, \ell-2, \ell-2, \ell-1, \ell-1\}$
$\ell = 3, 4, 5, 6$	$\{\ell-1, \ell-1, \ell-1, \ell-1, \ell-1, \ell-1\}$

We can also find all interior weight vectors for $\ell = 6$ on the list for $\ell = 7$. This pattern suggests that an inductive approach might be useful for finding a formula for interior weight vectors.

We also used `polymake` to create the polytopes formed by the vectors interior to the Fakhruddin cone. However, this did not yield any immediately recognizable patterns.

11. Future Directions

Our work suggests numerous future directions toward proving Fakhruddin's conjecture [10, Conjecture 4.2]. The most important first step in doing this seems to be sorting the facets of the Fakhruddin cone up to symmetry. This will yield a list of facets of manageable size, realistically allowing one to test vectors against these facets. With a list of facets of the Fakhruddin cone sorted up to symmetry, one may be able to develop inequalities similar to those of Lemmas 8.3 and 8.1. This may allow us to prove Theorem 8.5 for the Fakhruddin cone. It may also be useful to test which facets of the Fakhruddin cone intersect a conformal block on the boundary, which will be easier with this reduced list of facets.

Proofs of the patterns we noticed with regard to the weight sums 10.2 would also work toward proving Fakhruddin's conjecture. This, in combination with finding a pattern for the new weights at each level, could work to inductively prove a formula for weights giving first Chern class vectors interior to the Fakhruddin cone. It may also be useful to investigate why $6\ell - 8$ does not appear to be a viable weight sum for interior conformal blocks. For a fixed weight, it may be interesting to know the levels for which that weight is interior to the Fakhruddin cone. The way in which the list of interior weight vectors contains the same list for smaller levels may be helpful in showing showing this, and a pattern here may be useful.

There may also be useful information to be gained from studying the polytopes defined by the first Chern class vectors interior to the Fakhruddin cone but, as stated above, we did not find such a pattern.

12. Conclusions

In our work with $\mathfrak{sl}(2)$ conformal blocks associated to $\overline{M}_{0,6}$, we made some progress towards proving Fakhruddin's conjecture. Besides computationally proving it for $\ell = 1, 2, \ldots, 10$, we found some interesting patterns which may be useful in completing the proof in the future. By studying whether there are ample conformal blocks on the boundary of the Fakhruddin cone, we may be able to apply our proven results for the ampleness of conformal blocks with symmetric weights to the Fakhruddin cone. Although we were not able to prove Fakhruddin's conjecture, we did increase the tools available toward such a proof in the hope that it will eventually be proven.

References

[1] Pierre Deligne and David Mumford, *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. **36** (1969), 75–109.

[2] Carel Faber, *The nef cone of $\overline{M}_0, 6$: a proof by inequalities only* (2000). unpublished.

[3] Najmuddin Fakhruddin, *Chern classes of conformal blocks*, Compact moduli spaces and vector bundles, Contemp. Math., vol. 564, Amer. Math. Soc., Providence, RI, 2012, pp. 145–176, DOI 10.1090/conm/564/11148, (to appear in print).

[4] Dan Grayson and Mike Stillman, `Macaulay2`*: a software system for research in algebraic geometry* (2012). Version 1.4.

[5] Ewgenij and Joswig Gawrilow Michael, `polymake`*: a framework for analyzing convex polytopes* (2012). Version 2.12.

[6] Finn Faye Knudsen and David Mumford, *The projectivity of the moduli space of stable curves. I. Preliminaries on "det" and "Div"*, Math. Scand. **39** (1976), no. 1, 19–55.

[7] Finn F. Knudsen, *The projectivity of the moduli space of stable curves. II. The stacks $M_{g,n}$*, Math. Scand. **52** (1983), no. 2, 161–199.

[8] Finn F. Knudsen, *The projectivity of the moduli space of stable curves. III. The line bundles on $M_{g,n}$, and a proof of the projectivity of $\overline{M}_{g,n}$ in characteristic* 0, Math. Scand. **52** (1983), no. 2, 200–212.

[9] David Swinarski, `ConformalBlocks`*: software for computing conformal block divisors in* `Macaulay2` (2011). Version 1.1.

[10] David Swinarski, *sl_2 conformal block divisors and the nef cone of $\bar{M}_{0,n}$*, 2011. arxiv:1107.5331.

[11] Kenji Ueno, *Conformal field theory with gauge symmetry*, Fields Institute Monographs, vol. 24, American Mathematical Society, Providence, RI, 2008.

[12] Günter M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995.