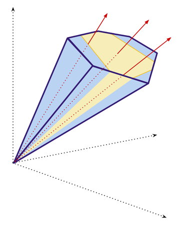


Investigating behavior of first Chern class vectors of $\mathfrak{sl}(2)$ conformal blocks

Zachery Wills and Dr. David Swinarski

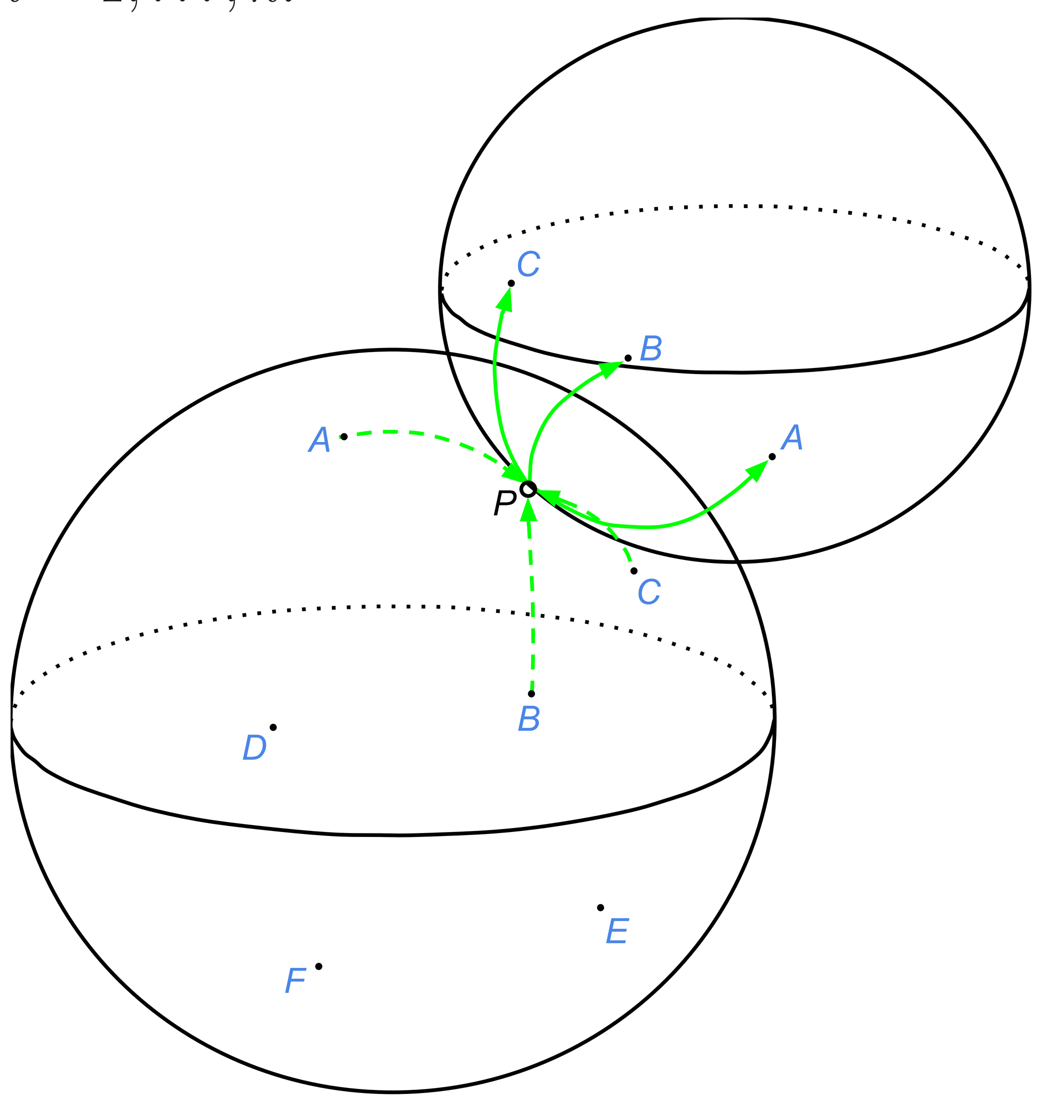
Fordham University
The Jesuit University of New York

Main Questions

We investigated the behavior of the first Chern class vectors of $\mathfrak{sl}(2)$ conformal blocks associated with $\overline{M}_{0,6}$. Najmuddin Fakhruddin conjectured that all of these vectors fall within a specific cone with 128 extremal rays and 20,516 facets, which we call the Fakhruddin cone. We tested Fakhruddin's conjecture for conformal blocks of level $\ell = 1, 2, \ldots, 10$. We also investigated how conformal blocks with wieghts of the form $(\ell-j, \ell-j, \ell-j, \ell-j, \ell-j, \ell-j)$ behave in the nef cone.

$\overline{M}_{0,n}$

$\overline{M}_{0,n}$ is the moduli space of the ordered set of distinct configurations of n points on a sphere. Sets of points $(p_1 \ldots p_n)$ and $(q_1 \ldots q_n)$ are equivalent if there exists a 2×2 matrix M such that $Mp_i = q_i$ for $i = 1, \ldots, n$.



When points approach the same point P in $\overline{M}_{0,n}$, we describe their behavior by having them move through the point P onto another sphere attached at P, as shown above.

First Chern class vectors

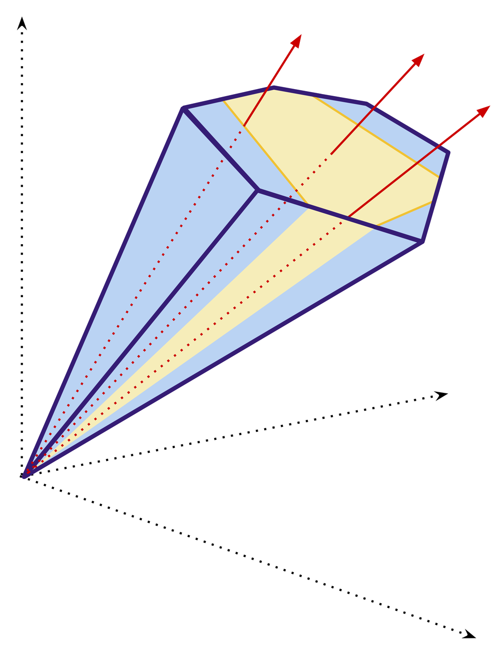
When $\vec{\lambda}$ has six weights, the first Chern class of $\mathbb{V}(\mathfrak{sl}(2), \ell, \vec{\lambda})$ is a 16-coordinate vector in a vector space called the Picard space. We denote the first Chern class of $\mathbb{V}(\mathfrak{sl}(2), \ell, \vec{\lambda})$ as $c_1\mathbb{V}(\mathfrak{sl}(2), \ell, \vec{\lambda})$. Fakhruddin provides a method of computing these first Chern class vectors [1, Proposition 2.5], and additionally proves that these vectors are nef [1, Lemma 1.5]. We used `Macaulay2` to quickly compute these vectors.

Conformal Blocks

The work of Deligne, Mumford, Mayer, and Knudsen demonstrates that $\overline{M}_{0,n}$ (see below) can be described using polynomial equations. To each configuration of points in $\overline{M}_{0,n}$ we may assign a vector space called a *conformal block*. Conformal blocks have three ingredients: a Lie algebra $\mathfrak{g}$, a level ℓ, and a weight vector $\vec{\lambda}$. A conformal block is denoted $\mathbb{V}(\mathfrak{g}, \ell, \vec{\lambda})$. We are working with the Lie algebra $\mathfrak{sl}(2)$, the set of 2×2 matrices with trace equal to zero. The level ℓ is an integer. The coordinates of the weight vector $\vec{\lambda}$ are positive integers less than or equal to ℓ. $\vec{\lambda}$ has six coordinates because we are studying conformal blocks related to $\overline{M}_{0,6}$.

Computational Results

Result. Using `Macaulay2` we tested the first Chern class vectors of $\mathfrak{sl}(2)$ conformal blocks against the facets of the Fakhruddin cone, and found that for levels $\ell = 1, 2, \ldots, 10$ Fakhruddin's conjecture is true.

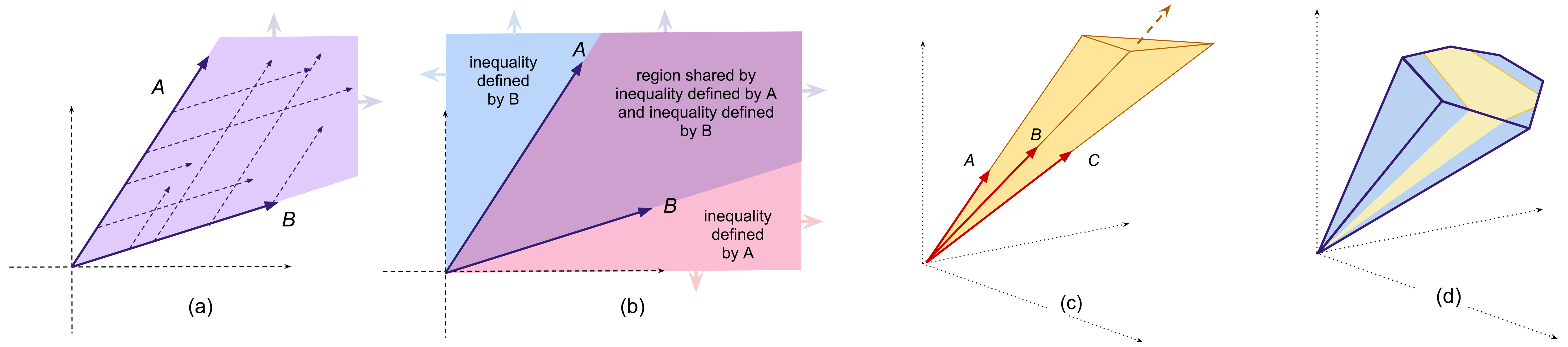


A condensed representation of first Chern class vectors in the Fakhruddin cone

We also noticed that, for a given level, all conformal blocks interior to the Fakhruddin cone from previous levels are interior at that level.

Cones

For a set of n vectors, X, the *cone over* X is the region in $\mathbb{Q}^n$ consisting of vectors of the form $\sum_{i=1}^{n} c_i x_i$ for all $c_i \geq 0$, where $c_i \in \mathbb{Q}$. A cone can also be described in terms of linear inequalities, which serve as the walls of the cones. These inequalities are called the *facets* of the cone.



(a) 2-dimensional cone described by linear combinations (b) 2-dimensional cone described by linear inequalities (c) 3-dimensional cone (d) 3-dimensional condensed representation of the nef cone and the Fakhruddin cone

The condensed representation (d) describes the relationship between the nef cone (blue) and the Fakhruddin cone (yellow). (d) shows that the Fakhruddin cone is a subcone of the nef cone, but it is important to realize that these cones are 16-dimensional. The darker color corresponding to each cone represents the boundary of that cone. We call a vector that lies on the boundary of the nef cone *strictly nef* and a vector that lies within the nef cone *ample*.

Proven Results

Result. Let $D = c_1\mathbb{V}(\mathfrak{sl}(2), \ell, (\ell-j, \ell-j, \ell-j, \ell-j, \ell-j, \ell-j))$.
If $\ell \geq 3$ and j is an integer such that $1 \leq j \leq \frac{2}{3}(\ell-1)$, then D is ample.

This means that we can pick any level $\ell \geq 3$ and integer j, and $c_1\mathbb{V}(\mathfrak{sl}(2), \ell, (\ell-j, \ell-j, \ell-j, \ell-j, \ell-j, \ell-j))$ will be in the interior of the nef cone.

Result. If $\ell \geq 3$ then $c_1\mathbb{V}(\mathfrak{sl}(2), \ell, (\ell-1, \ell-1, \ell-1, \ell-1, \ell-1, \ell-1))$
$= (5\ell-11, 2, 5\ell-11, 5\ell-11, 2, 5\ell-11, 5\ell-11, 2, 5\ell-11, 2, 2, 2, 10\ell-18, 2, 2, 2)$

Note that these results apply to conformal blocks at any level ℓ. Their proofs rely on lemmas proven by Swinarski [4]. Our calculations show that these conformal blocks are also interior to the Fakhruddin cone for $\ell = 3, 4, \ldots, 10$. This may be useful in proving Fakhruddin's conjecture.

Future Directions

We observed additional patterns in our computational data which we have not yet proven, such as how the conformal blocks of a given level interior to the Fakhruddin cone include those from previous levels. We also suspect that sorting the facets of the Fakhruddin cone up to S_6 symmetry may yield a more manageable list of facets. These intermediate results could be useful in proving Fakhruddin's conjecture.

References

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[5] Günter M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995.