

Computational Studies of $\mathfrak{sl}(2)$ Conformal Blocks

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Zachery Jan Wills, FCLC '14, is a mathematics major in the 3-2 Columbia Combined Plan. This is his first research work, and he enjoyed researching something completely new in mathematics.

Zachery most admires Thomas Wolfe, Neal Cassady, and Alyosha Karamozov.

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INTRODUCTION

Conformal field theory plays a key role within the study of string theory and mathematical physics. This paper investigates a class of conformal blocks, which are important in the Wess-Zumino-Witten model of conformal field theory. Conformal blocks also have useful applications in algebraic geometry because for every conformal block there is an associated polynomial map on $M_{0,6}$, the moduli space of configurations of six points on a sphere.

A conformal block is determined by three ingredients: a Lie algebra, a positive integer called the level, and a weight vector. A conformal block is denoted as $V(\mathfrak{g}, \ell, \vec{\lambda})$, where \mathfrak{g} , ℓ , and $\vec{\lambda}$ are the Lie algebra, level, and weight vector, respectively. Specifically we have been investigating the behavior of conformal blocks for the Lie algebra $\mathfrak{sl}(2)$, the set of 2×2 matrices with $\text{trace}=0$, with six-coordinate weight vectors while varying the level. The level, ℓ , within the conformal block provides the greatest possible integer value for the coordinates of the weight vector. In this way, a conformal block of level 2 can only have weight vector coordinates of 0, 1, or 2. For example, at $\ell=2$, one could have a weight vector $(2, 2, 2, 1, 1, 0)$ or $(0, 0, 0, 2, 2, 2)$, but not $(3, 2, 2, 1, 1, 0)$, because $3 > 2 = \ell$. In this way, for any fixed ℓ , there will be a finite number of weights. This paper does not investigate weights which have the same elements in a different order, the reason for which I stated in section 3. We may assume that the weights in the weight vector are (weakly) increasing, and so in choosing these vectors, of $(1,1,1,1,1,2)$ and $(1,2,1,1,1,1)$, only $(1,1,1,1,1,2)$ will be investigated. For lower levels, these calculations can be carried out using a computer to consider all possible weights. It would be impractical to carry these calculations out by hand. However, as the level increases, the number of weight vectors becomes so large that it is not feasible to perform these calculations, even by computer. For this reason, we are using the computer for calculations up to $\ell=10$.

We are concerned with a characteristic of conformal blocks called the first Chern class. The first Chern class of a conformal block is a vector associated to a

polynomial map from $M_{0,6}$ to another space, and it encodes information about the curvature of the image of $M_{0,6}$ under this polynomial map (for a more detailed treatment, see Griffiths 1994). The first Chern class of an $\mathfrak{sl}(2)$ conformal block takes the form of a sixteen-dimensional vector, and can be calculated using Fakhruddin's formula (Swinarski 2011a; Fakhruddin 2012, Prop. 2.5) as explained by Swinarski in Swinarski 2011a, 3-3.5. We have been employing Swinarski's `ConformalBlocks` package for `Macaulay2` (Swinarski 2011b; Grayson 2012) for the computation of these first Chern class vectors. The `ConformalBlocks` package allows us to compute large numbers of first Chern class vectors with relative ease given a specified Lie algebra, level, and weight vector.

FAKHRUDDIN'S CONJECTURE

Fakhruddin conjectures (see Fakhruddin 2012, Section 6) that for all $\mathfrak{sl}(2)$ $n=6$ conformal blocks with six weights, every first Chern class vector will fall within a specific cone, which we call the Fakhruddin cone. This cone has 192 extremal rays. The rays which generate this Fakhruddin cone are called Fakhruddin rays (for more details on the generation of this cone and more specific criteria denoting a Fakhruddin ray see (Swinarski 2011a, 4.1). A refinement of Fakhruddin's Conjecture is to study how these first Chern class vectors land within the Fakhruddin cone. For example, although (conjecturally) they do not fall outside of the cone, some may land in the interior of the cone or on the boundary of the cone. This paper further investigates specifically where in the Fakhruddin cone—either on the boundary or in the interior of the cone—these first Chern class vectors lie.

FIRST CHERN CLASS VECTORS IN THE FAKHRUDDIN CONE

We have calculated the first Chern class vectors for $\mathfrak{sl}(2)$ conformal blocks for $\ell=1, 2, \dots, 9$ using `ConformalBlocks` (Swinarski 2011b). We have also constructed the Fakhruddin cone from the Fakhruddin rays. For low levels ($\ell=1, 2, \dots, 9$) our calculations show that Fakhruddin's Conjecture is true. We also determined exactly whether each first

Chern class vector lies on the boundary or in the interior of the Fakhruddin cone. The construction of the Fakhruddin cone was done through the program `Polymake`, and the determination of where the first Chern class vectors fall in the Fakhruddin cone was computed in `Macaulay2` (Gawrilow 2012).

It is useful to note here that, as stated above, for a given weight vector the distinguishable permutations of its coordinates are not being considered as separate cases for study. This is because the location of the first Chern class vector for a conformal block with a given weight vector in the Fakhruddin cone is preserved when the coordinates of that weight vector are permuted. In other words, if one arrangement of the coordinates of a weight vector leads that conformal block's first Chern class vector to lie in the interior of the Fakhruddin cone, then all other arrangements of the coordinates of that weight vector will still land the first Chern class vector of that conformal block into the interior of the Fakhruddin cone. This results because Fakhruddin's cone is preserved under permutations, so all permutations take the boundary of the cone to the boundary of the cone and likewise take the interior of the cone to the interior of the cone. For these reasons, it is unnecessary to run calculations for all permutations of weight vector coordinates; the location in the Fakhruddin cone of a first Chern class vector will be the same regardless of which permutation of the weight vector coordinates is being considered.

All first Chern class vectors of conformal blocks at $\ell=1$ and $\ell=2$ lie on the boundary of the Fakhruddin cone. We have found that at $\ell=3$, one first Chern class vector, with weight $(1,2,2,3,3,3)$, lies in the interior of the cone, while the rest fall on the boundary. At $\ell=4$, there are four first Chern class vectors and at $\ell=5$ there are eight first Chern class vectors in the interior of the cone; all others lie on the boundary. For $\ell=4$ and $\ell=5$, the weights listed in Table 1 correspond to conformal blocks whose first Chern class vectors lie in the interior of the Fakhruddin cone.

All other first Chern class vectors for these levels lie on the boundary of the Fakhruddin cone.

We have also calculated that the first Chern classes for $\ell=1, 2, \dots, 9$ fall in the interior or on the boundary with the frequencies listed in Table 2 below.

ℓ	1	2	3	4	5	6	7	8	9
interior	0	0	1	4	8	17	35	62	109
boundary	2	8	24	65	150	310	587	1050	1774

Table 2

Due to the way in which the Fakhruddin cone is constructed (see Swinarski 2011a, 4.1) it is not surprising that levels 1 and 2 have no interior first Chern class vectors. The reasons that so many first Chern class vectors lie on the boundary of the Fakhruddin cone is an area of further investigation.

ℓ	weight vector
4	$(2,2,2,2,2,2)$
4	$(2,2,2,2,2,4)$
4	$(2,2,2,2,3,3)$
4	$(3,3,3,3,3,3)$
5	$(1,3,3,3,3,3)$
5	$(2,2,3,3,3,3)$
5	$(2,3,3,3,3,4)$
5	$(3,3,3,3,3,3)$
5	$(3,3,3,3,3,5)$
5	$(3,3,3,3,4,4)$
5	$(3,3,3,3,5,5)$
5	$(4,4,4,4,4,4)$

Table 1

We do not yet know why the majority of first Chern class vectors lie on the boundary of the Fakhruddin cone. We hope to notice and prove a rule or pattern governing the location of a first Chern class vector based on the weights of the conformal block. Due to the relatively low number of interior first Chern class vectors, we are looking for a rule governing why one such vector lands within the interior. Such a rule may be useful in proving Fakhruddin's conjecture, and it is the goal of ongoing research.

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