FORDHAM UNIVERSITY
THE JESUIT UNIVERSITY OF NEW YORK

Ranks and first Chern classes of $\mathfrak{sl}_2$ conformal blocks

Muhammad Ahsan and Dr. David Swinarski ~ April 10, 2014

Introduction

Conformal blocks have applications in string theory and mathematical physics. A conformal block is determined by a Lie algebra, a level, and a set of weights. We study conformal blocks for the Lie algebra $\mathfrak{sl}_2$, which is the set of all 2×2 matrices with trace zero. A weight for $\mathfrak{sl}_2$ is a nonnegative integer. Conformal blocks for $\mathfrak{sl}_2$ with up to five weights have been studied previously. We study two important properties called the *rank* and *first Chern class* of $\mathfrak{sl}_2$ conformal blocks of arbitrarily high level with six equal weights.

Rank formulas

The rank of a conformal block can be computed using recursive formulas. The formulas for computing ranks when there are two or three weights are shown below:

$$r_\ell(a,b) = \begin{cases} 1 \text{ if } a = b \\ 0 \text{ otherwise.} \end{cases}$$

$$r_\ell(a,b,c) = \begin{cases} 1 \text{ if } & a+b+c \text{ is even} \\ & a \le b+c \\ & b \le a+c \\ & c \le a+b \\ & a+b+c \le 2\ell \\ 0 \text{ otherwise} & \end{cases}$$

A recursive formula for the rank when there are n weights with $n \ge 4$ is

$$r_\ell(\lambda_1, \ldots, \lambda_n) = \sum_{i=0}^{\ell} r_\ell(\lambda_1, \ldots, \lambda_{n-2}, i) r_\ell(\lambda_{n-1}, \lambda_n, i).$$

In principle, these formulas can be used to obtain the rank for any $\mathfrak{sl}_2$ conformal block, but in practice, this is slow if the level is very large. For this reason, we want to find closed formulas for the ranks that will work even when the level is large.

B. Alexeev found closed formulas for ranks when $n = 4$. We found closed formulas for the rank when $n = 6$ and the weights are equal to each other.

There are eight expressions altogether. We divide into cases as follows: Let the set of weights be (w, w, w, w, w, w).

Case A: $0 \le w \le 2l - 3w \le l$
Case B: $0 \le 2l - 3w \le w \le l$
Case C: $0 \le w \le l \le 2l - 3w$
Case D: $2l - 3w \le 0 \le w \le l$

The results also depend on whether the weight is even or odd.

Theorem (Ahsan, 2013). *The rank of a conformal block for $\mathfrak{sl}_2$ and six equal weights $\lambda_1 = \cdots = \lambda_6 = w$ is given by the following formulas.*

When the weight is odd, $w = 2k+1$:

Case A: $\frac{1}{6}l^3 - 3l^2k + 18lk^2 - 32k^3 - 2l^2 + 24lk - 60k^2 + \frac{47}{6}l - 34k - 5$
Case B: $\frac{-5}{6}l^3 + 9l^2k - 30lk^2 + 32k^3 + 4l^2 - 24lk + 36k^2 - \frac{19}{6}l + 10k + 1$
Case C: $4k^3 + 12k^2 + 13k + 5$
Case D: $\frac{1}{2}l^3 - 3l^2k + 6lk^2 - 4k^3 + \frac{1}{2}l - k$

When the weight is even, $w = 2k$:

Case A: $\frac{1}{6}l^3 - 3l^2k + 18lk^2 - 32k^3 - \frac{1}{2}l^2 + 6lk - 12k^2 + \frac{1}{3}l + 2k + 1$
Case B: $\frac{-5}{6}l^3 + 9l^2k - 30lk^2 + 32k^3 - \frac{1}{2}l^2 + 6lk - 12k^2 + \frac{4}{3}l - 2k + 1$
Case C: $4k^3 + 6k^2 + 4k + 1$
Case D: $\frac{1}{2}l^3 - 3l^2k + 6lk^2 - 4k^3 + \frac{3}{2}l^2 - 6lk + 6k^2 + 2l - 4k + 1$

The formulas above are *piecewise polynomial* (that is, in each case, the expression is a polynomial). These formulas allow us to compute ranks of conformal blocks even when the level ℓ is large.

First Chern class formulas

In 2008, Fakhruddin described a method for computing the first Chern class of a conformal block [1]. When there are six equal weights, the first Chern class is a vector of two nonnegative integers. We wanted to know which points in the first quadrant are first Chern classes of a conformal block for $\mathfrak{sl}_2$.

For this analysis, we assumed that the set of weights is of the form

$$(\ell - j, \ell - j, \ell - j, \ell - j, \ell - j, \ell - j).$$

Fakhruddin's formulas for the first Chern class are recursive. The x coordinate can be computed by the formula

$$\sum_{i=0}^{\ell} \deg(\ell - j, \ell - j, \ell - j, i)\, \mathrm{rank}(\ell - j, \ell - j, \ell - j, i)$$

and the y coordinate can be computed by the formula

$$\sum_{i=0}^{\ell}\sum_{k=0}^{\ell} \deg(\ell - j, \ell - j, i, k)\, \mathrm{rank}(\ell - j, \ell - j, i)\, \mathrm{rank}(\ell - j, \ell - j, k),$$

where

$$\deg(a,b,c,d) = \mathrm{rank}(a,b,c,d) \max\left\{0, \frac{a+b+c+d-2\ell}{2}\right\}.$$

These formulas are programmed in Swinarski's `ConformalBlocks` package for `Macaulay2` [2,3]. We used them to obtain first Chern classes for many $\mathfrak{sl}_2$ conformal blocks for small levels, but for large levels, these recursive formulas are too slow.

We predict that the x and y coordinates of the first Chern class are given by piecewise polynomials of degree less than or equal to four in the variables ℓ and j. We interpolated the data produced by our program for small levels to find these polynomials.

As an example, here are some of the formulas for the x coordinates that we obtained by interpolation.

Conjecture. *If j is even, the x coordinate of the first Chern class of the $\mathfrak{sl}_2$ conformal block with weights $(\ell - j, \ell - j, \ell - j, \ell - j, \ell - j, \ell - j)$ is given by the formulas below.*

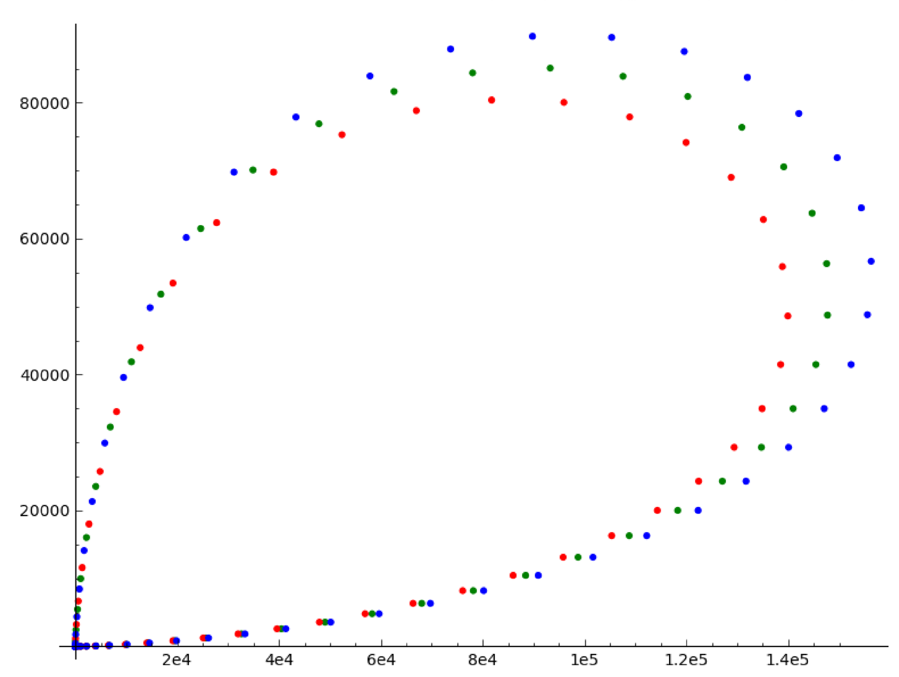
Case	*x coordinate formula*
If $j \le \frac{1}{3}\ell$	$-\frac{17}{16}j^4 + \frac{1}{2}j^3l - \frac{13}{4}j^3 + \frac{3}{2}j^2l - \frac{35}{8}j^2 + 2jl - \frac{9}{4}j + l$
If $\frac{1}{3}\ell < j \le \frac{1}{2}\ell$	$\frac{91}{16}j^4 - \frac{17}{2}j^3l + \frac{9}{2}j^2l^2 - jl^3 + \frac{1}{12}l^4 + \frac{23}{4}j^3 - \frac{15}{2}j^2l + 3jl^2$ $-\frac{1}{3}l^3 - \frac{5}{8}j^2 - \frac{1}{2}jl + \frac{5}{12}l^2 - \frac{7}{4}j + \frac{5}{6}l$
If $\frac{1}{2}\ell < j \le \frac{2}{3}\ell$	$\frac{27}{16}j^4 - \frac{9}{2}j^3l + \frac{9}{2}j^2l^2 - 2jl^3 + \frac{1}{3}l^4 - \frac{9}{4}j^3$ $+\frac{9}{2}j^2l - 3jl^2 + \frac{2}{3}l^3 + \frac{3}{8}j^2 - \frac{1}{2}jl + \frac{1}{6}l^2 + \frac{1}{4}j - \frac{1}{6}l$
If $\frac{2}{3}\ell < j$	*0.*

We obtain similar formulas if j is odd, and for the y coodinate of the first Chern class. We have proven the first formula shown above, and believe that our techniques will allow us to prove the others as well.

The distribution of first Chern classes

These formulas allow us to compute first Chern classes for large levels for the first time. We discovered that the first Chern classes are distributed throughout the first quadrant in an interesting pattern. For each level, the first Chern classes form a 'leaf' with one corner at the origin:

First Chern classes for $\ell = 69$ (red), 70 (green), 71 (blue)



If the formulas we obtained by interpolation are correct, this suggests that a vector with nonnegative integer coordinates can only be a first Chern class of a conformal block if it lies on one of these leaves.

References

[1] Najmuddin Fakhruddin, *Chern classes of conformal blocks*, Compact moduli spaces and vector bundles, Contemp. Math., vol. 564, Amer. Math. Soc., Providence, RI, 2012, pp. 145–176.
[2] Dan Grayson and Mike Stillman, `Macaulay2`*: a software system for research in algebraic geometry* (2012). Version 1.4.
[3] David Swinarski, `ConformalBlocks`*: software for computing conformal block divisors in* `Macaulay2` (2011). Version 1.1.
[4] David Swinarski, *sl_2 conformal block divisors and the nef cone of $\bar{M}_{0,n}$*, 2011. arxiv:1107.5331.