FORMULAS FOR WEIGHT DIAGRAMS FOR THE LIE ALGEBRA $\mathfrak{sp}(4)$

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Introduction

Lie algebras are vector spaces and one of three components of conformal blocks, which are used in string theory in physics. Each Lie algebra has a rule for drawing weight diagrams, and these weight diagrams can be used to compute the fusion rules of conformal blocks. There are three simple Lie algebras whose weight diagrams are two-dimensional: $\mathfrak{sl}(3)$, $\mathfrak{sp}(4)$, and $\mathfrak{g}_2$. Their weight diagrams have differing amounts of symmetries (6-fold, 8-fold, or 12-fold symmetry, respectively). The Lie algebra $\mathfrak{sl}(3)$ has been studied extensively in previous work, and we sought to study $\mathfrak{sp}(4)$ next. We obtain piecewise polynomial formulas for the entries in weight diagrams, which are called multiplicities. Our approach is to split $\mathfrak{sp}(4)$ weight diagrams into different regions where we observe patterns in the multiplicities, and then to use these patterns to interpolate polynomial formulas for the multiplicities. We prove our formulas by showing that they match the description of the multiplicities given by a theorem of Dehy. Finally, we use our formulas to compute the fusion rules of $\mathfrak{sp}(4)$ conformal blocks.

Procedure

We divide the weight diagram into regions with differing patterns in the multiplicities. For each region, we claim that the formula for the multiplicity will be a polynomial of the form

$$k_1 u^2 + k_2 uv + k_3 v^2 + k_4 u + k_5 v + k_6$$

To find the coefficients $k_1, \ldots, k_6$, we select six points within the region and solve the resulting system of equations. A theorem by Dehy states that the multiplicity of $\mu = (u, v)$ in a weight diagram is equal to the number of integer points in a polytope that we call the "Dehy slice" [2]. For $\mathfrak{sp}(4)$, each Dehy slice is defined by up to eight inequalities. We show that the number of integer points within the region defined by these inequalities is equal to the proposed polynomial formula, thus proving its validity.

Regions of the weight diagram
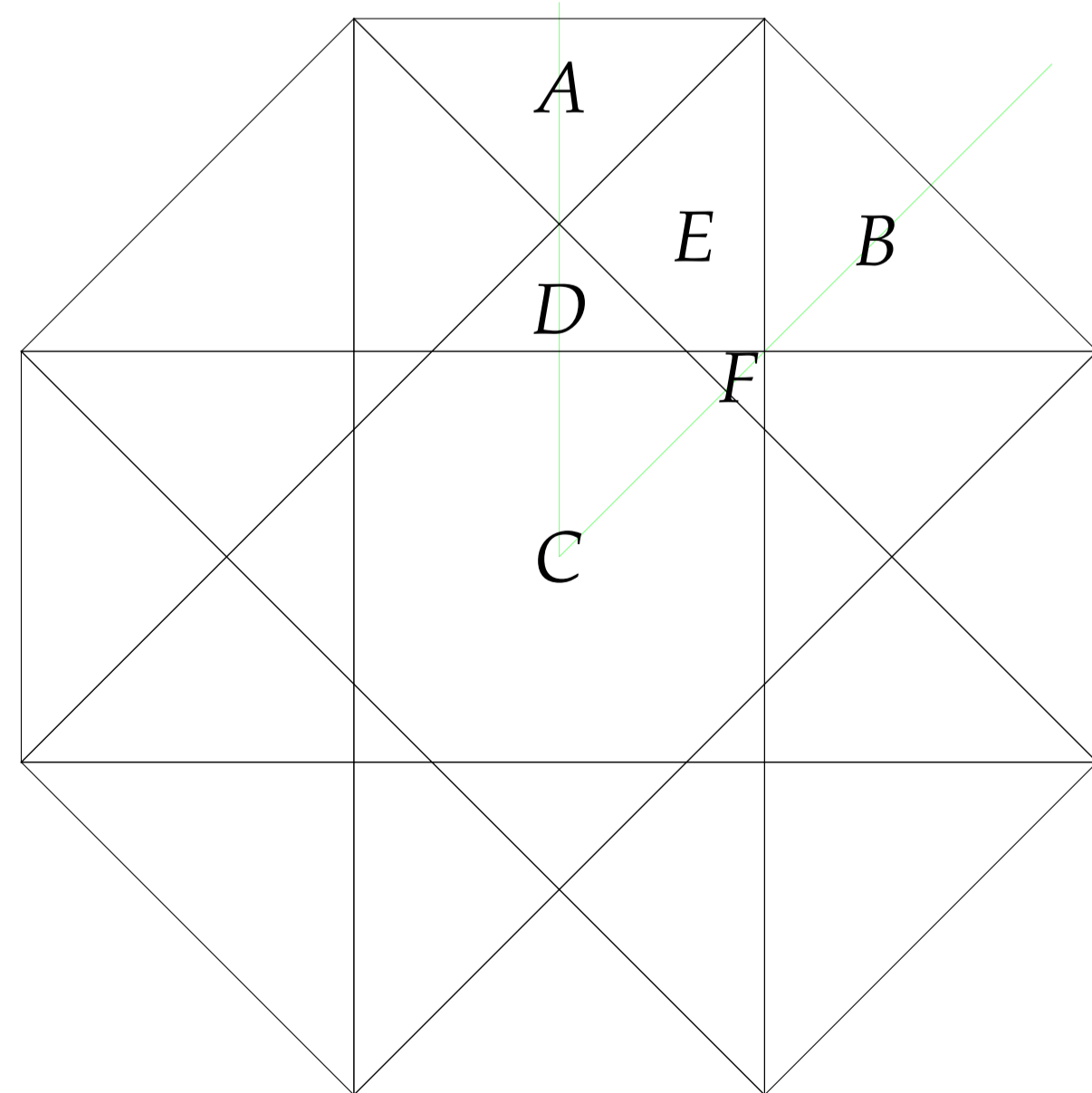
We describe nine basic regions $A - H$. These regions and their symmetric images cover the entire weight diagram. Not all regions appear in every weight diagram; for instance, G and H do not appear in the example above.

Weight diagram example
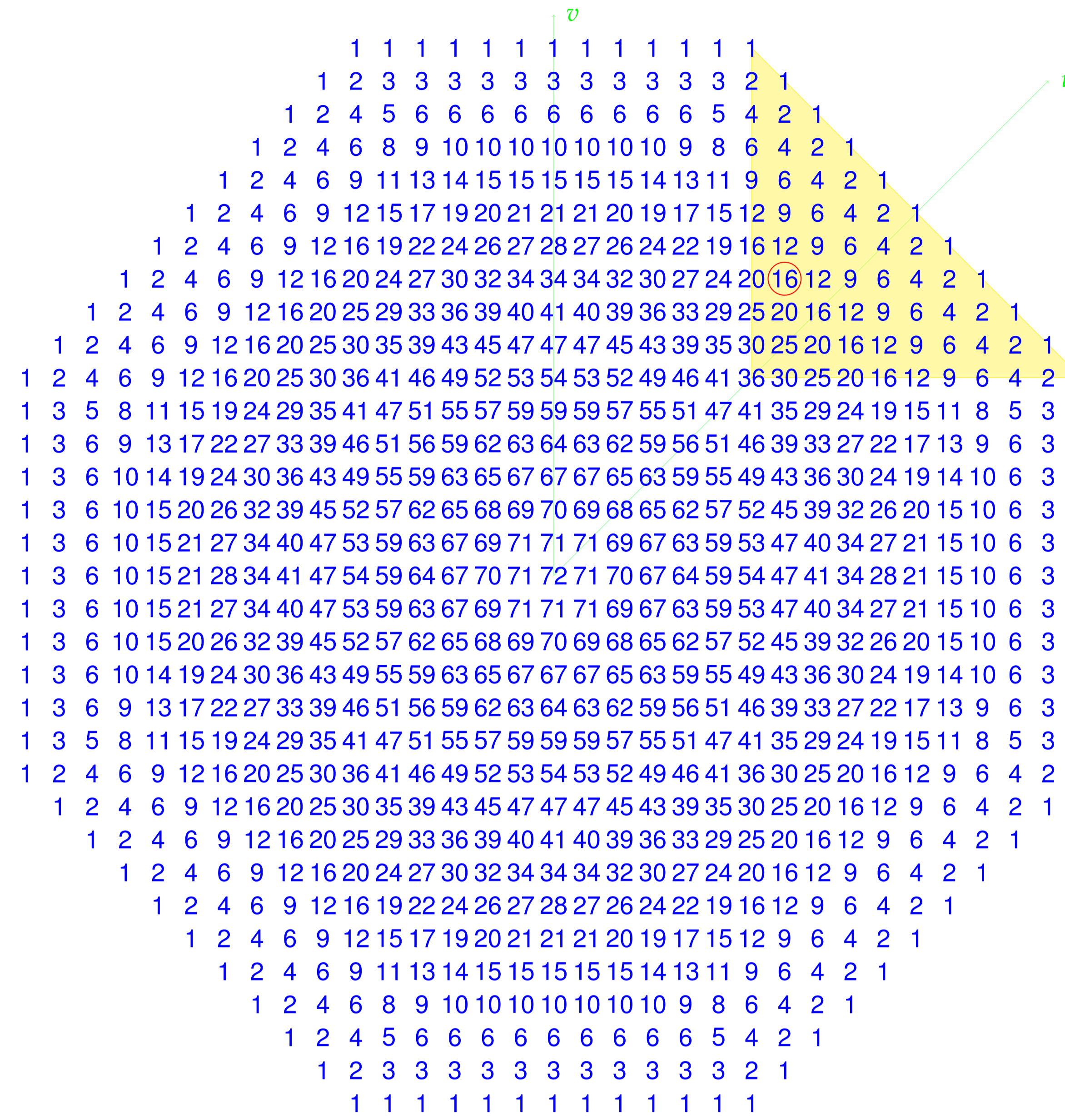
The coordinate system we use when drawing weight diagrams is not the usual x,y-coordinate system with horizontal and vertical axes. Instead, we use a diagonal axis and a vertical axis, labelled u and v instead of x and y.

Each weight diagram is determined by the position of its upper right corner. For the example above, the upper corner is placed at $(a, b) = (12, 10)$ in the u,v-coordinate system.

Once the position of the upper corner is given, Freudenthal's algorithm can be used to compute the multiplicities in the weight diagram [3].

Example: Region B

Consider a weight diagram whose upper right corner is positioned at (a, b). Starting from the outer edge on the upper right, and moving toward the center, we see a pattern: the multiplicities increase by either $+3, +5, +7, \ldots$ or $+4, +6, +8, \ldots$

This pattern holds inside the region highlighted in yellow. It is defined by the inequalities

$$a + b - u - v + 1 \geq 0 \quad -a + u + 2v \geq 0 \quad \quad -a + u \geq 0$$
$$b - v + 1 \geq 0 \quad \quad b + v + 1 \geq 0 \quad -b + u + v - 1 \geq 0$$

We interpolate using this pattern to find:

the multiplicity at a point (u, v) in Region B is given by the formula
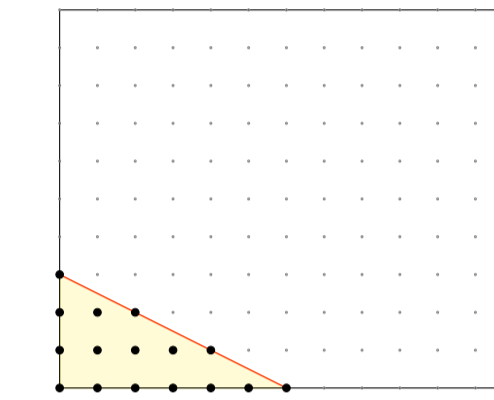
$$\frac{1}{4}u^2 + \frac{1}{2}uv + \frac{1}{4}v^2 - \frac{a+b+2}{2}(u+v) + \frac{(a+b)^2 + 4(a+b) + 4}{4}$$

if $v - b$ is even, and by the formula

$$\frac{1}{4}u^2 + \frac{1}{2}uv + \frac{1}{4}v^2 - \frac{a+b+2}{2}(u+v) + \frac{(a+b)^2 + 4(a+b) + 3}{4}$$

if $v - b$ is odd.

Next, we study Dehy slices. Here is an example of the Dehy slice for $(u, v) = (14, 2)$, which is circled in red in the weight diagram:

We can prove that when (u, v) is in Region B, the Dehy slice is a triangle with boundaries $x \geq 0$, $y \geq 0$, and $x + 2y \leq a + b - u - v$. To compute the number of integer points in this triangle, we compute the number of integer points in each horizontal row and sum.

The top vertex of the triangle is $(0, \frac{1}{2}(a + b - u - v))$. Write $n = \lfloor \frac{1}{2}(a + b - u - v) \rfloor$ for the number of rows of integer points. The number of integer points on the row of height k is $a + b - u - v - 2k + 1$. Therefore, the number of integer points in the triangle is

$$\sum_{k=0}^{n}(a + b - u - v - 2k + 1) = (n+1)(a + b - u - v + 1) - n(n+1).$$

Substituting the formula for n into the equation above, we find that the number of integer points in the Dehy slice equals our polynomial formula.

Similar proofs can be given for the other regions.

Application: fusion rules

Conformal blocks are a mathematical concept used in some versions of string theory. An algorithm due to Kac and Walton [3] allows us to compute the fusion rules of conformal blocks starting from weight diagrams.

For the Lie algebra $\mathfrak{sl}(3)$, Barker, Swinarski, Vogelstein, and Wu wrote a computer program that computes the fusion rules starting from a piecewise polynomial formula for the multiplicities in an $\mathfrak{sl}(3)$ weight diagram [1, 5]. We modified their program to work for $\mathfrak{sp}(4)$. Its input is our formula for the multiplicities, and it computes piecewise polynomial formulas for the fusion rules for $\mathfrak{sp}(4)$. To our knowledge, no closed formula for the fusion rules of $\mathfrak{sp}(4)$ conformal blocks has appeared in the literature before.

References

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[5] Dan Grayson and Mike Stillman, Macaulay2: *a software system for research in algebraic geometry* (2014). Version 1.7.