FORDHAM UNIVERSITY
THE JESUIT UNIVERSITY OF NEW YORK

Nonvanishing fusion rules for the Lie algebra $\mathfrak{g}_2$

John Wu ~ April 11, 2013

Introduction

Conformal blocks have many applications in mathematical physics, and much research has been done on the conformal blocks of the Lie algebra $\mathfrak{sl}_n$, especially for $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ ([6, Ch. 3]). An important part of this research is to describe *fusion rules* (defined below), which are used to compute a property of a conformal block called its rank. For the Lie algebra $\mathfrak{g}_2$ no formula for the fusion rules is known. One of the first steps toward finding such a formula is to find out when the fusion rules for $\mathfrak{g}_2$ are nonzero.

Components of conformal blocks

The components of a conformal block are a Lie algebra, an integer called a level, and a vector containing dominant integral weights.

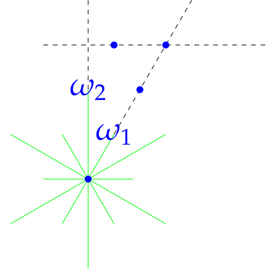
Lie algebras are named after the Norwegian mathematician Sophus Lie (pronounced “Lee”).

Definition. [5, Section 1.1] A vector space $\mathfrak{g}$ over the field $\mathbb{C}$ with an operation $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, denoted $(x, y) \mapsto [x, y]$, is called a *Lie algebra* if the following axioms are satisfied:

1. The bracket operation is bilinear.
2. $[x, x] = 0$ for all x in $\mathfrak{g}$.
3. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for $x, y, z \in \mathfrak{g}$.

Mathematicians have classified all the simple Lie algebras by their type $A - G$. There is only one simple Lie algebra of type G. We call it $\mathfrak{g}_2$ because its root system is two-dimensional. For the general definition of roots, weights, and the Weyl alcove, see [2, Section 5]. As an example, the Weyl alcove of $\mathfrak{g}_2$ at level 3 is shown below. It contains six integral weights: $(0,0), (1,0), (2,0), (3,0), (0,1), (1,1)$. (Here (a, b) represents $a\omega_1 + b\omega_2$.)

Weyl alcove of $\mathfrak{g}_2$ for level 3



We use the notation $\mathbb{V}(\mathfrak{g}, l, \vec{\lambda})$ for the conformal block for the Lie algebra $\mathfrak{g}$, level l, and weights $\vec{\lambda} = (\lambda_1, \ldots \lambda_n)$ (where each λ_i is in the Weyl alcove). The *fusion rules for the Lie algebra* $\mathfrak{g}$ *at level* l are the ranks when $\vec{\lambda}$ has three elements; that is, the set of numbers $\{(\lambda, \mu, \nu, \operatorname{rank} \mathbb{V}(\mathfrak{g}, l, (\lambda, \mu, \nu)))\}$.

The Kac-Walton algorithm

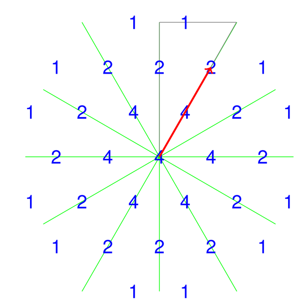
Several techniques to compute ranks have already been discovered. For any Lie algebra, any level, and $\vec{\lambda}$ with four or more elements, two formulas (*factorization* and *propagation*, see [6, Theorems 3.15 and 3.19]) can be used to reduce the number of elements to three. Also, the fusion rules for any $\vec{\lambda}$ with three elements containing any zero valued weights are known. For $\vec{\lambda}$ of three elements and with only nonzero weights, there is an algorithm, but no general formula for $\mathfrak{g}_2$. The Kac-Walton algorithm [1, Section 16.2.2] can be used to compute $\operatorname{rank} \mathbb{V}(\mathfrak{g}, l, (\lambda, \mu, \nu))$. The algorithm is as follows:

1. Generate the weight diagram of λ using Freudenthal’s algorithm.
2. Translate the weight diagram by μ.
3. Reflect the points now outside the Weyl alcove back into the Weyl alcove.

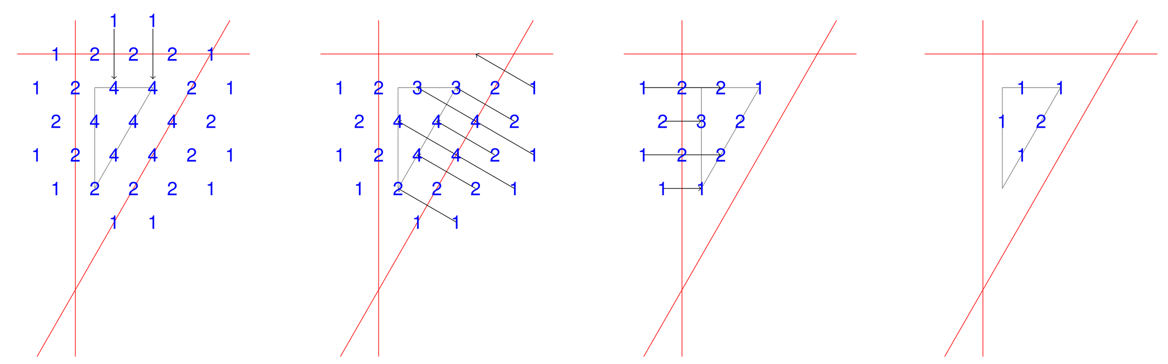
The number in position ν in the resulting diagram is $\operatorname{rank} \mathbb{V}(\mathfrak{g}, l, (\lambda, \mu, \nu))$. The algorithm can be used to compute any specific fusion rule; we would like to find a closed formula for the fusion rules.

Example

The series of figures below illustrates the Kac-Walton algorithm for $\lambda = (1, 1)$ and $\mu = (2, 0)$ at level 3. First we generate the weight diagram for $\lambda = (1, 1)$, and translate by the vector $\mu = (2, 0)$(represented by a red arrow).



The next three pictures illustrate the reflections across all three sides of the Weyl alcove. The ranks can be read from the final picture.



From this we read that

$$\operatorname{rank} \mathbb{V}(\mathfrak{g}_2, 3, ((1,1), (2,0), (1,0)) = 1$$
$$\operatorname{rank} \mathbb{V}(\mathfrak{g}_2, 3, ((1,1), (2,0), (2,0)) = 2$$
$$\operatorname{rank} \mathbb{V}(\mathfrak{g}_2, 3, ((1,1), (2,0), (3,0)) = 1$$
$$\operatorname{rank} \mathbb{V}(\mathfrak{g}_2, 3, ((1,1), (2,0), (0,1)) = 1$$
$$\operatorname{rank} \mathbb{V}(\mathfrak{g}_2, 3, ((1,1), (2,0), (1,1)) = 1$$
$$\text{and } \operatorname{rank} \mathbb{V}(\mathfrak{g}_2, 3, ((1,1), (2,0), \nu)) = 0$$
$$\text{for any weight } \nu \notin \{(1,0), (2,0), (3,0), (0,1), (1,1)\}.$$

Current work

In order to find a more general formula for the fusion rules, I used `Macaulay2`, a computer algebra system, to compute for each $1 \leq l \leq 13$ which conformal blocks $\mathbb{V}(\mathfrak{g}_2, l, (\lambda, \mu, \nu))$ have nonzero ranks. I then used a second software package, `polymake`, to compute linear inequalities that ensure $\operatorname{rank} \mathbb{V}(\mathfrak{g}_2, l, (\lambda, \mu, \nu,)) \neq 0$ for each l.

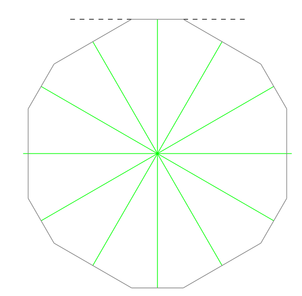
Below is a table with the number of inequalities that are associated with nonzero ranks at each level.

Level	Inequalities	Level	Inequalities	Level	Inequalities	Level	Inequalities
1	6	4	58	7	69	10	70
2	25	5	63	8	57	11	69
3	36	6	61	9	60	12	48
						13	69

Surprisingly, in contrast with the fusion rules for $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$, the number of inequalities for $\mathfrak{g}_2$ is different for different levels. Furthermore, there seem to be inequalties that are present at every level and inequalities that are dependent on a level.

One of the inequalities that I found to be independent of the level was $0 \leq a + 2b + c + 2d - e - 2f$. Here $\lambda = (a, b), \mu = (b, c)$, and $\nu = (e, f)$.

Now let us look at the outline of the weight diagram of λ.



The top boundary line connects the points $(-a, a+b)$ and (a, b). When we translate the weight diagram by μ, these points become $(-a+c, a+b+d)$ and $(a+c, b+d)$. The equation of the line through these points is $x + 2y = a + 2b + c + 2d$, and the inequality $x + 2y \leq a + 2b + c + 2d$ describes the region underneath this line. Therefore, $\nu = (e, f)$ must satisfy $e + 2f \leq a + 2b + c + 2d$ in order to be in the translated weight diagram. By the Kac-Walton algorithm, weights ν outside the translated weight diagram always give $\operatorname{rank} \mathbb{V}(\mathfrak{g}, \ell, (\lambda, \mu, \nu)) = 0$. Similar calculations for the other sides of the boundary of the translated weight diagram give all of the level-independent inequalities.

In the future I hope to be able to explain the level-dependent inequalties and eventually find a general formula that describes the fusion rules for $\mathfrak{g}_2$.

References

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[5] James E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, vol. 9, Springer-Verlag, New York, 1978. Second printing, revised.

[6] Kenji Ueno, *Conformal field theory with gauge symmetry*, Fields Institute Monographs, vol. 24, American Mathematical Society, Providence, RI, 2008.