The game of craps

We first recall here how the game of craps is played, ignoring secondary aspects that affect only betting on the game and not its outcome. Then we explain how the probability of winning it may be calculated, not because this is essential to the main theme of the paper, but because the arguments provide a very pretty application of basic ideas of finite probability to an infinite sample space and are increasingly rarely covered in contemporary probability courses.

The game is played with two standard cubical dice with faces numbered from 1 to 6. There are two stages, in both of which, the outcome of play is determined by the total of the numbers showing on the two dice. In the first stage, consisting of single *comeout* roll t, the player wins by throwing a 7 or 11 or *natural* and loses by throwing a 2, 3 or 12 or *crap*, when the name of the game. Rolls of 4–6 and 8–10 lead to a second stage in which this first roll becomes the players *point*. In this second stage, the player wins by re-rolling the comeout point t, loses by rolling a 7 and rolls again in all other cases. So the second stage contains outcomes involving any finite number of rolls.

Table 1 Probabilities arising in finding the chance of winning at craps.

Total t on first roll	2	3	4	5	6	7	8	9	10	11	12
$\mathbb{P}(t)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
$\mathbb{P}(w\|t)$	0	0	$\frac{3}{9}$	$\frac{4}{10}$	$\frac{5}{11}$	1	$\frac{5}{11}$	$\frac{4}{10}$	$\frac{3}{9}$	1	0
$\mathbb{P}(t \cap w)$	0	0	$\frac{9}{324}$	$\frac{16}{360}$	$\frac{25}{396}$	$\frac{6}{36}$	$\frac{25}{396}$	$\frac{16}{360}$	$\frac{9}{324}$	$\frac{2}{36}$	0

Table 1 summarizes the ingredients that go into finding the probability of a win for the player, an event that we denote w. Since the different totals are mutually exclusive, the probability that the player wins at craps can be computed from the table as

$$\mathbb{P}(w) = \sum_{t=2}^{12} \mathbb{P}(t \cap w) = \frac{244}{495} \simeq 0.4929\,.$$

How can we check the values in the table? Those in the first row are standard counts. Given the total t, the number i on the first die determines that on the second and i must be between 1 and $t-1$ if $t \leq 7$ and between $t-6$ and 6 if $t \geq 7$. Given the numbers in the second row, those in the third follow by applying the Intersection Formula for probabilities, $\mathbb{P}(t \cap w) = \mathbb{P}(t) \cdot \mathbb{P}(w|t)$.

The conditional probabilities in the second row are more interesting. They can be computed from a tree diagram, but this diagram has the feature, seldom found in the examples treated in finite probability courses, of being infinite. An initial segment of the branch of this tree, starting from the node on the left where a comeout point of $t = 9$ has just been rolled, is shown in Figure 2. The tree branches up to a winning leaf (shaded black) on a roll of 9, down to a losing leaf (shaded gray) on a roll of 7 and across on any other role. Each edge is labeled with the probability of following it from its left endpoint, and each leaf is labeled with the probability of reaching it from the root of the tree.

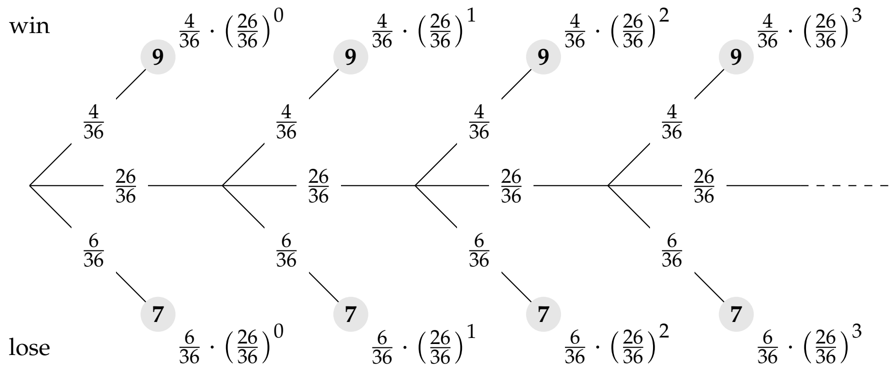

Figure 2 Tree diagram for the game of craps *after* a comeout roll of 9.

The tree makes visible a formula for $\mathbb{P}(w|9)$ as the sum $\frac{a}{1-r} = \frac{4}{10}$ of the geometric series with initial term $a = \frac{4}{36}$ and ratio $r = \frac{26}{36}$.

But there is a much easier way to see obtain this value from the tree, by noticing that we move to a leaf on any outcome in the event $(7 \text{ or } 9)$ and win when this outcome is a 9. Hence $\mathbb{P}(w|9) = \mathbb{P}\big(9|\,(7 \text{ or } 9)\big) = \frac{\mathbb{P}(9)}{\mathbb{P}(7 \text{ or } 9)}$ which the first row of Table 1 gives immediately as $\frac{4}{10}$. The other non-trivial entries in the second row of the table then follow analogously from the first row.

Exercise 3. The reader who wonders how many times the dice are rolled in a typical game of craps is invited to work this exercise.

(1) A Bernoulli trial with probability of success p and of failure $q = 1 - p$ is repeated until the first success is observed. Show that the expected number n_p of trials required is $n_p = pm_q$ where $m_q := \sum_{i=0}^{\infty}(i+1)q^i$.
(2) By considering $m_q - qm_q$, show that $m_q = \frac{1}{(1-q)^2} = \frac{1}{p^2}$ and that $n_p = \frac{1}{p}$.
(3) If, on the comeout roll, a point of t is made, play continues until the first t or 7 is rolled. Apply (2) to show that the expected number of subsequent rolls is $\frac{36}{9}$ for a point of 4 or 10, $\frac{36}{10}$ for a 5 or 9, and $\frac{36}{11}$ for a 6 or 8.
(4) Conclude that a typical game of craps lasts $\frac{557}{165} \simeq 3.38$ rolls.