The Mukai model of $M_7$

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ZAG Marathon
Introductory comments

Joint work with:

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Han-Bom Moon (Fordham)

as part of AIM Square “Computational aspects of GIT with a view of moduli spaces”
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Preliminary results: two examples (Fricke-Macbeath curve, and the 7-cuspidal curve with dihedral symmetry) are $T$-semistable in Mukai’s model of $M_7$ for a maximal torus $T$ in $SO(10)$
Outline

1. Mukai’s construction
2. Background on GIT
3. Three examples and three strategies
4. Future work
Moduli spaces of curves

$M_g$: moduli space of smooth genus $g$ curves/isomorphism

$M_g$ is not compact. Deligne-Mumford compactification:

$$\overline{M}_g = \left\{ \text{connected projective genus } g \text{ curves,} \right. \quad \text{only nodes as singularities,} \quad \left. \text{finite automorphism group} \right\} / \text{isomorphism}$$

Problem

*Understand the birational geometry of $\overline{M}_g$*

Today we'll approach this by constructing different compactifications of $M_g$. 
Gieseker’s construction of $\overline{M}_g$

Let $C$ be a smooth curve of genus $g$.

Fix a positive integer $\nu \geq 2$. Consider the embedding $C \to \mathbb{P}^N$ associated to the pluricanonical linear system $|\nu K|$.

Then $C$ is represented by a $\text{PGL}(N + 1)$ orbit in $\text{Hilb}(\mathbb{P}^N, dt - g + 1)$.

Idea: find the locus $J_{g,\nu} \subset \text{Hilb}$ parametrizing pluricanonically embedded smooth curves $C$.

Then, set-theoretically, we expect $J_{g,\nu}/\text{SL}(N + 1) = M_g$.

$$\text{Hilb} \quad \to \quad \text{Gr}(Q(m), \binom{N+m}{m})$$

$X \subset \mathbb{P}^N \quad \leftrightarrow \quad l_m \hookrightarrow S_m$

When $g = 7, \nu = 1, m = 2$ get $\text{Gr}(10, 28)$
Mukai’s model of $\overline{M}_7$

For a general curve $g \geq 3$, the canonical ideal is generated by $\binom{g-2}{2}$ quadrics. For $g = 7$ need 10 quadrics in $\mathbb{P}^6$.

Mukai: for a general smooth genus 7 curve (no $g_2^1$, $g_3^1$, or $g_4^1$),

$$\text{Sym}^2(l_2) \rightarrow l_4$$

has one-dimensional kernel. Let $Q$ be a generator.

$\Rightarrow (l_2, Q)$ is a quadratic vector space.

Furthermore for each $p \in C$, the row space of the Jacobian matrix evaluated at $p$

$$\left[ \frac{\partial f_j}{\partial x_i}(p) \right]_{j=1,\ldots,10}^{i=0,\ldots,6}$$

is a Lagrangian of $(l_2, Q)$, denoted $W_p^\perp$. 
Mukai’s model of $\overline{M}_7$, continued

Theorem (Mukai, 1995)

- When $C$ is general,
  \[
  C \rightarrow \text{OG}(5, 10) \rightarrow \mathbb{P}^{15} \\
  p \mapsto W_p^\perp
  \]
  is an embedding of $C$

- Its image is the intersection $(P \cap \text{OG}(5, 10))$ where $P$ is a 6-dimensional linear subspace of $\mathbb{P}^{15}$

- Let $U \subseteq \text{Gr}(9, 16)$ be the set of such $P$ that intersect $\text{OG}(5, 10)$ transversally. Then
  \[
  U / \text{SO}(10) \rightarrow \overline{M}_7.
  \]
ABC’s of GIT

- Geometric invariant theory (GIT) quotients require two ingredients:
  1. a *parameter space* \(X\) with group action
  2. a *linearization* of the group action (a lifting of the group action to a line bundle \(L\))

- The GIT quotient scheme, denoted \(X//_L G\), is \(\text{Proj}(R)\), where \(R := \bigoplus_{n \in \mathbb{N}} H^0(X, L^\otimes n)^G\), the ring of invariant sections.

- There is a rational map from \(X\) to \(X//_L G\).

- Points \(x \in X\) for which there exists a nonvanishing invariant section are called *semistable*; the quotient map is actually a morphism at such points.
The Hilbert-Mumford numerical criterion

\[ x \text{ is } G\text{-semistable } \iff \text{ } x \text{ is } \lambda\text{-semistable for every } 1\text{-PS } \lambda : \mathbb{G}_m \to G. \]

- There is a beautiful way to picture \( T\)-semistability for a maximal torus \( T \subset G \).
- It is sometimes possible to compute \( T\)-semistability and then extend it to \( G\)-semistability.
Hilbert-Mumford numerical criterion

Setup for this section:
- \( T \): a torus
- \( W \): a finite-dimensional representation of \( T \)
- \( S \): set of weights of \( T \) on \( W \)

\[
T \text{ abelian} \Rightarrow W = \bigoplus_{\chi \in S} W_{\chi}
\]

\[
w = \bigoplus w_{\chi}
\]

Hilbert-Mumford numerical criterion:
1. \( w \in W \) is \( T \)-stable \( \iff \) \( w \) is \( \lambda \)-stable for every 1-PS \( \lambda : \mathbb{G}_m \hookrightarrow T \)

2. \( w \in W \) is \( \lambda \)-stable \( \iff \) \( \exists \chi \in S \) such that \( \langle \lambda, \chi \rangle < 0 \) and \( w_{\chi} \neq 0 \)
**Definition**

A *state* is a subset $\Xi \subseteq S$.

The *state of* $w \in W$ is $\text{St}(w) = \{ \chi : w_\chi \neq 0 \}$

- **Stable**: $\chi_0$ in interior of convex hull
- **Strictly semistable**: $\chi_0$ on boundary of convex hull
- **Nonsemistable**: $\chi_0$ outside convex hull
Definition

A state is a subset $\Xi \subseteq S$. The state of $w \in W$ is $\text{St}(w) = \{\chi : w_\chi \neq 0\}$.
The instability fan

Definition

1. For any 1-PS $\lambda$, let

$$\text{St}(\lambda) = \{ \chi \in S : \langle \lambda, \chi \rangle \geq 0 \}.$$ 

2. For any state $\Xi \subseteq S$, let

$$C(\Xi) = \{ \lambda : \langle \lambda, \chi \rangle \geq 0 \forall \chi \in \Xi, \langle \lambda, \chi \rangle \leq 0 \forall \chi \in S \setminus \Xi \}$$

Lemma

1. $C(\Xi)$ is a closed, finitely generated polyhedral cone.

2. The cones $C(\text{St}(\lambda))$ form a fan as $\lambda$ varies (I call this the instability fan).
Example: Pencils of plane quadrics

Example point coordinates:

- $(1, 0, -1)$
- $(2, 1, -3)$
- $(1, 2, -3)$
- $(3, -1, -2)$
- $(3, -2, -1)$
Two algorithms to compute the instability fan

Algorithm 1: Breadthfirst search
1. Choose a random 1-PS $\lambda$ and the cone $C$ of its state
2. Compute the change in state as you cross each facet of $C$. Use these new states to compute the cones that are neighbors of $C$.
3. Compute the neighbors of all cones obtained this way until all neighbors of all cones have been enumerated.

Algorithm 2: (only gives the rays of the instability fan)
1. Let $r = \text{rank } T$. Test each subset of $r - 1$ characters in $S$ to see if the associated hyperplanes are linearly independent. If so, compute a complementary ray.

Proposition (G-MG-M-S.)

*Up to Weyl group symmetry, the instability fan for Mukai’s model has 4,875,339 rays.*
From $T$-semistability to $G$-semistability

Approach 1: Find a coordinate-free geometric interpretation of each maximal non-semistable state. (See the plane cubic example in Harris and Morrison, *Moduli of Curves*.)

Approach 2: use results of Kempf
Kempf’s results

Theorem (Kempf, 1978)

- When $x$ is unstable, there exists a “worst 1-PS” $\lambda$
- There exists a proper parabolic subgroup associated to $\lambda$:

$$P(\lambda) = \{ g \in G : \lim_{t \to 0} (\lambda(t) \cdot g \cdot \lambda^{-1}(t)) \text{ exists in } G \}$$

- Each maximal torus $T$ in $P(\lambda)$ contains one worst 1-PS
- $P(\lambda)$ contains $\text{Stab}(x)$.

Corollary (Kempf): if $\text{Stab}(x)$ is not contained in any proper parabolic, then $x$ is $G$-semistable.

Morrison-Swinarski: Determine $G$-semistability by describing the parabolic(s) that contain $\text{Stab}(x)$, and computing $T$-semistability for a maximal torus in each such parabolic subgroup.
Breuer and Conder: computer searches listing groups $G$ such that $G \subseteq \text{Aut}(C)$ for some curve of genus $g$, for $g = 2, \ldots, 101$

Magaard, Shasta, Shpectorov, and Völklein: list $\text{Aut}(C)$ for each $g$, for $g = 2, \ldots, 10$, when $|\text{Aut}(C)| > 4(g - 1)$.

Swinarski (2018): Equations of curves with large automorphism groups for $4 \leq g \leq 7$
Genus 7 isolated curves with large automorphism groups

For \( g = 7 \) there are 13 isolated curves with large automorphism groups.

<table>
<thead>
<tr>
<th>Group ( (504,156) )</th>
<th>Signature ( (2,3,7) )</th>
<th>Gonality ( g_1^4 )</th>
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<tbody>
<tr>
<td>( (64,41) )</td>
<td>( (2,4,16) )</td>
<td>( g_1^4 )</td>
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<tr>
<td>( (56,4) )</td>
<td>( (2,4,28) )</td>
<td>( g_2^2 )</td>
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<td>( (54,6) )</td>
<td>( (2,6,9) )</td>
<td>( g_1^2 )</td>
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<td>( (48,32) )</td>
<td>( (3,4,6) )</td>
<td>( g_1^4 )</td>
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<td>( (32,11) )</td>
<td>( (4,4,8) )</td>
<td>( g_1^4 )</td>
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<tr>
<td>( (30,4) )</td>
<td>( (2,15,30) )</td>
<td>( g_2^2 )</td>
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<tr>
<th>Group ( (144,127) )</th>
<th>Signature ( (2,3,12) )</th>
<th>Gonality ( g_1^4 )</th>
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<tr>
<td>( (64,38) )</td>
<td>( (2,4,16) )</td>
<td>( g_1^2 )</td>
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<tr>
<td>( (54,6) )</td>
<td>( (2,6,9) )</td>
<td>( g_1^3 )</td>
</tr>
<tr>
<td>( (54,3) )</td>
<td>( (2,6,9) )</td>
<td>( g_1^3 )</td>
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<tr>
<td>( (42,4) )</td>
<td>( (2,6,21) )</td>
<td>( g_1^3 )</td>
</tr>
<tr>
<td>( (32,10) )</td>
<td>( (4,4,8) )</td>
<td>( g_1^4 )</td>
</tr>
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</table>

There are also eight 1-dimensional families of curves with large automorphism groups. None of them are general, either.
Example 1: The Fricke-Macbeath curve

\[
\begin{align*}
    f_1 &= x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2, \\
    f_2 &= \sum_{i=0}^{7} \zeta_7^i x_i^2, \\
    f_3 &= \sum_{i=0}^{7} \zeta_7^{-i} x_i^2, \\
    f_4 &= -a_3 x_0 x_6 + a_2 x_1 x_4 + a_1 x_3 x_5, \\
    f_5 &= -a_3 x_1 x_0 + a_2 x_2 x_5 + a_1 x_4 x_6, \\
    f_6 &= -a_3 x_2 x_1 + a_2 x_3 x_6 + a_1 x_5 x_0, \\
    f_7 &= -a_3 x_3 x_2 + a_2 x_4 x_0 + a_1 x_6 x_1, \\
    f_8 &= -a_3 x_4 x_3 + a_2 x_5 x_1 + a_1 x_0 x_2, \\
    f_9 &= -a_3 x_5 x_4 + a_2 x_6 x_2 + a_1 x_1 x_3, \\
    f_{10} &= -a_3 x_6 x_5 + a_2 x_0 x_3 + a_1 x_2 x_4,
\end{align*}
\]

where \(a_j = \zeta_7^j - \zeta_7^{-j}\). Here \(\text{Aut}(C) \cong \text{PSL}(2, 8)\)
Finding the Fricke-Macbeath curve in Mukai’s model

- Compute \( \ker(\text{Sym}^2 I_2 \to I_4) \). (Almost the standard quadratic form \( q_1^2 + \cdots + q_{10}^2 \), easily diagonalized).

- Compute the induced action of \( \text{Aut}(C) \) on quadrics. Naturally get a subgroup of \( \text{SO}(10) \).

- Choose a pair of complementary Lagrangians \( U_0 \) and \( U_\infty \) in \( I_2 \) and compute the action of \( \text{Aut}(C) \) on \( U_0 \oplus U_\infty \).
\( U_0 \) and \( U_\infty \) for different \( Q \)

- When working with \( Q = q_1^2 + \cdots + q_{10}^2 \): choose

\[
U_0 = \frac{1}{2} \begin{bmatrix}
1, 0, 0, 0, 0, -i, 0, 0, 0, 0 \\
0, 1, 0, 0, 0, 0, -i, 0, 0, 0 \\
0, 0, 1, 0, 0, 0, 0, -i, 0, 0 \\
0, 0, 0, 1, 0, 0, 0, 0, -i, 0 \\
0, 0, 0, 0, 1, 0, 0, 0, 0, -i
\end{bmatrix}
\]

and \( U_\infty = \overline{U}_0 \).

- When working with \( Q = q_1 q_6 + \cdots + q_5 q_{10} \), the first five and last five rows of the identity matrix span complementary Lagrangians.
Finding the Fricke-Macbeath curve in Mukai’s model

- Compute ker(Sym$^2 I_2 \rightarrow I_4$). (Almost the standard quadratic form $q_1^2 + \cdots + q_{10}^2$, easily diagonalized).
- Compute the induced action of Aut$(C)$ on quadrics. Naturally get a subgroup of SO(10).
- Choose a pair of complementary Lagrangians $U_0$ and $U_\infty$ in $I_2$ and compute the action of Aut$(C)$ on $U_0 \oplus U_\infty$.
- Compute the spin representation of Aut$(C)$ on $S^+ = \bigwedge^{\text{even}} U_\infty$.
- Have a linear representation of a double cover of Aut$(C)$ (or a projective representation of Aut$(C)$). It splits! Can find Aut$(C) \subset GL(16)$.
- This representation is the direct sum of two irreducible representations, of dimensions 9 and 7. Projecting to the 7-dimensional representation gives the Fricke-Macbeath curve. (Mukai, 1992)
- The 9 hyperplanes we seek are given by the kernel of the projection matrix.
Theorem (S., 2020)

The Fricke-Macbeath curve is $T$-semistable for an explicit maximal torus $T$.

Question: Can we extend $T$-semistability to SO(10)-semistability here?
Example 2: the 7-cuspidal curve with dihedral symmetry

Let $C$ be the rational cuspidal curve with cusps at the 7th roots of unity.

- Obtain a parametrization and equations of $C$ by taking the hyperplane section $x_0 = x_7$ of the tangent developable of the rational normal curve.
- Compute $\ker(\text{Sym}^2 I_2 \rightarrow I_4)$ and diagonalize the resulting quadratic form.
- Choose $U_0$ and $U_\infty$.
- Produce 7 linearly independent nonsingular points $p_i$ on $C$ such that $\dim(W_{p_i}^\perp \cap U_\infty) = 0$.
- Compute the spinors $s_1, \ldots, s_7$ of the points $p_1, \ldots, p_7$.
- Compute the hyperplanes orthogonal to $\text{Span}(s_1, \ldots, s_7)$.

Theorem (S., 2020)

The 7-cuspidal curve with dihedral symmetry is $T$-semistable for an explicit maximal torus $T$. 
What goes wrong for special curves?

Genus 7 curve with 144 automorphisms: tetragonal and has $g_6^2$.

$$\text{ker}(\text{Sym}^2 I_2 \rightarrow I_4)$$

is three-dimensional, and every quadratic form in it is degenerate.
The balanced ribbon

The *balanced ribbon* of genus 7 is a nonreduced dimension 1 scheme supported on $\mathbb{P}^1$.

Equations:

\[
\begin{align*}
    x_3^2 - 2x_2x_4 + x_1x_5, & \quad x_2x_4 - 2x_1x_5 + x_0x_6, \\
    x_2x_3 - 2x_1x_4 + x_0x_5, & \quad x_3x_4 - 2x_2x_5 + x_1x_6, \\
    -x_1^2 + x_0x_2, & \quad -x_1x_2 + x_0x_3, & \quad -x_2^2 + x_1x_3, \\
    -x_4^2 + x_3x_5, & \quad -x_4x_5 + x_3x_6, & \quad -x_5^2 + x_4x_6
\end{align*}
\]

Automorphisms: $\mathbb{G}_m$ acting as $(t^{-3}, t^{-2}, t^{-1}, 1, t, t^2, t^3)$ and $(x_0, \ldots, x_6) \rightarrow (x_6, \ldots, x_0)$

Question: does it appear in Mukai’s model?
Balanced ribbon

- The balanced ribbon has the Betti table of a general genus 7 curve
- \( \ker(\text{Sym}^2 I_2 \to I_4) \) is one-dimensional, generated by a full rank quadric
- The second strategy won’t work—there are no nonsingular points
  - \( W_p^\perp \) has dimension 4, not 5
  - as \( p \) varies, these four-dimensional spaces have pairwise dim 1 intersection
- The first strategy gets farther than I expected—get 5 of the 9 hyperplanes this way
Third strategy/Future work

- Fix an ordering of the 28 degree 2 monomials in $x_0, \ldots, x_6$
- Every ideal in $x_0, \ldots, x_6$ generated by 10 quadrics has a unique representation: write each quadric as a row in this $10 \times 28$ matrix, and echelonize
- For any of the $\binom{16}{9}$ subsets of the variables on $\mathbb{P}^{15}$, invert that $9 \times 9$ minor, and write 7 unknown coefficients in each row for the remaining variables. Substitute into Mukai’s equations and echelonize. Set equal to the echelonized form of the desired ideal to obtain an affine algebraic variety. If nonempty, any solution yields a representative in Mukai’s model.
- Could potentially work for families of curves, too
Future work: geometric invariant theory

- Can we evaluate any $\text{SO}(10)$- or $\text{SL}(16)$-invariant polynomials on $x \in \bigwedge^9 \mathbb{C}^{16}$?

- If an invariant polynomial doesn’t vanish at $x$, then $x$ is $\text{SO}(10)$-semistable.