Singular curves in Mukai's model of $\overline{M}_7$

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Nearly 30 years later, very little is known about this space.

Mukai's model of $\overline{M}_7$

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Mukai: for a general smooth genus 7 curve (no g_2^1, g_3^1, or g_4^1),

$$\mathrm{Sym}^2(I_2) \to I_4$$

has one-dimensional kernel. Let Q be a generator.
$\Rightarrow (I_2, Q)$ is a quadratic vector space.

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For each (smooth) $p \in C$, the row space of the Jacobian matrix

$$\left[\frac{\partial f_j}{\partial x_i}(p) \right]_{i=0,\ldots,6}^{j=1,\ldots,10}$$

is a Lagrangian of (I_2, Q), denoted $W_p^\perp$

Mukai's model of $\overline{M}_7$

Theorem [Mukai, 1995]

▶ When C is a general smooth genus 7 curve,

$$\text{Spin}: \quad \begin{array}{ccccc} C & \to & OG(5,10) & \to & \mathbb{P}^{15} \\ p & \mapsto & W_p^\perp & & \end{array}$$

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- $\text{Spin}(C) = P \cap \text{OG}(5,10)$, where $P \cong \mathbb{P}^6 \subset \mathbb{P}^{15}$

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► $\text{Spin}(C) = P \cap \text{OG}(5,10)$, where $P \cong \mathbb{P}^6 \subset \mathbb{P}^{15}$

► The GIT quotient is a birational model of $\overline{M}_7$:

$$\text{Gr}(7,16) /\!\!/ \text{Spin}(10) \dashrightarrow \overline{M}_7.$$

Geometric invariant theory (GIT)

GIT is a technique to construct the quotient of a scheme by the action of an algebraic group.

X: projective scheme

L: ample line bundle on X

G: reductive group acting on X

$$X /\!/_L G := \mathrm{Proj}\left(\left(\bigoplus_{n \geq 0} H^0(X, L^{\otimes n}) \right)^G \right)$$

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Issue 1: the quotient map $X \dashrightarrow X /\!/_L G$ has a base point at $x \in X$ if all G-invariant sections vanish at x.

Semistability and stability

Definition
$x \in X$ is *semistable* if there exists $s \in H^0(X, L^{\otimes n})^G$ such that $s(x) \neq 0$.

Then there is a morphism

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Issue 2: The GIT quotient is not necessarily an orbit space: we have

$$x \sim y \Leftrightarrow \overline{G.x} \cap \overline{G.y} \neq \emptyset$$

Open questions

1. Which schemes occur as $P \cap \mathrm{OG}(5, 10)$ for $P \in \mathrm{Gr}(7, 16)$, and how do these spaces intersect?
2. What are the GIT semistable points?
3. Which orbits are identified in the GIT quotient?
4. What is its cohomology ring?
5. What is its homogeneous coordinate ring?
6. Describe the birational map $\mathrm{Gr}(7, 16) /\!/ \mathrm{Spin}(10) \dashrightarrow \overline{M}_7$
7. Is $\mathrm{Gr}(7, 16) /\!/ \mathrm{Spin}(10)$ the moduli space of some interesting moduli stack?

Outline of the talk

I. Explore three examples of singular curves:
  1. 7-cuspidal curve with heptagonal symmetry
  2. The genus 7 balanced ribbon
  3. A family of nodal curves

For each example, we'll discuss how to locate these examples in $Gr(7, 16)$

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II. Discuss $\mathrm{Spin}(10)$-semistability of these examples

Why these examples?

- Sought to study examples with pure Betti tables
  - Many such examples were proposed in the 1990s as part of efforts to study Green's Conjecture
- Have previously studied these examples in connection with the Hassett-Keel program (log minimal model program for $(\overline{M}_g, \Delta)$)

Example 1: the 7-cuspidal curve with heptagonal symmetry

Let C be the rational cuspidal curve with cusps at the 7th roots of unity.

- ▶ Obtain a parametrization and equations of C by taking the hyperplane section $x_0 = x_7$ of the tangent developable of the rational normal curve.

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- ▶ Produce 8 linearly independent nonsingular points p_i on C

- ▶ Compute the spinors s_i of the points p_i. Define $P = \mathrm{Span}(s_1, \ldots, s_7)$. Then P is the row span of the matrix shown below.

0	0	0	0	$-3/5$	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	$1/5$	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	$3/4$	1	0	0	0	0	0	0
30	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	$8/9$	1	0	0	0
0	-2	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	$-15/2$	0	0	0	0	0	0	0	0	0	0	0	0	1

Example 1: the 7-cuspidal curve with heptagonal symmetry

Is $P \cap \mathrm{OG}(5, 10) = C$?

- Let J be the ideal of $P \cap \mathrm{OG}(5, 10)$ to P.
- Construct an element of $F \in \mathrm{PGL}(7)$ mapping $s_i \mapsto p_i$
- Check that $I = F(J)$.

Example 1: the 7-cuspidal curve with heptagonal symmetry

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We'll see in Part II that it is $\mathrm{Spin}(10)$-semistable.

Example 2: The balanced ribbon in genus 7

A *ribbon* is a dimension 1, generically nonreduced scheme that is a double structure on the underlying reduced curve.

Ribbons are limits of the canonical models of smooth curves as they degenerate to a hyperelliptic curve.

Example 2: The balanced ribbon in genus 7

In each odd genus $g = 2k + 1$ with $g \geq 5$ there is a ribbon called the *balanced ribbon* with

- $\mathbb{G}_m$ action with weights $-k, \ldots, +k$
- involution interchanging the positive and negative weight spaces.

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- ▶ involution interchanging the positive and negative weight spaces.

Genus 7 balanced ribbon equations:

$$y_2 y_3 - 2y_1 y_4 + y_0 y_5,$$
$$y_2 y_4 - 2y_1 y_5 + y_0 y_6,$$
$$y_3^2 - 2y_2 y_4 + y_1 y_5,$$
$$y_3 y_4 - 2y_2 y_5 + y_1 y_6,$$
$$-y_1^2 + y_0 y_2,$$
$$-y_1 y_2 + y_0 y_3,$$
$$-y_2^2 + y_1 y_3,$$
$$-y_4^2 + y_3 y_5,$$
$$-y_4 y_5 + y_3 y_6,$$
$$-y_5^2 + y_4 y_6$$

Does the ribbon arise as $P \cap \mathrm{OG}(5, 10)$?

The ribbon has no smooth points—we cannot proceed as we did for the 7-cuspidal curve.

Use the automorphisms instead.

- Compute $\ker(\mathrm{Sym}^2 I_2 \to I_4)$ and "diagonalize" the resulting quadratic form.

Does the ribbon arise as $P \cap \mathrm{OG}(5, 10)$?

The ribbon has no smooth points—we cannot proceed as we did for the 7-cuspidal curve.

Use the automorphisms instead.

- ▶ Compute $\ker(\mathrm{Sym}^2 I_2 \to I_4)$ and "diagonalize" the resulting quadratic form.
- ▶ Compute the action of $\mathbb{G}_m$ and the involution on these new quadrics
- ▶ Lift these actions to $\mathrm{Spin}(10)$ and compute their actions on the half-spin representation S^+

Does the ribbon arise as $P \cap \mathrm{OG}(5, 10)$?

- The $\mathbb{G}_m$ action on S^+ has weights

$$-5, -4, -3, -2, -2, -1, -1, 0, 0, 1, 1, 2, 2, 3, 4, 5$$

- Know we need to kill the ± 4 and ± 5 weight spaces

- The involution acts trivially on the 0 weight space of the ribbon. So kill the part on which the involution acts nontrivially

- Next, we'll study the ± 1 and ± 2 weight spaces to get four more linear equations.

The ± 1 weight spaces

The ± 1 weight spaces form a multiplicity 2 module under the $\mathbb{G}_m \times Z_2$ action.

Choose bases of these two summands having the same $\mathbb{G}_m \times Z_2$ action:

	x_{13}	$\frac{1}{2}x_{24}$	x_{2345}	x_{15}
x_{13}	0	1	0	0
$\frac{1}{2}x_{24}$	1	0	0	0
x_{2345}	0	0	0	1
x_{15}	0	0	1	0

Then a multiplicity one submodule of this multiplicity two module can be written in the form $\mathrm{Span}\langle c_1 x_{13} + c_2 x_{2345}, \frac{1}{2}c_1 x_{24} + c_2 x_{15}\rangle$ for some constants c_1 and c_2.

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Assume $c_1 \neq 0$ and scale to make $c_1 = 1$.

Does the ribbon arise as $P \cap \mathrm{OG}(5, 10)$?

In a similar way, from the weight ± 2 spaces, we get two more linear equations, for a total of 9 linear equations.

These nine equations have two unknown parameters c_2 and c_4.

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In a similar way, from the weight ± 2 spaces, we get two more linear equations, for a total of 9 linear equations.

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Compute the intersection with $\mathrm{OG}(5, 10)$ and compare to the balanced ribbon equations. We're so close that we can guess values of c_2 and c_4 giving the balanced ribbon.

Example 2: The balanced ribbon in genus 7

Proposition (S., 2022)

The genus 7 balanced ribbon can be realized as $P \cap \mathrm{OG}(5, 10)$.

We'll see in Part II that it is $\mathrm{Spin}(10)$-semistable.

Graph curves (in the sense of Bayer-Eisenbud)

Bayer-Eisenbud: A graph curve C is a connected, projective algebraic curve which is a union of projective lines, each meeting exactly three others, transversely at distinct points.

- ▶ Such a curve is determined up to isomorphism by its dual graph
- ▶ Graph curves are the 0-strata of the boundary of $\overline{M}_g$.

Searching for genus 7 graph curves with pure Betti tables

[BE, Prop. 2.5]: A (trivalent) graph curve has very ample ω iff G is 3-edge-connected.

In `Sage`, I got a list of connected genus 7 trivalent graphs (85 of these).
I selected the 3-edge-connected ones (57 of these).

I computed the Betti tables of these graph curves in `Macaulay2` and found two genus 7 graph curves with pure Betti tables.

I selected the one that had the larger automorphism group for further study.

A genus 7 trivalent graph with pure Betti table (first view)
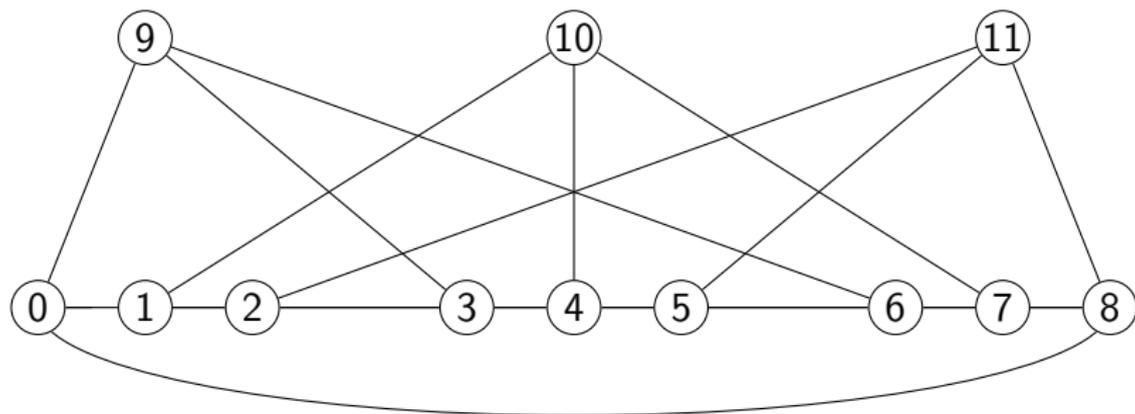
A genus 7 trivalent graph with pure Betti table (second view)
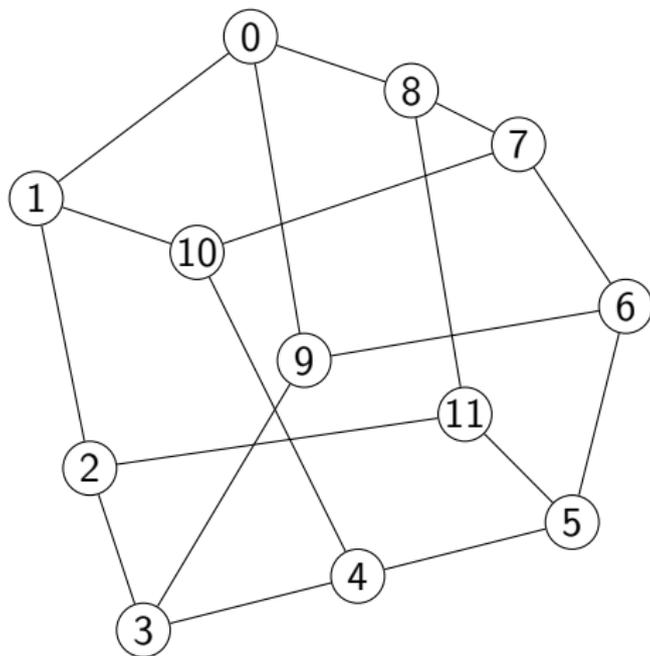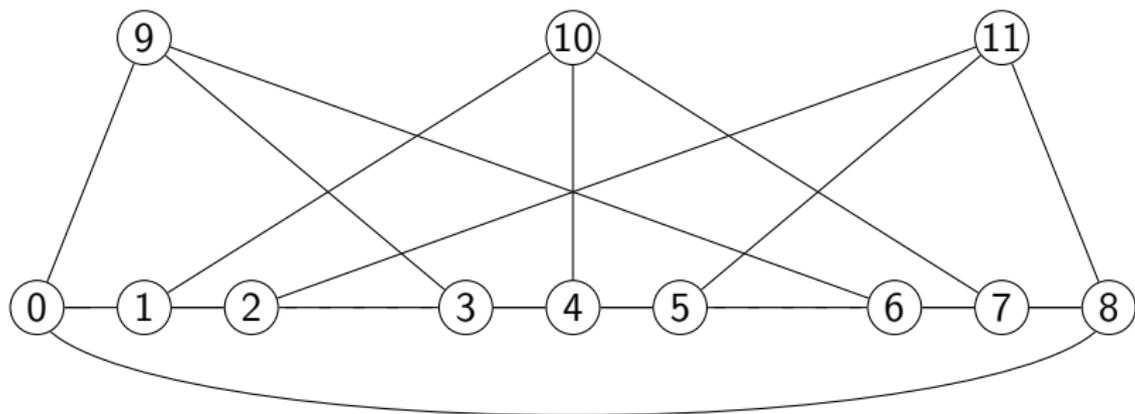
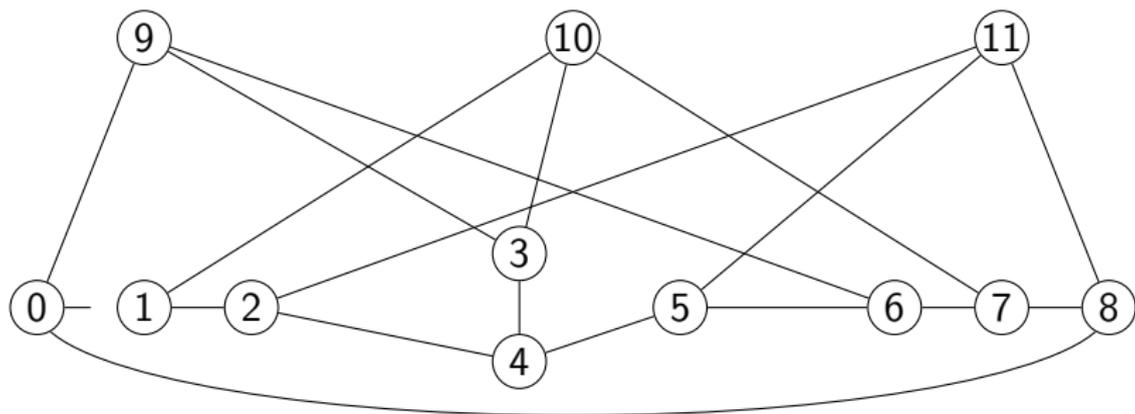# Example 3: A family of genus 7 nodal curves

Consider the 1-dimensional family of genus 7 nodal curves with the following dual graph:
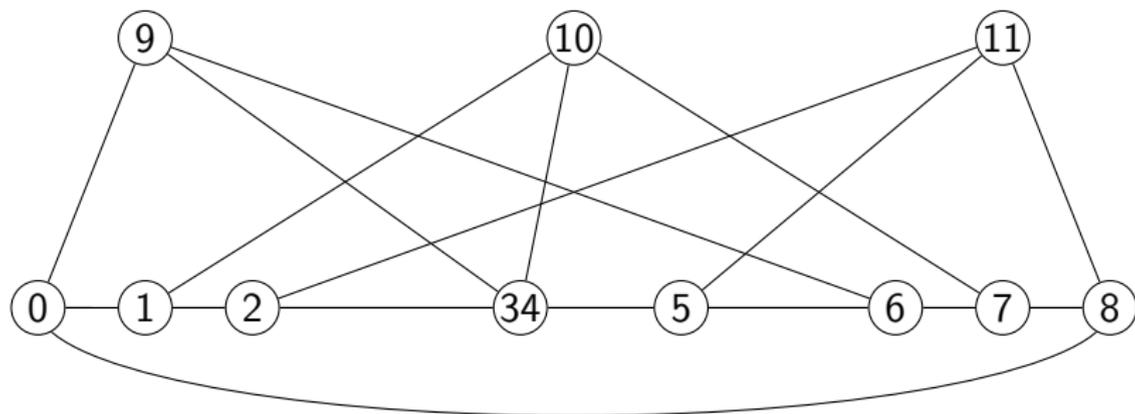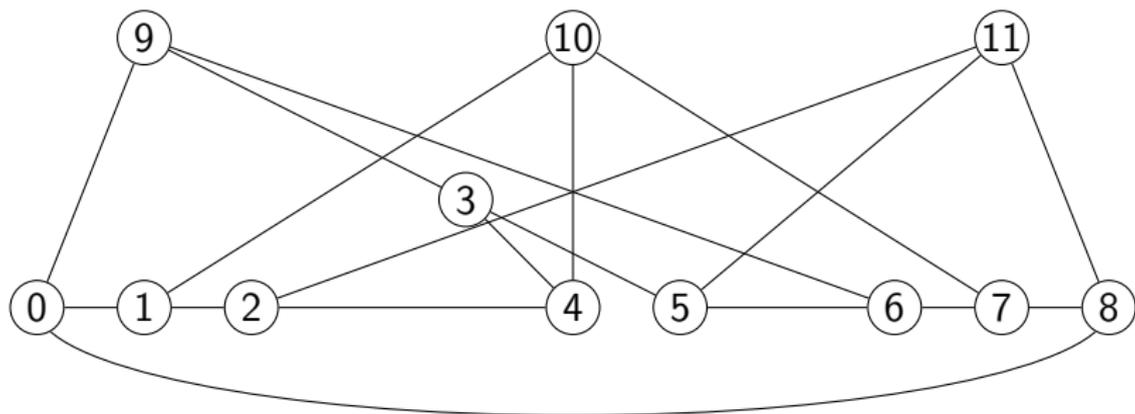

- ▶ Trivalent except at the vertex labelled 34, which is 4-valent
- ▶ Get a family by varying the cross ratio of the four nodes on component 34
- ▶ Has three specializations to trivalent graphs. (Bayer-Eisenbud graph curves). Put these at $t = 0, 1, \infty$

G_∞

Why study this family?

Goal: Gain insight into how Mukai's model completes a family that is acquiring a g_4^1.

- The dualizing sheaf ω is very ample on every member of this family

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- The dualizing sheaf ω is very ample on every member of this family
- A general member of this family has no g_2^1, g_3^1, or g_4^1. (equivalently: canonical ideal has a pure Betti table)
  - G has girth 4, and the 4-cycles pass through the 4-valent vertex
  - G_0 has girth 5
- Two of the trivalent specializations ($t = 1, \infty$) have a g_4^1.
  - G_1 and G_∞ have girth 4

Do these curves arise as $P_t \cap \mathrm{OG}(5,10)$?

I have code in `Macaulay2`/`Magma` to:

- ▶ Write equations for this family in $\mathbb{P}^6$
- ▶ Compute the spin embedding in $\mathbb{P}^{15}$ of a general member of the family
- ▶ Compute the linear space in $\mathrm{Gr}(7,16)$ whose intersection with $\mathrm{OG}(5,10)$ yields a general member
- ▶ Study the limits in $\mathrm{Gr}(7,16)$ as $t \to 0, 1, \infty$

Proposition (S., 2023)

Let $t \in \mathbb{P}^1 \smallsetminus 1, \infty$.

1. *The map*

$$\text{Spin}: \quad \begin{array}{ccccc} C_t & \to & \text{OG}(5, 10) & \to & \mathbb{P}^{15} \\ p & \mapsto & W_p^{\perp} & & \end{array}$$

   is an embedding.

2. $\text{Spin}(C_t) = P_t \cap \text{OG}(5, 10)$, *where P_t is a six-dimensional projective linear subspace given by explicit equations.*

3. $[P_t] \in Gr(7, 16)$ *is $\text{Spin}(10)$-semistable*

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3. $[P_t] \in Gr(7, 16)$ *is* $\text{Spin}(10)$-*semistable*

What happens for $t = 1, \infty$?

Flat limits in the Hilbert scheme $\text{Hilb}(\mathbb{P}^{15}, 12x - 6)$

We study the family $\text{Spin}(C_t)$ as $t \to 1, \infty$.

Proposition (S., 2022)

1. As $t \to 1$, the flat limit of the family $\text{Spin}(C_t)$ is the graph curve with the expected dual graph.
   However, this ideal is not of the form $P \cap \text{OG}(5, 10)$.
   (The quadrics don't give $\text{OG}(5, 10)$.)

2. As $t \to \infty$, the flat limit of the family $\text{Spin}(C_t)$ is not a graph curve. Instead:
   ▶ Three pairs of nodes collide to form three spatial triple points
   ▶ The component corresponding to vertex 34 remains an irreducible quadric.

   This ideal is also not of the form $P \cap \text{OG}(5, 10)$.

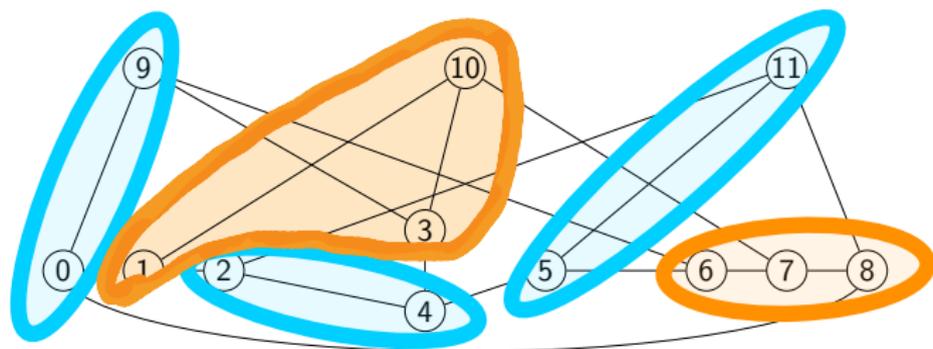# The limit in $Gr(7, 16)$ as $t \to 1$

Proposition (S., 2022)

$P_1 \cap OG(5, 10)$ is a union of five irreducible components.

- ▶ *Lines 0 and 9 in the flat limit are replaced by their span*
- ▶ *Lines 1, 10, and 3 in the flat limit are replaced by the join of lines 1 and 3*
- ▶ *Lines 2 and 4 in the flat limit are replaced by their span*
- ▶ *Lines 5 and 11 in the flat limit are replaced by their span*
- ▶ *Lines 6,7,8 in the flat limit are replaced by the join of lines 6 and 8*

This limit is T-unstable with respect to the diagonal maximal torus T in $SO(10)$.

The limit in Gr(7, 16) as $t \to 1$

The limit in $\mathrm{Gr}(7,16)$ as $t \to \infty$

Proposition (S., 2022)

$P_\infty \cap \mathrm{OG}$ *is a union of eight irreducible components.*

- ▶ *Lines 0, 1, 2, 7, 10, 11 in the flat limit appear in the Grassmannian limit*
- ▶ *Component 34 (an irreducible quadric) in the flat limit appears in the Grassmannian limit*
- ▶ *Lines 5, 6, 8, 9 are replaced by their span, a $\mathbb{P}^2$*

This limit is T-unstable with respect to the diagonal maximal torus T in $\mathrm{SO}(10)$.

The limit in $\mathrm{Gr}(7, 16)$ as $t \to \infty$

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- *Lines 0, 1, 2, 7, 10, 11 in the flat limit appear in the Grassmannian limit*
- *Component 34 (an irreducible quadric) in the flat limit appears in the Grassmannian limit*
- *Lines 5, 6, 8, 9 are replaced by their span, a $\mathbb{P}^2$*

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Question: How does Mukai's model complete this family?

Part II: Spin(10)-semistability

How can we establish Spin(10)-semistability for these examples?

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The previous proofs of GIT stability for moduli spaces of curves that I know do not apply:

1. Gieseker's construction of $\overline{M}_g$
2. Alper, Fedorchuk, and Smyth's proof of finite Hilbert stability
3. GIT stability for hyperplanes (e.g. plane cubics)

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Idea: can we find a Spin(10) invariant polynomial that doesn't vanish on these examples?
~~geometric~~ invariant theory

"Invariant theory has already been pronounced dead several times, and like the phoenix it has been again and again rising from its ashes.... It might be worthwhile to push the 19th century computations of invariants a little further along, with the help of modern computers."

—Dieudonné and Carrell
Invariant Theory: Old and New (1970)

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See also
Sturmfels, *Algorithms in Invariant Theory* (1993/2008)
Derksen and Kemper, *Computational Invariant Theory* (2002/2015)

Invariants for Mukai's model

Let S^+ be the half-spin representation of $\mathrm{Spin}(10)$.
$\dim S^+ = 16$.

Mukai's model is the quotient $\mathrm{Gr}(7, 16) /\!/ \mathrm{Spin}(10)$, i.e. we seek an element of

$$(\mathrm{Sym}^d \Lambda^7 S^+)^{\mathrm{Spin}(10)}$$

for some $d > 0$.

Computing G invariants in a fixed degree

We can use the Casimir operator on $\mathfrak{g}$.

Definition
Let $\delta_1, \ldots, \delta_m$ be a basis of $\mathfrak{g}$
Let $\gamma_1, \ldots, \gamma_m$ be the dual basis of $\mathfrak{g}$ with respect to κ. Then

$$c = \sum \delta_i * \gamma_i$$

Computing G invariants in a fixed degree

Proposition

If $V(\lambda)$ is an irreducible representation with highest weight λ, then c acts as multiplication by the scalar $(\lambda, \lambda + 2\rho)$.

This suggests the following strategy for computing invariants.

Proposition

v is invariant under G if and only if $v \in \ker(c)$.

Caveat

This is not easy to apply to Mukai's problem.

$$\dim \Lambda^7 S^+ = \binom{16}{7} = 11,440.$$

A character calculation shows that the lowest degree invariants are in degree 4, and

$$\dim \operatorname{Sym}^4 \Lambda^7 S^+ = \binom{11440 + 4 - 1}{4} = 714,036,824,189,260.$$

Don't give up yet

Observation:

$$\Lambda^7 S^+ \cong V_1 \oplus V_2$$

where
V_1 has highest weight $(1, 0, 1, 0, 1)$, dim $V_1 = 8800$
V_2 has highest weight $(3, 0, 0, 1, 0)$, dim $V_2 = 2640$

Thus

$$\text{Sym}^d(V_1 \oplus V_2) \cong \sum_{d_1 + d_2 = d} \text{Sym}^{d_1} V_1 \otimes \text{Sym}^{d_2} V_2$$

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V_2 has highest weight $(3, 0, 0, 1, 0)$, dim $V_2 = 2640$

Thus

$$\text{Sym}^d(V_1 \oplus V_2) \cong \sum_{d_1 + d_2 = d} \text{Sym}^{d_1} V_1 \otimes \text{Sym}^{d_2} V_2$$

I studied the portion

$$\text{Sym}^2 V_1 \otimes \text{Sym}^2 V_2$$

Don't give up yet

There are 89 Spin(10) invariants in

$$\text{Sym}^2 V_1 \otimes \text{Sym}^2 V_2,$$

and they are all of the form

$$(V(\lambda) \otimes V(\lambda^*))^G$$

Step 1: I analyzed the irreducible decompositions of $\text{Sym}^2 V_1$ and $\text{Sym}^2 V_2$ and selected one pair of summands for further study.

Specifically, I chose the summand with highest weight $(5, 0, 0, 0, 0)$. $\dim V(5\omega_1) = 1782$.

Computing the invariant of $V(5\omega_1) \otimes V(5\omega_1)$

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I computed $\ker(c)$ on this space to obtain an invariant polynomial, $F_{5\omega_1}$.

Toward Spin(10)-semistability

Step 4: I computed highest weight vectors v_1 and v_2 generating $V(5\omega_1)$ in $\mathrm{Sym}^2 V_1$ and $\mathrm{Sym}^2 V_2$.

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on the linear spaces P giving the examples discussed above

► This is nontrivial, because each basis element has a large number of terms.

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Step 6: I substituted these values into the invariant polynomial $F_{5\omega_1}$.

Results

Theorem (S., Feb 27, 2023)

The points $P \in \mathrm{Gr}(7, 16)$ parametrizing the following curves are $\mathrm{Spin}(10)$*-semistable.*

1. *The 7-cuspidal curve with heptagonal symmetry*
2. *The genus 7 balanced ribbon*
3. *The nodal curves in the family presented above for $t \neq 1, \infty$*

Proof.

The $\mathrm{Spin}(10)$-invariant polynomial $F_{5\omega_1}$ does not vanish on these points. $\qquad\square$

Application: lighting