IVRG WEEK 3
A BRIEF INTRODUCTION TO RIEMANN SURFACES

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The goal of today's lecture is to give you a glimpse of the theory of Riemann surfaces. Riemann surfaces arise in many different contexts in mathematics, and have been the subject of a great deal of research over the past 150 years. I study them both because they are beautiful, and because they are useful—they are the strings in string theory.

I'll begin by stating the definition, and then try to explain what each word in it means.

Definition 0.1 *A* Riemann surface *is a closed, compact, connected complex manifold of dimension 1.*

Note: not all authors require the conditions closed and compact.

1. What is a complex manifold?

Before we define manifolds, I want to define something called a topological space.

1.1. **Topological spaces.**

Definition 1.1 *A* topological space *is a pair* (X, τ)*.* X *is called the set of points, and* τ *is the set of open sets.* τ *is required to satisfy three conditions:*

(1) *The whole set* X *and the empty set* $\emptyset$ *are open.*
(2) *If* U, V *are open, then* $U \cap V$ *is open.*
(3) *If* $\{U_i : i \in I\}$ *is any collection of open sets, then the union* $\bigcup_{i \in I} U_i$ *is open.*

A good reference on topological spaces is Armstrong's book [1]. This is the book I used when I first learned topology, and I have very good memories of it. Chapter 1 (Introduction) is a good place to start. Chapters 4 (Identification spaces) and 7 (Surfaces) might be relevant, too.

Example 1.2 $X = \mathbb{R}$. We define open sets as follows: a set $U \in \mathbb{R}$ is open if for every point $p \in U$, there exists an interval (a_p, b_p) containing p such that $(a_p, b_p) \subseteq U$.

Notice that this topology is *generated* by the open intervals (a, b) because every open set is the union of open intervals:

$$U = \bigcup_{p \in U} (a_p, b_p)$$

Example 1.3 $X = \mathbb{R}^2$. We define open sets as follows: a set $U \in \mathbb{R}^2$ is open if for every point $p = (x_0, y_0) \in U$, there exists an open disk $D(p, r) = \{(x, y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - y_0)^2 < r^2\}$ containing p such that $D(p, r) \in U$.

Once again, we can say this topology is *generated by* open disks.

There's really no reason to stop at dimension 2: we can put a topology on $\mathbb{R}^n$ for any n using open disks defined by the n-dimensional distance formula. In fact, if you look closely, this definition coincides exactly with what we did in dimension 1 in the first example, too.

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Example 1.4 $X = \mathbb{C}$. As usual, we identify the complex plane $\mathbb{C}$ with $\mathbb{R}^2$ by the identification $x + iy \mapsto (x, y)$. We define open sets in $\mathbb{C}$ the same way we did for $\mathbb{R}^2$ as follows: a set $U \in \mathbb{R}^2$ is open if for every point $p = x_0 + iy_0 \in U$, there exists an open disk $D(p, r) = \{x + iy \in \mathbb{C} : (x - x_0)^2 + (y - y_0)^2 < r^2\}$ containing p such that $D(p, r) \in U$.

Again, we can generalize this definition and define a topology on $\mathbb{C}^n$.

1.2. Complex manifolds.

Definition 1.5 *A* neighborhood *of a point $p \in X$ is an open set U containing p.*

A chart *containing p is a map $f : U \to V$. Here U is a neighborhood of p, $V \subseteq \mathbb{C}^n$ is an open subset of $\mathbb{C}^n$, and we require that f is continuous and 1:1. Then the inverse map f^{-1} also exists, and we require this to be continuous, too.*

Next we define compatibility of charts. *See Figure 1.2 below. Suppose $f_1 : U_1 \to V_1$ and $f_2 : U_2 \to V_2$ are two charts such that $U_1 \cap U_2 \neq \emptyset$. Let $W_1 = f_1(U_1 \cap U_2)$ and let $W_2 = f_2(U_1 \cap U_2)$. Then we can form two new maps*

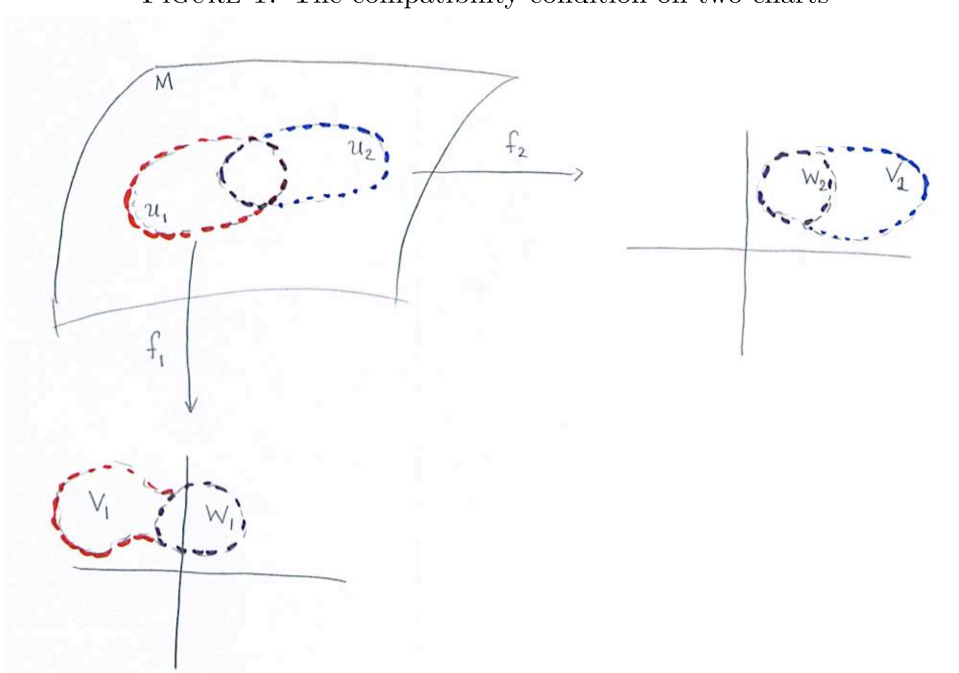
$$g_{12} := f_2 \circ f_1^{-1} : W_1 \to W_2$$
$$g_{21} := f_1 \circ f_2^{-1} : W_2 \to W_1$$

Note that W_1 and W_2 are both subsets of $\mathbb{C}^n$. Then we say that the charts f_1 and f_2 are compatible if g_{12} and g_{21} are complex analytic *maps, i.e. around every point in W_1 or W_2, g_{12} and g_{21} can be represented by a power series.*

An n-dimensional complex manifold *M is a topological space*[1] *such that*

(1) *for every point $p \in M$, there is at least one chart containing p;*
(2) *if $f_1 : U_1 \to V_1$ and $f_2 : U_2 \to V_2$ are two charts such that $U_1 \cap U_2 \neq \emptyset$, then f_1 and f_2 are compatible.*

Figure 1. The compatibility condition on two charts



[1]Some additional technical hypotheses are probably also needed, such as M is Hausdorff and has a countable basis, but we will skip over this for now

1.3. **Closed, compact, connected.** Let's go in reverse order. *Connected* means what you think it means: the space cannot be separated into two disjoint nonempty open sets. Now, I can always partition any space X any space into two disjoint nonempty subsets A and B. But if X is connected, you'd expect that there would be a boundary dividing these two subsets, and each of these boundary points must be in either A or B. So either A or B would fail to be open. On the other hand, if X is disconnected, you could call its connected components A and B. Then there do not have to be any boundary points between them, so there is no contradiction.

The word "closed" gets used a lot in mathematics, and here there are two usages in play which conflict pretty badly with each other. One usage is: a subset of a topological space is closed if its complement is an open set. But the operative usage in the definition of a Riemann surface is as follows: we call a compact manifold *closed* if it has no boundary. So for instance, the unit disk $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ is a closed subset of $\mathbb{R}^2$, since its complement is open set, but not closed as a complex manifold, since it has a boundary (the unit circle). An example of a closed surface would be the (hollow) unit sphere in $\mathbb{R}^3$.

A subset of $\mathbb{R}^n$ is compact if it is closed (i.e. its complement is open) and bounded. The definition of compactness for a space that isn't sitting inside R^n is a little more abstract: we say a topological space is *compact* if every open cover has a finite subcover. That is, anytime I have a collection of open sets which cover X, actually, I can find finitely many open sets from this collection that cover X. I would say it is highly unobvious why these two properties are related.

2. The Riemann sphere

In this section we study one of the most basic and important examples: the Riemann sphere.

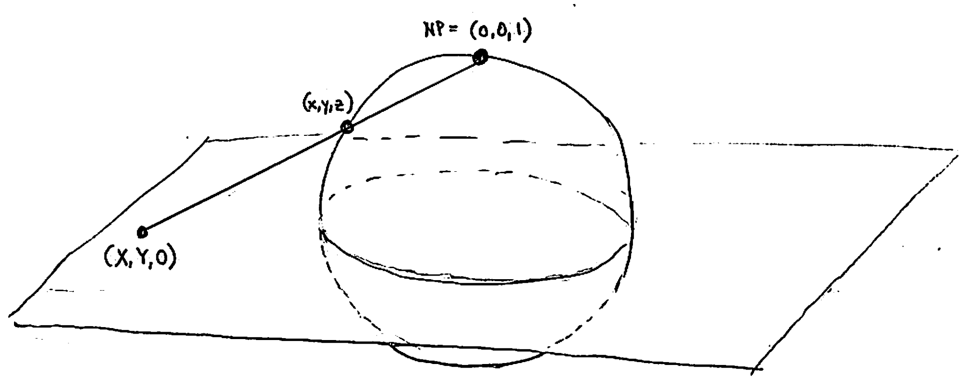
Let S^2 be the unit sphere in $\mathbb{R}^3$. We identify the x, y-plane in $\mathbb{R}^3$ with $\mathbb{C}$ as usual.

I will define charts on S^2 and show that S^2 together with these charts satisfies the definition of a 1-dimensional complex manifold.

2.1. **Two charts.** Let NP $= (0, 0, 1)$ and SP $= (0, 0, -1)$ be the North and South Poles, respectively. Let $U_1 = S^2 \setminus \{\text{NP}\}$ and $U_2 = S^2 \setminus \{\text{SP}\}$. I declare U_1 and U_2 to be open sets.

Next, I will define two maps by *stereographic projection*. Let *sn* be stereographic projection from the North Pole. In words, if $(x, y, z) \in U_1$, then we draw the ray R which starts at the North Pole and goes through (x, y, z). We define $sn(x, y, z)$ to be the point $(X, Y, 0)$ where R intersects the x, y-plane.

Figure 2. Stereographic projection from the North Pole



We find formulas for sn: The ray through $(0,0,1)$ and (x,y,z) can be parametrized as $(0,0,1)+t(x,y,z-1)$. We can solve to find t such that $1+t(z-1)=0$ to get $t=\frac{-1}{z-1}$. Then

$$sn(x,y,z)=\left(\frac{-x}{z-1},\frac{-y}{z-1},0\right).$$

We are going to need a formula for the inverse map sn^{-1}, so we compute this now: Let $(X,Y,0)$ be a point in the x,y-plane. We can parametrize the ray through this point and $(0,0,1)$ as $(0,0,1)+t(X,Y,-1)=(tX,tY,-t+1)$, and we seek t that $(tX,tY,-t+1)\in S^2$. We can solve $(tX)^2+(tY)^2+(-t+1)^2=1$ to get $t=\frac{2}{X^2+Y^2+1}$. Then

$$sn^{-1}(X,Y,0)=\left(\frac{2X}{X^2+Y^2+1},\frac{2Y}{X^2+Y^2+1},1-\frac{2}{X^2+Y^2+1}\right).$$

Exercise 2.1 *Check that $sn^{-1}(sn(x,y,z))=(x,y,z)$. You may need to use the fact that $(x,y,z)\in S^2$, i.e. $x^2+y^2+z^2=1$.*

We define our first chart $f_1:U_1\to\mathbb{C}$ by $f_1=sn$.

We can define stereographic projection from the South Pole analogously and find a formula for it by calculations similar to those above.

Exercise 2.2 *Show that ss, the stereographic projection from the South Pole is given by*

$$ss(x,y,z)=\left(\frac{x}{z+1},\frac{y}{z+1},0\right). \tag{2.3}$$

Actually I don't want to use ss for the chart f_2. I want to modify it a little bit first. Here's why:

The normal vector $\frac{\partial f_1^{-1}}{\partial X}\times\frac{\partial f_1^{-1}}{\partial Y}$ for the map f_1^{-1} is inward pointing on U_1. We can see this by a direct calculation, or just by visualizing how the map f_1^{-1} parametrizes U_1. In contrast, the normal vector for the map ss^{-1} is outward pointing. I want my chart f_2^{-1} to be inward pointing on U_2 so that it matches f_1^{-1}. To do this, I will compose ss in Formula (2.3) with a 180 degree rotation of the x,y-plane in $\mathbb{R}^3$ along the x-axis. This rotation is given by the map $(X,Y,0)\mapsto(X,-Y,0)$. Then I obtain

$$f_2(x,y,z)=\left(\frac{x}{z+1},\frac{-y}{z+1},0\right). \tag{2.4}$$

Exercise 2.5 *Show that*

$$f_2^{-1}(X,Y,0)=\left(\frac{2X}{X^2+Y^2+1},\frac{-2Y}{X^2+Y^2+1},-1+\frac{2}{X^2+Y^2+1}\right). \tag{2.6}$$

and show that the normal vector $\frac{\partial f_2^{-1}}{\partial X}\times\frac{\partial f_2^{-1}}{\partial Y}$ is inward pointing, as desired.

2.2. **The topology.** So far, I have declared that the following sets are open sets of S^2: $\emptyset, U_1\cap U_2, U_1, U_2, S^2$. If you look up the definition of a continuous map in Armstrong's book, you'll see that a map $f:X\to Y$ is continuous if $f^{-1}(V)$ is open in X for every open set V in Y. So to make my maps f_1 and f_2 above continuous, I declare that every set of the form $f_1^{-1}(V)$ or $f_2^{-1}(V)$ is also open. Now I have told you all the open sets of S^2, and my maps f_1 and f_2 are continuous.

2.3. **Checking the compatibility condition.** We need to check the compatibility condition for our two charts. We want to show that g_{12} and g_{21} can be represented by power series.

Proposition 2.7 $g_{12}(\zeta)=\frac{1}{\zeta}$.

Proof. Recall that $g_{12} : \mathbb{C} \to \mathbb{C}$ is the composition $f_2 \circ f_1^{-1}$.

We start with a complex number ζ. Then to apply f_1^{-1}, we want to split $\zeta = X + iY$ into its real and imaginary parts, and identify ζ with the point $(X, Y, 0)$ in the x, y-plane in $\mathbb{R}^3$. We apply f_1^{-1} and get

$$\left(\frac{2X}{X^2+Y^2+1}, \frac{2Y}{X^2+Y^2+1}, \frac{X^2+Y^2-1}{X^2+Y^2+1} \right).$$

We apply f_2 to this point and get

$$\left(\frac{\frac{2X}{X^2+Y^2+1}}{\frac{X^2+Y^2-1}{X^2+Y^2+1}+1}, \frac{\frac{-2Y}{X^2+Y^2+1}}{\frac{X^2+Y^2-1}{X^2+Y^2+1}+1}, 0 \right)$$

We can simplify the first and second coordinates to get

$$\left(\frac{X}{X^2+Y^2}, \frac{-Y}{X^2+Y^2}, 0 \right).$$

To finish, we need to show that the above expression is $\frac{1}{\zeta}$. As a complex number, it is $\frac{X}{X^2+Y^2} - i\frac{Y}{X^2+Y^2}$. We have that

$$(X+iY)\left(\frac{X}{X^2+Y^2} - i\frac{Y}{X^2+Y^2} \right) = \frac{X^2+Y^2}{X^2+Y^2} = 1,$$

and hence this is the multiplicative inverse of the complex number we started with. □

Why is the map $\zeta \mapsto \frac{1}{\zeta}$ a power series? It looks like a rational function to me. When $\zeta \neq 0$, we can take as many derivatives of $f(\zeta) = \frac{1}{\zeta}$ as we want. Thus, near a point $\zeta_0 \neq 0$, we can compute the Taylor series for $f(\zeta)$ near ζ_0, and it will converge on a disk of radius $|\zeta_0|$ centered at ζ_0.

3. Fundamental facts about Riemann surfaces

In the title of this section, I am using the word "fundamental" in the sense that lots of the rest of the theory of Riemann surfaces rests on these facts. I don't mean to imply that they are easily proved.

Proposition 3.1
1. *Every Riemann surface looks like a donut with a certain number of holes. The number of holes is called the* genus.
2. *Every Riemann surface can be represented by the zeroes of a system of polynomial equations.*

I won't prove these statements, but I can at least point you to some references.

For the first statement: First, we need to know that complex manifolds are orientable (see [3, pp.8-9] for 1-dimensional complex manifolds, or [4, p. 18] for any dimension). Then we just need to know the classification of orientable surfaces; this can be found in [1, Ch. 7].

For the second statement: see the discussion in [5, Appendix B3].

In closing, I will elaborate on the kinds of polynomials that arise in the proposition.

3.1. Genus 0.
There is a unique Riemann surface of genus 0, namely, the Riemann sphere we explored in Section 2 above.

3.2. Genus 1.
There exist infinitely many different Riemann surfaces of genus 1. (In fact, for any $g \geq 1$, there exist infinitely many different Riemann surfaces of genus g.) Every genus 1 Riemann surface can be obtained in the following way: Choose a complex number ζ such that $\operatorname{Im}\zeta \neq 0$. Draw the parallelogram in the complex plane with vertices at 0, 1, ζ, and $1+\zeta$. Then if we roll up this parallelogram and glue together opposite sides, we will get a donut or torus C_ζ with one hole.

If you do this starting with two different complex numbers ζ_1 and ζ_2 with $|\zeta_1| = |\zeta_2| = 1$ but $\zeta_1 \neq \zeta_2$, then it turns out that there is no power series map between the two tori C_{ζ_1} and C_{ζ_2}.

This explains what I meant by "different": I think that in order for two complex manifolds to be considered "the same," there ought to be a power series map between them, with an inverse map that's also given by a power series.

Every genus 1 Riemann surface can be represented by an equation of the form $y^2 = x^3 + ax + b$.

3.3. **Genus 2.** There exist infinitely many different Riemann surfaces of genus 2.

Every genus 2 Riemann surface can be represented by an equation of the form $y^2 = f(x)$, where f is a polynomial of degree 5 or 6.

Next week I will try to give you feeling for how the polynomial $y^2 = x^5 - x$ gives you a genus 2 Riemann surface.

3.4. **Genus 3.** There exist infinitely many different Riemann surfaces of genus 3.

Every genus 3 Riemann surface C can be represented by an equation of exactly one of the following two forms:

(1) (General case) If C is nonhyperelliptic: $F(x, y, z)$, where F is a degree four homogeneous polynomial in three variables.
(2) If C is hyperelliptic: $y^2 = f(x)$, where f is a polynomial of degree 7 or 8.

3.5. **Genus $g \geq 4$.** As noted above, for any $g \geq 1$, there exist infinitely many different Riemann surfaces of genus g.

Here we describe the kinds of polynomials that are involved. I will give some modern references for the claims below. Much of the information below is classical, and I apologize that I lack sufficient knowledge of the history of mathematics to credit the first discoverers.

Definition 3.2 *A polynomial F in several variables is* homogeneous of degree d *if every monomial in F has degree d; that is, there are no lower-order terms.*

We call a homogeneous polynomial of degree 2 a quadric*, and a homogeneous polynomial of degree 3 a* cubic*.*

Here is the general pattern:

Dave: find a reference for the second claim

Proposition 3.3 *Let C be a Riemann surface of genus $g \geq 4$. Suppose if $g = 6$ that C is not a plane quintic. Then C can be represented by a set of equations of exactly one of the following three forms:*

(1) *(General case) If C is nonhyperelliptic and not trigonal: $(g-2)(g-3)/2$ quadrics.*
(2) *If C is nonhyperelliptic and not trigonal: $(g-2)(g-3)/2$ quadrics and $g-3$ cubics.*
(3) *If C is hyperelliptic: $y^2 = f(x)$, where f is a polynomial of degree $2g+1$ or $2g+2$.*

If $g = 6$ and C is a plane quintic, then the equations are just like those for a trigonal curve, i.e. six quadrics and three cubics.

Here are some references for the claims made above: For the number of quadrics in the nonhyperelliptic cases, see [2, Cor. 9.4]. For the number of cubics required to define a trigonal curve: (need a reference). For the case of a plane quintic, see [6, p. 107]. For hyperelliptic curves, see [3, §IV.4].

We write out the details for a few small values of g. This information can be found in [5, §IV.4] and [6, p. 107].

Corollary 3.4

Genus 4: *Every genus 4 Riemann surface C can be represented by a set of equations of exactly one of the following two forms:*

(a) *(General case) If C is nonhyperelliptic: one irreducible quadric and one irreducible cubic, i.e. $\{F_1(a,b,c,d), F_2(a,b,c,d)\}$, where F_1 and F_2 are homogeneous polynomials in four variables of degree two and three, respectively.*

(b) *If C is hyperelliptic: $y^2 = f(x)$, where f is a polynomial of degree 9 or 10.*

Genus 5: *Every genus 5 Riemann surface C can be represented by a set of equations of exactly one of the following three forms:*

(a) *(General case) If C is nonhyperelliptic and not trigonal: three quadrics, i.e. $\{F_1(a,b,c,d,e), F_2(a,b,c,d,e), F_3(a,b,c,d,e)\}$, where F_i are homogeneous polynomials in five variables of degree two for $i = 1, 2, 3$.*

(b) *If C is nonhyperelliptic and trigonal: three quadrics and two cubics, i.e. $\{F_1(a,b,c,d,e), F_2(a,b,c,d,e), F_3(a,b,c,d,e), F_4(a,b,c,d,e), F_5(a,b,c,d,e),\}$, where F_i are homogeneous polynomials in five variables of degree two for $i = 1, 2, 3$ and degree three for $i = 4, 5$.*

(c) *If C is hyperelliptic: $y^2 = f(x)$, where f is a polynomial of degree 11 or 12.*

Genus 6: *Every genus 6 Riemann surface C can be represented by a set of equations of exactly one of the following three forms:*

(a) *(General case) If C is nonhyperelliptic and not trigonal and not a plane quintic: six quadrics, i.e. $\{F_i(a,b,c,d,e,f) : i = 1, \ldots, 6\}$, where F_i are homogeneous polynomials in six variables of degree two for $i = 1, \ldots, 6$.*

(b) *If C is nonhyperelliptic and either trigonal or a plane quintic: six quadrics and three cubics, i.e. $\{F_i(a,b,c,d,e,f) : i = 1, \ldots, 9\}$, where F_i are homogeneous polynomials in six variables of degree two for $i = 1, \ldots, 6$ and degree three for $i = 7, 8, 9$.*

(c) *If C is hyperelliptic: $y^2 = f(x)$, where f is a polynomial of degree 13 or 14.*

Genus 7: *Every genus 7 Riemann surface C can be represented by a set of equations of exactly one of the following three forms:*

(a) *(General case) If C is nonhyperelliptic and not trigonal: ten quadrics, i.e. $\{F_i(a,b,c,d,e,f) : i = 1, \ldots, 10\}$, where F_i are homogeneous polynomials in six variables of degree two for $i = 1, \ldots, 10$.*

(b) *If C is nonhyperelliptic and trigonal: ten quadrics and four cubics, i.e. $\{F_i(a,b,c,d,e,f) : i = 1, \ldots, 14\}$, where F_i are homogeneous polynomials in five variables of degree two for $i = 1, \ldots, 10$ and degree three for $i = 11, 12, 13, 14$.*

(c) *If C is hyperelliptic: $y^2 = f(x)$, where f is a polynomial of degree 15 or 16.*

Genus 8: *Every genus 8 Riemann surface C can be represented by a set of equations of exactly one of the following three forms:*

(a) *(General case) If C is nonhyperelliptic and not trigonal: fifteen quadrics, i.e. $\{F_i(a,b,c,d,e,f) : i = 1, \ldots, 15\}$, where F_i are homogeneous polynomials in six variables of degree two for $i = 1, \ldots, 15$.*

(b) *If C is nonhyperelliptic and trigonal: fifteen quadrics and five cubics, i.e. $\{F_i(a,b,c,d,e,f) : i = 1, \ldots, 20\}$, where F_i are homogeneous polynomials in five variables of degree two for $i = 1, \ldots, 15$ and degree three for $i = 16, \ldots, 20$.*

(c) *If C is hyperelliptic: $y^2 = f(x)$, where f is a polynomial of degree 17 or 18.*

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