

# THE WORST DESTABILIZING 1-PARAMETER SUBGROUP FOR TORIC RATIONAL CURVES WITH ONE UNIBRANCH SINGULARITY

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**ABSTRACT.** Kempf proved that when a point is unstable in the sense of Geometric Invariant Theory, there is a “worst” destabilizing 1-parameter subgroup  $\lambda^*$ . It is natural to ask: what are the worst 1-PS for the unstable points in the GIT problems used to construct the moduli space of curves  $\overline{M}_g$ ? Here we consider Chow points of toric rational curves with one unibranch singular point. We translate the problem as an explicit problem in convex geometry (finding the closest point on a polyhedral cone to a point outside it). We prove that the worst 1-PS has a combinatorial description that persists once the embedding dimension is sufficiently large, and present some examples.

## 1. INTRODUCTION

There are several GIT results concerning the Hilbert or Chow stability of embedded singular curves. When the dimension of the linear system is sufficiently small compared to the arithmetic genus, many singularities are GIT stable. For example, cusps are semistable in several GIT problems, including plane quartics and 2-, 3-, or 4-canonical curves [1, 10, 13, 24, 29]. In contrast, when the dimension of the linear system is sufficiently large compared to the genus, the only singularities that are GIT semistable are nodes; see [7] for asymptotic Hilbert semistability or [23] for Chow semistability.

Kempf showed in [17] that when  $x \in X$  is GIT unstable, there is a “worst” destabilizing 1-parameter subgroup  $\lambda^*$  associated to  $x$ . It is the worst in the sense that it maximizes  $\mu(x, \lambda)/\|\lambda\|$ , where  $\mu(x, \lambda)$  is the Hilbert-Mumford function. Thus, it is natural to ask: when a point  $[C]$  parametrising a singular curve is GIT unstable, what is the worst 1-PS  $\lambda^*$ ?

Knowledge of worst 1-PS has important applications in moduli theory. Hesselink and Kempf-Ness describe a locally closed stratification that is invariant under the group action and arises from indexing each point by its worst 1-PS; Kirwan uses this stratification to compute the cohomology of the GIT quotient [11, 19, 25]. More recently, this whole picture has been greatly generalised by Halpern-Leistern’s Beyond GIT program, in which the HN filtration of an unstable point is a generalization of the worst 1-PS [8]. On the moduli side, it has been shown that these unstable Hesselink-Kempf-Kirwan-Ness strata can themselves be quotiented using results from Non-Reductive GIT [12], allowing one to construct moduli spaces of *unstable* objects. Although we perform no such quotients in this paper, a principal motivation for our work here is the construction of new moduli spaces of unstable (i.e. singular) curves.

We study the problem of Chow stability for a toric rational curve  $C$  with one unibranch singularity. There are two reasons we begin our study with these examples. The first reason is that we know where to look: when  $X$  has an automorphism group whose action is multiplicity-free, the Kempf-Morrison Lemma [22, Prop. 4.7] guarantees that the worst 1-PS will appear in the maximal torus diagonalizing this action. The second reason is that when  $X$  is toric, the vertices of the Chow polytope can be identified with coherent triangulations of the polytope of  $X$ . This allows us to interpret the value of the Hilbert-Mumford function  $\mu([X], \lambda)$  as the volume of a convex region. We can go even further when  $X$  is a rational curve with one unibranch singularity and restate the problem of finding the worst 1-PS as finding the closest point on a polyhedral cone  $W$  to a vector  $a$  that lies outside  $W$ . The cone  $W$  and vector  $a$  are completely explicit once the singularity and embedding dimension are given.

Using these ideas, we wrote software that uses `GAP`, `Macaulay2`, and `MATLAB` or `Octave` [4, 20, 21, 27] to compute many examples. We observed that, for a fixed singularity, the worst 1-PS behaves in a predictable way once the embedding dimension  $N$  is sufficiently large. There exists an integer  $\ell$  such that the first  $\ell$  weights in the worst 1-PS are constant with respect to  $N$ , and the remaining weights are given by an explicit formula that arises from a least squares linear regression calculation. We call this phenomenon the *persistence of the worst 1-PS for all  $N$  sufficiently large*.

We state this more precisely as follows. Let  $C$  be a rational curve with one unibranch singularity at  $p$ . Let  $\Gamma = \{\gamma_0, \gamma_1, \dots\}$  be the semigroup of the singularity. For all  $N$ , consider the map  $C \rightarrow \mathbb{P}^N$  given by

$$t \mapsto (1, t^{\gamma_1}, t^{\gamma_2}, \dots, t^{\gamma_N})$$

Let  $C_\Gamma$  be the image of  $C$  under this map, and let  $w^*$  be the weights of the worst 1-PS.

**Main Theorem** (Persistence of the worst 1-PS). *There exist an integer  $N_0$  and an integer  $\ell$  (both depending on  $\Gamma$ ) such that for all  $N \geq N_0$ , the coordinates  $w_0^*, \dots, w_{\ell-1}^*$  are constant with respect to  $N$ , and the coordinates  $w_\ell^*, \dots, w_N^*$  are given by the formula*

$$w_i^* = mi + b$$

where

$$m = -\frac{6}{(N - \ell + 1)(N - \ell + 2)}$$

$$b = 2 + \frac{2(N + 2\ell - 1)}{(N - \ell + 1)(N - \ell + 2)}$$

To prove this theorem, we first prove a similar statement for a closely related optimisation problem that we call the Simplified Problem. For each face  $F$  of  $W$ , we use the Karush-Kuhn-Tucker (KKT) conditions in nonlinear optimisation to study the proximum  $\text{Prox}(a, \text{Span}(F))$ , and show that the face containing the global optimum has the same combinatorial description in all sufficiently large degrees. We then use the solution to the Simplified Problem to construct the solution to the original problem.

**1.1. Outline of the paper.** In Section 2, we use the identification of the vertices of the Chow polytope with coherent triangulations to translate the problem of finding the worst 1-PS into an explicit convex optimisation problem. In Section 3, we recall the KKT conditions in convex optimisation and apply them to describe how the stationary points on each face of the cone  $W$  vary with the embedding dimension  $N$ . In Section 4, we describe the behavior of the stationary points when all the corners are at or below the conductor. In Section 5, we describe the behavior of the stationary points when there is at least one corner above the conductor. In Section 6, we prove the main theorem. Finally, in Section 7, we present several examples.

To keep the main body of the paper short, we moved some technical proofs to two appendices. Appendix A gives a detailed proof of Lemmas 3.9 and 3.11. Appendix B gives a detailed proof of Proposition 7.4 describing the worst 1-PS for cusps.

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**Software.** We used several mathematical software packages for experimentation related to this project, including GAP, `gfan`, `Macaulay2`, `MATLAB`, `Octave`, `QEPcad`, and `SageMath` [4, 5, 20, 21, 27, 28, 30]. This experimentation was essential to the project in that it led us to conjecture the main result, though ultimately, the proof of the main result is a “pencil-and-paper” proof that does not rely on any computer calculations. The examples listed in Table 1 were computed using software. We have posted our code and some demonstrations for the interested reader on our website: see [15].

## 2. THE WORST 1-PS FOR CHOW POINTS OF TORIC CURVES AS A CONVEX OPTIMISATION PROBLEM

In this section, we translate the problem of finding the worst 1-PS for Chow points of certain toric curves into an explicit convex optimisation problem. See Theorem 2.18.

First, we recall Kempf’s description of the worst 1-PS. We follow the conventions of [9, Ch. 4A] and [17].

**2.1. The worst 1-PS.** Let  $G$  be a connected reductive algebraic group over a field  $\mathbf{k}$ , and let  $W$  be a finite-dimensional  $k$ -vector space. Suppose that  $G$  acts on  $X \subset \mathbb{P}(W)$  by a representation  $G \rightarrow \mathrm{GL}(W)$ .

Let  $\lambda : \mathbb{G}_m \rightarrow G$  be a 1-parameter subgroup. Write  $W_i$  for the weight  $i$  subspace, and write  $S_\lambda(W)$  for the set of weights  $\{i : W_i \neq 0\}$ . Then

$$W = \bigoplus_{i \in S_\lambda(W)} W_i$$

**Definition 2.1.** The Hilbert-Mumford function  $\mu(x, \lambda)$  is given by

$$\mu(x, \lambda) = \min\{i \mid x_i \neq 0\}$$

**Definition 2.2.** We call  $\lambda^*$  a *worst 1-PS* for  $x$  if

$$\frac{\mu(x, \lambda^*)}{\|\lambda^*\|} = \sup_{\lambda} \frac{\mu(x, \lambda)}{\|\lambda\|}$$

*A priori* it is not clear that this supremum is finite, or that it is achieved, but Mumford showed that both statements are true. Kempf proved that when  $x$  is unstable, the supremum is achieved on a unique parabolic conjugacy class of 1-PS.

**Theorem 2.3** ([24, Prop. 2.17], [17, Theorem 3.4]). *Suppose that  $x \in X$  is unstable for the action of  $G$ . Then there exists a worst 1-PS for  $x$ .*

In the following subsections, we recall the definitions of Chow points, with the goal of understanding their worst one-parameter subgroups.

**2.2. Chow forms and Chow polytopes.** Given a curve  $C \in \mathbb{P}^N$  of degree  $d$ , there is a corresponding point  $[C]$  in the Chow variety  $\mathrm{Chow}_d(\mathbb{P}^N)$ . We briefly recall the relevant definitions; see [6] Chapters 3 and 4 for more details.

Let  $X \subset \mathbb{P}^{n-1}$  be an irreducible subscheme of dimension  $k-1$  and degree  $d$ . A generic  $(n-k-1)$ -dimensional projective subspace  $L \subset \mathbb{P}^{n-1}$  will miss  $X$ . Let

$$\mathcal{Z}(X) = \{L \mid \dim(L) = n-k-1, X \cap L \neq \emptyset\}$$

**Theorem 2.4.** [6, Ch. 3, Prop. 2.1 and 2.2]

- (1)  $\mathcal{Z}(X)$  is an irreducible hypersurface of degree  $d$  in  $\mathrm{Gr}(n-k, n)$ .
- (2) Up to a scalar,  $\mathcal{Z}(X)$  is defined by a polynomial  $R_X$  called the Chow form of  $X$ .

Thus,  $X \subset \mathbb{P}^{n-1}$  corresponds to the point  $[R_X] \in \mathbb{P}(\mathrm{Sym}^d \Lambda^{n-k} \mathbf{k}^n)$ . We will write  $[X]$  for  $[R_X]$ .

The closure of the set of points of the form  $[X]$  has the structure of an algebraic variety called the *Chow variety*.

**2.3. Numerical semigroups and monomial curves.** We seek the worst 1-PS for the Chow points of curves  $[C] \in \mathrm{Chow}_d(\mathbb{P}^N)$ . In general, this is difficult, so in this work we only consider the special case where  $C$  is a toric rational curve with a single unibranch singularity at  $p \in C$ . One can associate to such a curve its semigroup of values.

**Definition 2.5.** The semigroup of values of  $C$  is the numerical semigroup

$$\Gamma_C := \{n \in \mathbb{N} \mid \exists f \in \mathcal{O}_{C,p} \text{ with } \nu_p(f) = n\} \subset \mathbb{N}$$

where  $\nu_p$  is the valuation at the singular point  $p \in C$ .

**Example 2.6.** The simplest example is a cuspidal curve, that analytically looks like  $y^2 = x^3$  and has semigroup of values  $\Gamma = \langle 2, 3 \rangle = \{0, 2, 3, 4, \dots\}$ . More generally we may consider higher order cusps  $y^2 = x^{2r+1}$ , and get  $\Gamma = \langle 2, 2r+1 \rangle = \{0, 2, 4, \dots, 2r, 2r+1, 2r+2, \dots\}$

For each numerical semigroup  $\Gamma$ , there exists a positive integer  $\mathrm{cond}(\Gamma)$  called the *conductor* of  $\Gamma$  such that  $\mathbb{N}_{\geq \mathrm{cond}(\Gamma)} \subseteq \Gamma$ . Thus one has  $|\mathbb{N} \setminus \Gamma| < \infty$ , and the size of that set, i.e. the number of gaps, is equal to the arithmetic genus of the curve  $C$  (or equivalently, the  $\delta$ -invariant of the singularity). We list the elements of  $\Gamma$  as  $\gamma_0, \gamma_1, \gamma_2, \dots$  with  $\gamma_0 = 0$ , and write  $\mathrm{c.i.}(\Gamma)$  for the index of the conductor. That is,  $\gamma_{\mathrm{c.i.}(\Gamma)} = \mathrm{cond}(\Gamma)$ .

Conversely, given a numerical semigroup  $\Gamma \subset \mathbb{N}$ , we can write down a curve  $C_\Gamma \subset \mathbb{P}^N$  of degree  $d$  that has semigroup of values  $\Gamma$ .

**Definition 2.7.** Let  $\Gamma \subset \mathbb{N}$ , and let  $d > \text{cond}(\Gamma)$ . The *monomial curve*  $C_\Gamma$  is the closure of the image of the parametrisation

$$t \longmapsto (1, t^{\gamma_1}, t^{\gamma_2}, \dots, t^d)$$

where  $\gamma_i$  are the elements of  $\Gamma_{\leq d}$ .

**2.4. The  $\mathbb{G}_m$ -action on  $C_\Gamma$ .** Let  $\Gamma$  be a numerical semigroup, and let  $N > \text{c.i.}(\Gamma)$ . Let  $C_\Gamma \subset \mathbb{P}^N$  be the monomial curve as in Definition 2.7. Our goal is to find the worst 1-PS for this action of  $\text{GL}_{N+1}$  on the Chow point  $[C_\Gamma]$ .

Hyeon and Park have shown that for an unstable point  $x$  in a GIT problem, a generic maximal torus will not contain a destabilizing 1-PS [14]. Fortunately, we have a clue where to look for the worst 1-PS:  $C_\Gamma$  has a  $\mathbb{G}_m$ -automorphism scaling the coordinate  $t$ , and it acts with distinct weights on  $\mathbb{P}^N$ . This allows us to apply the Kempf-Morrison Lemma [22, Prop. 4.7] and conclude that if  $C_\Gamma$  is unstable for the  $\text{GL}_{N+1}$ , then a worst 1-PS will appear in the maximal torus diagonalizing the  $\mathbb{G}_m$ -action. (Note: the statement in the cited work is for a finite automorphism group, but the proof works for  $\mathbb{G}_m$  as well.)

$C_\Gamma$  is a non-normal toric variety. Its associated polytope is just the interval  $[0, d]$ . Because the variety is not normal, some of the interior lattice points of the polytope are missing—these are exactly the gaps in the semigroup.

This observation is useful because the Chow polytope of a toric variety has a second description due to Kapranov, Sturmfels, and Zelevinsky: it is the *secondary polytope* of the polytope of  $X$  ([6, Ch. 8, Thm. 3.1]). In the next section, we briefly review part of this theory.

**2.5. The Hilbert-Mumford function for Chow points of toric varieties.** Consider a polytope  $Q = \text{conv}(A)$ , where  $A = \{A_0, \dots, A_n\} \subset \mathbb{R}^{k-1}$  is a finite set of vectors.

**Definition 2.8.** The *secondary polytope*  $\Sigma(A)$  of  $Q$  is the convex hull of the vectors  $\phi_T$  as  $T$  runs over all triangulations of  $Q$ .

Let  $\mathbb{R}^A$  be the set of functions  $A \rightarrow \mathbb{R}$ . Given a triangulation  $T$  of  $Q$ , one can associate a vector  $\phi_T \in \mathbb{R}^A$ , whose  $i$ th entry is the real number

$$\phi_T(i) = \sum_{\sigma: A_i \in \text{Vert}(\sigma)} \text{Vol}(\sigma)$$

where  $\text{Vol}$  is a translation-invariant volume form and the sum is over all maximal simplices of  $T$  for which  $A_i$  is a vertex. If  $A_i$  is not a vertex of any maximal simplex of  $T$ , the entry is zero.

The vertices of the secondary polytope are those vectors  $\phi_T$  corresponding to coherent triangulations of  $(Q, A)$  [6]Ch. 7, Thm. 1.7 (We omit the definition of coherence, because we won't need it: in dimension one, all triangulations are coherent.)

The following definitions and lemma are adapted from [6, Ch. 7 Lem. 1.9(c)]. (The cited result gives a formula for the maximum, but the minimum appears in our GIT calculations.)

**Definition 2.9.** Given a function  $f \in \mathbb{R}^A$ , let  $H_f$  be the convex hull of the vertical half-lines

$$\{(x, y) \mid y \geq f(x), x \in A, y \in \mathbb{R}\}.$$

Let the *lower convex hull* of  $f$ , denoted  $\text{lch}(f) : Q \rightarrow \mathbb{R}$ , be the piecewise linear function

$$\text{lch}(f)(x) = \min\{y \mid (x, y) \in H_f\}.$$

**Lemma 2.10.** [6, Ch. 7 Lem. 1.9(c)] *Given a polytope  $Q = \text{conv}(A)$  and a vector  $w \in \mathbb{R}^{n+1}$  we have*

$$\min_{\phi \in \Sigma(A)} \langle w, \phi \rangle = k \int_Q \text{lch}(f_w)(x) dx,$$

where  $f_w : A \rightarrow \mathbb{R}$  is the function  $f_w(A_i) = w_i$ .

In summary: to compute the minimal pairing of a vertex of the secondary polytope with a vector  $w \in \mathbb{R}^{n+1}$ , we first take the lower convex hull, and then integrate it over the polytope.

The formula in Lemma 2.10 leads to the following expression for Hilbert-Mumford function. To our knowledge, this has not appeared in the literature before.

**Proposition 2.11.** *The value of the Hilbert-Mumford function for the point  $[X] \in \text{Chow}_d(\mathbb{P}^N)$  at a 1-PS  $\lambda : \mathbb{G}_m \rightarrow T$  with weight vector  $w$  is*

$$\mu(\lambda, [X]) = k \int_Q \text{lch}(f_w)$$

**2.6. Chow polytopes of monomial curves  $C_\Gamma$ .** Now we apply these results to monomial curves.

Let  $\Gamma$  be a numerical semigroup, and let  $N \geq \text{c.i.}\Gamma$ , and let  $C_\Gamma \subset \mathbb{P}^N$  be the corresponding monomial curve

Its corresponding polytope is  $Q = \text{conv}(A) = [0, d]$ , where  $A = (\gamma_0, \dots, \gamma_N)$ . The Chow polytope of  $C_\Gamma$  is the same as the secondary polytope  $\Sigma(A)$ , and hence its vertices correspond to triangulations of  $(Q, A)$ . A 1-dimensional triangulation is just a decomposition into intervals, and it is determined by the placement of the vertices; hence, such triangulations are in bijection with subsets of the finite set  $A \setminus \{0, \gamma_N\}$ .

**Example 2.12.** Consider a cuspidal cubic  $X^2Z - Y^3 \subset \mathbb{P}^2$ . The secondary polytope/Chow polytope has one vertex for each integral subdivision of  $[0, 3]$ , where we are not allowed to place a vertex at 1, because that value is missing from the semigroup of values. Thus there are only two possible subdivisions:  $[0, 2, 3]$  and  $[0, 3]$ , i.e. we either place a vertex at 2 or not. These two subdivisions correspond to the two vertices of the weight polytope, and hence to the two possible initial ideals: one is  $X^2Z$  and the other is  $Y^3$ .

Thus for curves, Proposition 2.11 says that the value of the Hilbert-Mumford function is the area under the graph of the lower convex hull of the piecewise linear function determined by the weight vector  $w$ .

**Definition 2.13.** We call a weight vector *convex* if it is equal to its lower convex hull.

Equivalently, a weight vector is convex if and only if the slopes of the corresponding piecewise linear function are increasing.

**Lemma 2.14.** *Let  $w^*$  be the weight vector of the worst 1-PS for  $C_\Gamma$ . Then  $w^*$  is nonnegative and convex.*

*Proof.* First, suppose that  $w$  has at least one negative coordinate. The vertices of the secondary polytope all lie in the positive orthant, and the Chow polytope coincides with the secondary polytope. Let  $w'$  be defined by  $w'_i = \max\{0, w_i\}$ . Then  $\langle w', \phi \rangle \geq \langle w, \phi \rangle$  for all  $\phi \in \Sigma(A)$ . Hence  $\mu([C_\Gamma], w') \geq \mu([C_\Gamma], w)$ . Also  $\|w'\| < \|w\|$ . Thus

$$\frac{\mu([C_\Gamma], w')}{\|w'\|} \geq \frac{\mu([C_\Gamma], w)}{\|w\|}.$$

Thus,  $w^*$  is nonnegative.

Now suppose  $w$  satisfies  $w_i \neq 0$  for all  $i$ , and for at least one index  $i$ , we have  $w_i > \text{lch}(f_w)(A_i)$ . Then define  $w'$  by lowering the  $i^{\text{th}}$  coordinate to the lower convex hull. We have  $\mu([C_\Gamma], w') = \mu([C_\Gamma], w)$  since they have the same lower convex hull, but  $\|w'\| < \|w\|$ . Thus

$$\frac{\mu([C_\Gamma], w')}{\|w'\|} \geq \frac{\mu([C_\Gamma], w)}{\|w\|}.$$

Thus,  $w^*$  is convex. □

**2.7. The optimisation problem.** In this section we present an explicit optimisation problem whose solution is a worst 1-PS for  $[C_\Gamma]$ . We need to introduce a little more notation first.

**2.7.1. The cone  $W$ .** By Lemma 2.14, in finding the worst 1-PS we are justified in restricting our attention to one-parameter subgroups whose weight vectors  $w$  are convex.

**Definition 2.15.** Let  $W \subset \mathbb{R}^{N+1}$  be the set of convex weight vectors.

Since a weight vector is convex if and only if the slopes of the corresponding piecewise linear function are increasing,  $W$  is a polyhedral cone generated by the inequalities that the slope of the  $i^{\text{th}}$  line is less than or equal to the slope of the  $(i+1)^{\text{th}}$  line. For more details about  $W$ , see Section 3.1.1 below.

2.7.2. *The vector  $a$ .*

**Lemma 2.16.** *Let  $C_\Gamma$  be a monomial curve, let  $A = (\gamma_0, \dots, \gamma_N)$ ,  $Q = \text{conv}(A)$ , and suppose that  $w$  is nonnegative and convex. Then*

$$2 \int_Q \text{lch}(f_w)(x) dx = a \cdot w$$

where

$$a_i = \begin{cases} \gamma_1 & \text{if } i = 1 \\ \gamma_i - \gamma_{i-2} & \text{if } 2 \leq i \leq N \\ 1 & \text{if } i = N + 1 \end{cases}$$

(Note: in this formula, we have indexed  $w = (w_0, \dots, w_N)$  and  $A = (\gamma_0, \dots, \gamma_N)$  starting at 0, and indexed  $a = (a_1, \dots, a_{N+1})$  starting at 1. This is because in the sequel,  $a$  will appear in a matrix equation where we index the rows beginning at 1.)

*Proof.* Since  $w$  is convex, we have  $\text{lch}(f_w)(\gamma_i) = w_i$ . Since  $w$  is nonnegative, the integral  $2 \int_{\gamma_{i-1}}^{\gamma_i} \text{lch}(f_w)(x) dx$  is twice the area of the trapezium with heights  $w_{i-1}$  and  $w_i$  and width  $(\gamma_i - \gamma_{i-1})$ . Then

$$\begin{aligned} 2 \int_0^{\gamma_N} \text{lch}(f_w) dx &= \sum_{i=1}^{N-1} (w_{i-1} + w_i)(\gamma_i - \gamma_{i-1}) \\ &= w_0(\gamma_1 - \gamma_0) + \sum_{i=2}^N w_{i-1}(\gamma_i - \gamma_{i-2}) + w_N(\gamma_N - \gamma_{N-1}) \\ &= w_0(\gamma_1) + \sum_{i=2}^N w_{i-1}(\gamma_i - \gamma_{i-2}) + w_N(1). \end{aligned}$$

Here the last line follows because  $\gamma_0 = 0$  and  $(\gamma_N - \gamma_{N-1}) = 1$ , since  $N \geq \text{c.i.}(\Gamma)$ . □

**Remark 2.17.** Once we are above the index of the conductor, all gaps between consecutive semigroup elements are 1, and hence, if the degree of the curve is large enough, the tail of  $a$  looks like  $(2, 2, \dots, 2, 1)$ . We can think of increasing the degree as merely adding an extra 2. This gives the moral reason why we might expect to see persistent optima, and motivates the Simplified Problem, discussed in the next subsection.

We are now ready for the main result of this section.

**Theorem 2.18.** *Let  $W$  and  $a$  be the cone and vector defined above for the monomial curve  $C_\Gamma$ .*

*Let  $\text{prox}(a, W)$  be the nearest point on the cone  $W$  to the vector  $a$  outside it.*

*Then  $\text{prox}(a, W)$  is the weight vector of a worst 1-PS for  $[C_\Gamma]$ .*

*Proof.* By definition, the worst 1-PS satisfies

$$\frac{\mu(x, \lambda^*)}{\|\lambda^*\|} = \sup_{\lambda} \frac{\mu(x, \lambda)}{\|\lambda\|}$$

By applying Lemma 2.14, then Proposition 2.11, and then Lemma 2.16, we have

$$\begin{aligned} \frac{\mu([C_\Gamma], w^*)}{\|w^*\|} &= \max_{w \in W} \frac{\mu([C_\Gamma], w)}{\|w\|} \\ &= \max_{w \in W} \frac{2 \int_Q \text{lch}(f_w)(x) dx}{\|w\|} \\ &= \max_{w \in W} \frac{a \cdot w}{\|w\|} \\ &= \max_{w \in W} \|a\| \cos(\theta(a, w)) \end{aligned}$$

Since  $\text{prox}(a, W)$  minimizes  $\theta(a, w)$ , the result follows. □

2.7.3. *The Simplified Problem.* In the sequel, it will be convenient to solve the optimisation problem for the vector  $a$  whose last coordinate is 2 rather than 1. We call this the Simplified Problem. We then use the solution to the Simplified Problem to construct a solution to the original problem, which we henceforth refer to as the Unsimplified Problem.

We illustrate an example in Figure 1. Let  $\Gamma = \langle 2, 3 \rangle$ , and consider the Simplified Problem when  $N = 10$ . Then we have  $a = (2, 3, 2, 2, 2, 2, 2, 2, 2, 2)$ , and the worst 1-PS has weights  $w = (\frac{33}{14}, \frac{157}{70}, \frac{153}{70}, \frac{149}{70}, \frac{29}{14}, \frac{141}{70}, 2, 2, 2, 2)$ . We plot the points  $(\gamma_i, a_i)$  in green, and plot a blue line graph through the points  $(\gamma_i, w_i)$ .

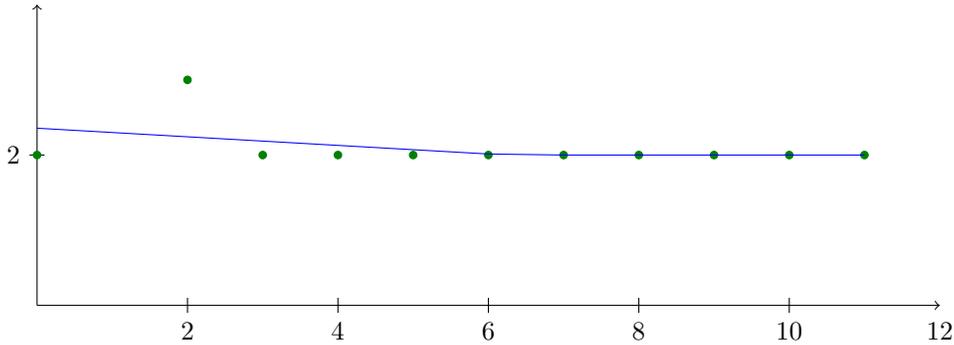


FIGURE 1. The vector  $a$  and the optimal weights  $w$  for the Simplified Problem for  $\Gamma = \langle 2, 3 \rangle$  and  $N = 10$

### 3. THE KKT MATRIX EQUATION AND ITS SOLUTIONS

To solve the optimisation problem described in the previous section, we will use the Karush-Kuhn-Tucker (KKT) conditions in nonlinear optimisation to study the closest point on the span of each face of  $W$  to the vector  $a$ .

Recall that we write  $\text{cond}(\Gamma)$  for the conductor of  $\Gamma$ , and  $\text{c.i.}(\Gamma)$  for its index. That is,  $\gamma_{\text{c.i.}(\Gamma)} = \text{cond}(\Gamma)$ . When we say there is a corner above the conductor, we mean that the set of corner indices  $I$  contains an element  $\ell$  greater than  $\text{c.i.}(\Gamma)$ , so that the point  $(\gamma_\ell, w_\ell)$  is to the right of the line  $x = \text{cond}(\Gamma)$ .

**3.1. The KKT conditions for convex optimisation.** The KKT conditions are necessary conditions satisfied by an optimal point for a broad class of nonlinear optimisation problems. See for instance [2, Section 5.5.3]. In many cases, including the problem considered here, they are also sufficient.

Suppose we want to minimize a function  $f(w)$  subject to the constraints  $g_i(w) \leq 0$  and  $h_j(w) = 0$ . The KKT conditions for the optimal point  $w^*$  are as follows.

- (1) Stationarity:  $\nabla f(w^*) + \sum \lambda_j \nabla h_j(w^*) + \sum \mu_i \nabla g_i(w^*) = 0$
- (2) Primal feasibility:  $g_i(w^*) \leq 0$  and  $h_j(w^*) = 0$  for all  $i, j$
- (3) Dual feasibility:  $\mu_i \geq 0$  for all  $i$
- (4) Complementary slackness:  $\mu_i g_i(w^*) = 0$  for all  $i$

In our case (computing the nearest point on a polyhedron to a point outside it), the objective function is a strongly convex quadratic function, and the constraints are given by affine inequalities. Therefore, the KKT conditions are also sufficient for obtaining an optimal  $w^*$  (see for instance [2, Section 5.5.3]). Finally, the optimal  $w^*$  is unique, since the objective function is strongly convex.

**3.1.1. The polyhedral cone  $W$  associated to  $\gamma$ .** For any  $\gamma$ , the cone  $W$  is defined by the conditions that the slopes of the line segments connecting the points  $(\gamma_i, w_i)$  are increasing. This yields  $N - 1$  halfspaces in  $\mathbb{R}^{N+1}$ . The polyhedral cone  $W$  therefore has a two-dimensional lineality space. The faces of  $W$  are easy to describe. A vector  $w$  lies on the facet  $m_i = m_{i+1}$  if and only there is no corner in the graph at  $(\gamma_i, w_i)$ .

The inequality

$$\frac{w_i - w_{i-1}}{\gamma_i - \gamma_{i-1}} \leq \frac{w_{i+1} - w_i}{\gamma_{i+1} - \gamma_i}$$

rearranges to

$$-(\gamma_{i+1} - \gamma_i)w_{i-1} + (\gamma_{i+1} - \gamma_{i-1})w_i - (\gamma_i - \gamma_{i-1})w_{i+1} \leq 0.$$

The following piecewise linear functions are useful when working with  $W$ .

**Definition 3.1.**

$$\begin{aligned} F_k(x) &:= \max\{-x + k, 0\} \\ L_{\gamma_N}(x) &:= \gamma_N - x \\ L_1(x) &:= 1 \end{aligned}$$

For a vector  $\gamma$ , we abuse notation and write  $F_k(\gamma)$  to denote the vector with coordinates  $F_k(\gamma_i)$  for all  $i$ . (We do this also for  $L_{\gamma_N}(x)$  and  $L_1(x)$ .)

**Proposition 3.2.** (1) Any piecewise linear function  $F(x)$  on  $[0, \gamma_N]$  with corners lying over integers  $0 \leq n \leq d$  is a linear combination of the functions  $\{F_k(x) : 1 \leq k \leq \gamma_n - 1\}$ ,  $L_{\gamma_N}(x)$ , and  $L_1(x)$ .  
(2) The lineality space of  $W$  is spanned by  $L_{\gamma_N}(\gamma)$  and  $L_1(\gamma)$ .  
(3) For each  $1 \leq i \leq N - 1$ , the vector  $F_{\gamma_i}(\gamma)$  spans a ray of  $W$ .  
(4) Let  $F(x)$  be a piecewise linear function on  $[0, \gamma_N]$  with corners lying over integers  $n \in \gamma$ . Suppose that the slopes of the line segments in the graph of  $F$  are negative and increasing, and  $F(x) \geq 0$ . Then  $F(x)$  is a nonnegative linear combination of the rays spanned by  $\{F_{\gamma_i}(\gamma) : 1 \leq i \leq N - 1\}$  and  $L_{\gamma_N}(\gamma)$  and  $L_1(\gamma)$ .

### 3.2. The KKT matrix equation.

**Definition 3.3.** We define the KKT matrix equation for  $\text{face}_{\gamma(I)}$  as follows.

First, we define an  $(N + 1) \times (N + 1)$  matrix  $A$  as follows. For  $1 \leq j \leq N - 1$ , if  $j \in I$ , then column  $j$  of  $A$  is given by  $2F_{\gamma_j}(\gamma)$ . If  $j \notin I$ , then column  $j$  of  $A$  is given by the coefficient vector of the inequalities  $g_j(w) \leq 0$ . The last two columns are given by  $2L_{\gamma_N}(\gamma)$  and  $2L_1(\gamma)$ .

Next, we define a vector  $x \in \mathbb{R}^{N+1}$  as follows. For  $1 \leq i \leq N - 1$ , if  $i \in I$ , then  $x_i$  is one of the parameters  $t_i$  used to parametrize  $\text{Spanface}_{\gamma(I)}$ . If  $i \notin I$ , then  $x_i$  is one of the KKT multipliers  $\mu_i$ . The last two coordinates  $x_N$  and  $x_{N+1}$  are the parameters used for the lineality space.

Then the KKT matrix equation for  $\text{face}_{\gamma(I)}$  is

$$Ax = 2a.$$

We have two versions of the vector  $a$ : one for the Simplified Problem, and one for the Unsimplified Problem. Most of our discussion will be dedicated to the Simplified Problem. Then, at the end, we will show how the result for the Simplified Problem implies the result for the Unsimplified Problem.

The matrices  $A$ ,  $x$ , and  $a$  all depend on  $N$ ,  $\Gamma$ , and  $I$ . However, since  $\Gamma$  and  $I$  are typically clear from context, we do not show them in our notation.

We record explicit formulas for the matrix  $A$  and the vector  $a$ .

**Proposition 3.4.** The entries of  $A$  are given by the following formula.

$$A_{i,j} = \begin{cases} \gamma_j - \gamma_{j+1} & \text{if } j \leq N - 1, j \notin I, \text{ and } i = j \\ \gamma_{j+1} - \gamma_{j-1} & \text{if } j \leq N - 1, j \notin I, \text{ and } i = j + 1 \\ \gamma_{j-1} - \gamma_j & \text{if } j \leq N - 1, j \notin I, \text{ and } i = j + 2 \\ 2(\gamma_j - \gamma_{i-1}) & \text{if } j \leq N - 1, j \in I, \text{ and } i \leq j \\ 2(\gamma_N - \gamma_{i-1}) & \text{if } j = N \\ 2 & \text{if } j = N + 1 \\ 0 & \text{otherwise} \end{cases}$$

The vector  $a$  on the right hand side of the KKT matrix equation is given by the following formula.

$$a_i = \begin{cases} \gamma_1 & \text{if } i = 1 \\ \gamma_i - \gamma_{i-2} & \text{if } 2 \leq i \leq N \\ 2 & \text{if } i = N + 1 \text{ (Simplified Problem)} \\ 1 & \text{if } i = N + 1 \text{ (Unsimplified Problem)} \end{cases}$$

The application to our problem is as follows.

**Proposition 3.5.** Let  $x$  be the solution of the KKT matrix equation for  $\text{face}_{\gamma(I)}$ . If  $x_i \geq 0$  for all  $1 \leq i \leq N - 1$ , then  $w(t_1, \dots, t_{k+2})$  is the closest point on  $W$  to  $a$ .

Moreover, if  $x_i > 0$  for all  $i \in I$ , then  $\text{face}_{\gamma(I)}$  is the smallest face of  $W$  containing this point.

**3.3. Persistence.** We begin with the following simple observation.

**Lemma 3.6.** *Fix  $\Gamma$ ,  $I$ , and  $N$ . Suppose that  $x$  is a solution to the KKT matrix equation for the Simplified Problem with  $x_N = 0$  and  $x_{N+1} = 2$ . Define  $x'$  by*

$$x'_i = \begin{cases} x_i, & i \leq N \\ 0, & i = N + 1 \\ 2, & i = N + 2 \end{cases}$$

*Then  $x'$  is a solution to the KKT matrix equation for  $\Gamma$  and  $I$  with embedding dimension  $N + 1$ .*

This motivates the following definition.

**Definition 3.7.** We call  $x$  a *persistent solution to the Simplified Problem* for the face associated to  $I$  if  $x_N = 0$  and  $x_{N+1} = 2$ .

We aim to prove the following result.

**Theorem** (Persistence of the global optimum for the Simplified Problem). *Let  $\Gamma$  be a numerical semigroup. There exists an integer  $N_0^{\text{simp}}$  and a set of corner indices  $I^{\text{simp}}$  (both depending on  $\Gamma$ ) such that for all  $N \geq N_0^{\text{simp}}$ , the global optimum for the Simplified Problem for  $\Gamma$  is the persistent solution for the face of  $W$  corresponding to  $I^{\text{simp}}$ .*

For small values of  $N$ , there can be non-persistent optima. However, our strategy is to show that such non-persistent optima obey bounds on  $N$ . Thus, for large  $N$ , the only remaining possibility is that there is a persistent optimum.

**3.4. How the stationary points vary with  $N$ .** Let  $I$  be a set of corners. For each  $N$ , let  $x(N)$  be the solution to the KKT matrix equation for the corresponding face of  $W$ .

By Cramer's Rule, we have a formula for each coordinate in the solution  $x(N)$  in terms of determinants.

$$x_j(N) = \frac{\det A(N, j)}{\det A(N)}.$$

The last three coordinates  $x_{N-1}$ ,  $x_N$ , and  $x_{N+1}$  play a special role in the discussion, and so we have special notation for their numerators.

Recall that under our notation conventions,  $x_N$  and  $x_{N+1}$  are parameters.  $x_{N+1} = w_N$  is the  $y$ -value of the last point on the graph of  $w$ , and  $-x_N$  is the slope of the last line segment in the graph of  $w$ . Neither of these two quantities is required to be nonnegative—they parametrize the lineality space of the polyhedron  $W$ .

**Definition 3.8.** Define functions

$$\begin{aligned} Q(N) &= (-1)^{N+1} \det A(N) \\ P_j(N) &= (-1)^{N+1} \det A(N, j) \\ \chi(N) &= (-1)^{N+1} \det A(N, N-1) \\ \psi(N) &= (-1)^{N+1} \det A(N, N) \\ \omega(N) &= (-1)^{N+1} \det A(N, N+1) \end{aligned}$$

(This notation is chosen as a mnemonic device. Recall that in the Greek alphabet, the last three letters in order are  $\chi$ ,  $\psi$ , and  $\omega$ .)

Then we have

$$\begin{aligned} x_j &= \frac{\det A(N, j)}{\det A} = \frac{(-1)^{N+1} \det A(N, j)}{(-1)^{N+1} \det A} = \frac{P_j}{Q} \\ x_{N-1} &= \frac{\det A(N, N-1)}{\det A} = \frac{(-1)^{N+1} \det A(N, N-1)}{(-1)^{N+1} \det A} = \frac{\chi}{Q} \\ x_N &= \frac{\det A(N, N)}{\det A} = \frac{(-1)^{N+1} \det A(N, N)}{(-1)^{N+1} \det A} = \frac{\psi}{Q} \\ x_{N+1} &= \frac{\det A(N, N+1)}{\det A} = \frac{(-1)^{N+1} \det A(N, N+1)}{(-1)^{N+1} \det A} = \frac{\omega}{Q} \end{aligned}$$

Finally, for  $j \leq N$ , we define  $A'(N, j)$  as the matrix obtained by replacing column  $j$  in  $A(N)$  by  $2a - \underline{2}$ . Since the last column of  $A(N, j)$  is  $\underline{2}$ , the matrices  $A(N, j)$  and  $A'(N, j)$  are column equivalent, and thus  $\det A(N, j) = \det A'(N, j)$ .

**Lemma 3.9.** *Suppose  $N \geq \max\{\text{c.i.}(\Gamma) + 2, \max(I) + 2\}$ .*

- (1) *The functions  $Q(N)$ ,  $\chi(N)$ ,  $\psi(N)$ , and  $\omega(N)$  are polynomials with the following degree bounds.*
  - (a)  $\deg Q = 4$
  - (b)  $\deg \chi \leq 3$
  - (c)  $\deg \psi \leq 2$
  - (d)  $\deg \omega = 4$ .
- (2) *If  $N \geq \max\{\text{c.i.}(\Gamma) + 2, \max(I) + 2\}$  and  $N \geq j + 1$ , the function  $P_j(N)$  is a polynomial of degree at most 4.*

We give a full proof in Appendix A. Here we only sketch the proof. As  $N$  grows, the matrices  $A(N)$  and  $A'(N, j)$  grow in a way that is easy to describe. We can write recurrences for the matrices, then show that the appropriate difference equations vanish to establish that each of these functions is a polynomial in  $N$  with the degree bounds claimed. (This is a paper-and-pencil proof checked by computer—the proof does not rely on computer calculations.)

**Corollary 3.10.** *Let  $x$  be the solution of the KKT matrix equation for face  $I$ . Then each coordinate of  $x$  is a rational function in  $N$ .*

Fix a positive integer  $k$ . We use the following notation for the Taylor expansions of these polynomials centered at  $k$ .

$$\begin{aligned}\chi(N) &= \chi_3(N - k)^3 + \chi_2(N - k)^2 + \chi_1(N - k) + \chi_0 \\ \psi(N) &= \psi_2(N - k)^2 + \psi_1(N - k) + \psi_0\end{aligned}$$

When  $k$  is sufficiently large, we have explicit formulas for these coefficients in terms of minors of the matrices  $A'$ .

**Lemma 3.11.** *Suppose that  $k \geq \max\{\text{c.i.}(\Gamma) + 2, \max(I) + 2\}$ .*

(1)

$$\begin{aligned}\chi_3 &= \frac{1}{3}(-1)^{k+1} \left( 2\delta_{k,k+1}^{k,k+1} A'(k, k-1) + 2\delta_{k-1,k+1}^{k,k+1} A'(k, k-1) \right) \\ \chi_2 &= (-1)^{k+1} \left( 4\delta_{k,k+1}^{k,k+1} A'(k, k-1) + 2\delta_{k-1,k+1}^{k,k+1} A'(k, k-1) \right) \\ \chi_1 &= \frac{1}{3}(-1)^{k+1} \left( -14\delta_{k,k+1}^{k,k+1} A'(k, k-1) - 8\delta_{k-1,k+1}^{k,k+1} A'(k, k-1) - 6\delta_{k+2}^{k+1} A'(k+1, k) \right) \\ \chi_0 &= (-1)^{k+1} \det A'(k, k-1).\end{aligned}$$

(2)

$$\begin{aligned}\psi_2 &= (-1)^{k+1} \left( \delta_{k+1}^{k+1} A'(k, k) + \delta_k^{k+1} A'(k, k) \right) \\ \psi_1 &= (-1)^{k+1} \left( 3\delta_{k+1}^{k+1} A'(k, k) + \delta_k^{k+1} A'(k, k) \right) \\ \psi_0 &= (-1)^{k+1} \det A'(k, k).\end{aligned}$$

We give a full proof in Appendix A. Here we only sketch the proof. We can get the leading coefficients  $\chi_3$  and  $\psi_2$  using difference equations via a procedure similar to the proof of 3.9. We can solve for the remaining coefficients by interpolating these polynomials using their values when  $N = k$ ,  $N = k + 1$ ,  $N = k + 2$ .

Next, we give a formula for  $x_j$ .

**Lemma 3.12.** *For each  $j$  satisfying  $j \geq \max\{\text{c.i.}(\Gamma) + 2, I + 2\}$  and  $j < N$ , we have*

$$x_j = (N - j)(N - j + 1) \left( \frac{1}{2}x_{N-1} + \frac{1}{3}(N - j - 1)x_N \right)$$

Hence, after clearing the common denominator  $Q$ , we have

$$(3.1) \quad P_j = (N - j)(N - j + 1) \left( \frac{1}{2}\chi + \frac{1}{3}(N - j - 1)\psi \right)$$



□

## 4. IF ALL THE CORNERS ARE AT OR BELOW THE CONDUCTOR

By Lemma 3.9, the polynomial  $\chi$  has degree at most 3. From a computer search of numerical semigroups of genus  $1 \leq g \leq 14$ , we see that in general,  $\chi$  has degree 3, but we also found examples where the degree drops. The next lemma describes what happens when the degree of  $\chi$  drops.

**Lemma 4.1.** *Suppose that  $\max(I) \leq \text{c.i.}(\Gamma)$ . Suppose  $k \geq \text{c.i.}(\Gamma) + 2$ , and consider the Taylor expansions of  $\chi$  and  $\psi$  centered at  $k$ .*

- (1) *If  $\chi_3 = 0$ , then  $\psi_2 = 0$ , and  $\chi_2 = -\psi_1$ .*
- (2) *If  $\chi_3 = \chi_2 = 0$ , then  $\chi_1 = 0$ , and  $\psi_2 = \psi_1 = \psi_0 = 0$ .*

*Proof.* Proof of Part (1): By Lemma 3.13, we have  $\psi_2 = -\frac{3}{2}\chi_3$ . For the second equation, we use the formulas from Lemma 3.11. Since  $\chi_3 = 0$ , we have

$$\delta_{k,k+1}^{k,k+1} A'(k, k-1) = -\delta_{k-1,k+1}^{k,k+1} A'(k, k-1),$$

and hence

$$\chi_2 = (-1)^{k+1} 2\delta_{k,k+1}^{k,k+1} A'(k, k-1).$$

Since  $\psi_2 = 0$ , we have

$$\delta_{k+1}^{k+1} A'(k, k) = -\delta_k^{k+1} A'(k, k),$$

and hence

$$\psi_1 = (-1)^{k+1} (-2)\delta_k^{k+1} A'(k, k).$$

Expanding  $d_k^{k+1} A'(k, k)$  along its bottom row yields

$$\delta_k^{k+1} A'(k, k) = \delta_{k,k+1}^{k-1,k+1} A'(k, k)$$

and the minors  $d_{k,k+1}^{k-1,k+1} A'(k, k)$  and  $d_{k,k+1}^{k,k+1} A'(k, k-1)$  are the same.

Hence, we obtain  $\chi_2 = -\psi_1$ .

Proof of Part (2): By Part (1), we have  $\psi_2 = \psi_1 = 0$ . First, we prove that  $\psi_0 = 0$ .

Similarly, from the formulas for  $\psi_2$  and  $\psi_1$  given above, we see that when both  $\psi_2$  and  $\psi_1$  vanish, we must have

$$\begin{aligned} \delta_{k+1}^{k+1} A'(k, k) &= 0 \\ \delta_k^{k+1} A'(k, k) &= 0. \end{aligned}$$

By expanding  $\delta_k^{k+1} A'(k, k)$  along its bottom row, we obtain

$$(4.1) \quad \delta_{k,k+1}^{k-1,k+1} A'(k, k) = 0.$$

Now we study  $A'(k+1, k+1)$ . By expanding along its bottom row, we obtain

$$(4.2) \quad \det A'(k+1, k+1) = -\delta_{k+2}^k A'(k+1, k+1) + 2\delta_{k+2}^{k+2} A'(k+1, k+1).$$

Expanding  $\delta_{k+2}^k A'(k+1, k+1)$  along its bottom row yields

$$\delta_{k+2}^k A'(k+1, k+1) = \delta_{k,k+2}^{k,k+2} A'(k+1, k+1)$$

Expanding this along its bottom row yields

$$\delta_{k,k+2}^{k,k+2} A'(k+1, k+1) = \delta_{k,k+1,k+2}^{k-1,k,k+2} A'(k+1, k+1)$$

But we have

$$D_{k,k+1,k+2}^{k-1,k,k+2} A'(k+1, k+1) = D_{k,k+1}^{k-1,k+1} A'(k, k)$$

and by (4.1), we have  $\delta_{k,k+1}^{k-1,k+1} A'(k, k) = 0$ . Thus, the first term on the right hand side of equation (4.2) vanishes.

Now consider the second term in equation (4.2). Expanding along its bottom row, we have

$$\delta_{k+2}^{k+2} A'(k+1, k+1) = -\delta_{k+1,k+2}^{k-1,k+2} A'(k+1, k+1) - 2\delta_{k+1,k+2}^{k,k+2} A'(k+1, k+1)$$

Both of these terms vanish. Indeed,

$$\begin{aligned}\delta_{k+1,k+2}^{k-1,k+2}A'(k+1,k+1) &= \delta_{k,k+1,k+2}^{k-1,k+1,k+2}A'(k+1,k+1) \\ &= \delta_{k,k+1}^{k-1,k+1}A'(k,k) \\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\delta_{k+1,k+2}^{k,k+2}A'(k+1,k+1) &= \delta_{k+1}^{k+1}A'(k,k) \\ &= 0.\end{aligned}$$

This completes the proof that  $\psi_0 = 0$ .

Finally, we prove that  $\chi_1 = 0$ . From the formulas for  $\chi_3$  and  $\chi_2$  given in Part (1) above, we see that when both  $\chi_3$  and  $\chi_2$  vanish, we must have

$$\begin{aligned}\delta_{k,k+1}^{k,k+1}A'(k,k-1) &= 0 \\ \delta_{k-1,k+1}^{k,k+1}A'(k,k-1) &= 0.\end{aligned}$$

These are the first two terms in the formula for  $\chi_1$  from Lemma 3.11. The remaining term is a multiple of  $\delta_{k+2}^{k+1}A'(k+1,k)$ . But we have

$$D_{k+2}^{k+1}A'(k+1,k) = A'(N,N)$$

and  $\det A'(N,N) = 0$ , since  $\psi_0 = 0$ . This shows that  $\chi_1 = 0$ . □

*Remark:* A computer search shows that for all numerical semigroups of genus  $g \leq 14$ , if  $\chi_3 = \chi_2 = 0$ , then  $\chi_0 = 0$ , too, so that  $\chi(N) \equiv 0$  and  $\psi(N) \equiv 0$ . However, we do not know whether this always holds.

**Definition 4.2.** We write  $\nu(\Gamma, I)$  for the smallest positive integer that is greater than the real roots of the polynomials  $\chi$  and  $(\chi + 2(N - \text{c.i.}(\Gamma) - 2)\psi)$ , and greater than  $\text{c.i.}(\Gamma) + 2$  and  $\max(I) + 2$ . We abbreviate this as  $\nu$  when  $\Gamma$  and  $I$  are clear from context.

**Proposition 4.3.** *Suppose that  $\max(I) \leq \text{c.i.}(\Gamma)$ . If  $N > \nu$ , then the global optimum for the Simplified Problem does not lie on face  $I$ .*

*Proof.* We will break into cases according to the degree of  $\chi(N)$ . We have  $\deg \chi(N) \leq 3$ . In the general case, when  $\deg \chi(N) = 3$ , we will show that for sufficiently large  $N$ , the last line segment in the graph corresponding to the stationary point on face  $I$  crosses the line  $y = 2$  past the conductor; hence, by Lemma 3.14, it will not be the global optimum. We will treat the cases where  $\deg \chi(N) < 3$  separately.

Write  $\ell = \max(I)$ . Consider the last line segment  $L$  in the graph. It connects the points  $(\gamma_\ell, w_\ell)$  and  $(\gamma_N, w_N)$ . Let  $k = \text{c.i.}(\Gamma) + 2$ . Since  $\ell \leq \text{c.i.}(\Gamma)$ , the point  $(\gamma_k, w_k)$  lies on the line segment  $L$ . We will show that if  $\deg \chi(N) \geq 2$ , the leading terms of  $2 - w_k$  and  $2 - w_N$  have opposite signs.

We have

$$\begin{aligned}2 - w_N &= 2 - x_{N+1} \\ &= -\frac{1}{2}x_{N-1} \\ &= -\frac{\chi}{2Q}.\end{aligned}$$

By our notation conventions, this line segment has slope  $-x_N$  and goes through the point  $(\gamma_N, w_N)$ . Hence, it has equation  $y - x_{N+1} = -x_N(x - \gamma_N)$ . We have  $w_k - x_{N+1} = -x_N(\gamma_k - \gamma_N)$ . Since  $k$  and  $N$  are both larger than  $\text{c.i.}(\Gamma)$ , we have  $\gamma_k - \gamma_N = k - N$ . Thus  $w_k = x_{N+1} - x_N(k - N)$ . Then

$$\begin{aligned}2 - w_k &= 2 - x_{N+1} - (N - k)x_N \\ &= -\frac{1}{2}x_{N-1} - (N - k)x_N \\ &= -\frac{1}{2}(x_{N-1} + 2(N - k)x_N) \\ &= -\frac{(\chi + 2(N - k)\psi)}{2Q}.\end{aligned}$$

Choose  $k$  as the center for the Taylor expansions of  $\chi$  and  $\psi$ . Then we have

$$2 - w_k = -\frac{((\chi_3 + 2\psi_2)(N - k)^3 + (\chi_2 + 2\psi_1)(N - k)^2 + (\chi_1 + 2\psi_0)(N - k) + \chi_0)}{2Q}$$

We now split into cases according to the degree of  $\chi(N)$ .

*Case 1:*  $\deg \chi = 3$ . Thus,  $\chi_3 \neq 0$ . Then by Lemma 3.13, we have  $\psi_2 = -\frac{3}{2}\chi_3$ . Thus, the leading coefficient of the numerator of  $2 - w_k$  is  $\chi_3 + 2\psi_2 = -2\chi_3$ , while the leading coefficient of the numerator of  $2 - w_N$  is  $\chi_3$ . When  $N > \nu$ , the signs of these polynomials are determined by the signs of their leading coefficients. Hence, for all  $N > \nu$ , the line segment  $L$  crosses the line  $y = 2$  after the conductor of  $\Gamma$ . By Lemma 3.14 it follows that the stationary point on face  $I$  will not be the global optimum for the Simplified Problem for  $N$  in this range.

*Case 2:*  $\deg \chi = 2$ . Thus,  $\chi_3 = 0$ , but  $\chi_2 \neq 0$ . By Lemma 4.1, we have  $\psi_2 = 0$  and  $\chi_2 = -\psi_1$ .

The leading coefficient of the numerator of  $2 - w_N$  is  $\chi_2$ . The leading coefficient of the numerator of  $2 - w_k$  is  $\chi_2 + 2\psi_1 = -\chi_2$ . Since these have opposite signs, we may argue as we did in Case 1. When  $N > \nu$ , the signs of these polynomials are determined by the signs of their leading coefficients. Hence, for all  $N > \nu$ , the line segment  $L$  crosses the line  $y = 2$  after the conductor of  $\Gamma$ . By Lemma 3.14 it follows that the stationary point on face  $I$  will not be the global optimum for the Simplified Problem for  $N$  in this range.

*Case 3:*  $\deg \chi = 1$ . By Lemma 4.1, this does not occur.

*Case 4:*  $\deg \chi = 0$ , but  $\chi_0 \neq 0$ .

By Lemma 4.1, we have  $\psi(N) \equiv 0$ , and  $\chi(N) = \chi_0$ .

Sum the KKT equations starting in row  $\ell + 2$ , where  $\ell = \max I$ . We omit  $x_N$ , since  $x_N = 0$ . We obtain

$$(\gamma_{\ell+2} - \gamma_{\ell+1})x_{\ell+1} + 2(N - \ell)x_{N+1} = 4(N - \ell)$$

Thus

$$\begin{aligned} (\gamma_{\ell+2} - \gamma_{\ell+1})x_{\ell+1} &= (N - \ell)(4 - 2x_{N+1}) \\ &= (N - \ell)(-x_{N-1}) \\ &= \frac{(N - \ell)(-\chi_0)}{Q(N)}. \end{aligned}$$

Thus, this stationary point cannot be the global optimum for any  $N$ , since one of the KKT multipliers  $x_{\ell+1}$  or  $x_{N-1}$  is negative.

*Case 5:*  $\chi(N) \equiv 0$ .

We will study rows c.i. +1,  $\dots$ ,  $N - 1$  of the KKT matrix equation, and eventually split into four further cases.

Since  $\chi \equiv 0$  and  $\psi \equiv 0$ , we have  $x_{N-1} = x_N = 0$ . The last row of the KKT matrix equation is  $2x_{N+1} = 4$ , so  $x_{N+1} = 2$ .

The penultimate row of the KKT matrix equation is  $-x_{N-2} + 2x_{N-1} + 2x_N + 2x_{N+1} = 4$ . With  $x_{N-1} = x_N = 0$  and  $x_{N+1} = 2$ , we have  $x_{N-2} = 0$ .

By induction, using rows c.i. +2,  $\dots$ ,  $N + 1$  of the KKT matrix equation, we may show that  $x_i = 0$  for c.i.  $\leq i \leq N - 1$ . In all these rows, the right hand side is 4.

In row c.i. +1, on the right hand side, we have  $2a_{c.i.+1} = 2(\gamma_{c.i.+1} - \gamma_{c.i.-1})$ . This is because  $\gamma_{c.i.+1} - \gamma_{c.i.-1} = (\gamma_{c.i.+1} - \gamma_{c.i.}) + (\gamma_{c.i.} - \gamma_{c.i.-1})$ , and  $\gamma_{c.i.+1} - \gamma_{c.i.} = 1$  while  $\gamma_{c.i.} - \gamma_{c.i.-1} > 1$  by the definition of the conductor.

We split into cases that determine the left hand side of row c.i. +1 of the KKT matrix equation.

*Case 5a:*  $\chi(N) \equiv 0$  and  $\ell < c.i. - 1$ .

Then row c.i. +1 says

$$(\gamma_{c.i.-2} - \gamma_{c.i.-1})x_{c.i.-1} + 4 = 2a_{c.i.+1}.$$

Since  $(\gamma_{c.i.-2} - \gamma_{c.i.-1}) < 0$  and  $2a_{c.i.+1} > 4$ , this implies  $x_{c.i.-1} < 0$ . Thus, this stationary point cannot be the global optimum for any  $N$ .

*Case 5b:*  $\chi(N) \equiv 0$  and  $\ell = c.i. - 1$ . In this case, row c.i. +1 says  $4 = 2a_{c.i.+1} > 4$ , a contradiction.

*Case 5c:*  $\chi(N) \equiv 0$ ,  $\ell = c.i.$ , and  $\ell - 1 \notin I$ . This case is similar to Case 5a. In this case, row c.i. +1 says

$$(\gamma_{c.i.-2} - \gamma_{c.i.-1})x_{c.i.-1} + 4 = 2a_{c.i.+1}.$$

Since  $(\gamma_{c.i.-2} - \gamma_{c.i.-1}) < 0$  and  $2a_{c.i.+1} > 4$ , this implies  $x_{c.i.-1} < 0$ . Thus, this stationary point cannot be the global optimum for any  $N$ .

*Case 5d:*  $\chi(N) \equiv 0$ ,  $\ell = c.i.$ , and  $\ell - 1 \in I$ . This case is imilar to Case 5b. In this case, row  $c.i.+1$  says  $4 = 2a_{c.i.+1} > 4$ , a contradiction.  $\square$

**Corollary 4.4.** *Let  $N_1 = \max_{I: \max(I) \leq c.i.(\Gamma)} \{\nu(\Gamma, I)\}$ . If  $N > N_1$ , then the global optimum for the Simplified Problem has at least one corner above the conductor.*

## 5. IF THERE IS AT LEAST ONE CORNER ABOVE THE CONDUCTOR

In this section, we consider the case where there is at least one corner above the conductor.

Let  $\ell = \max(I)$ . We must have  $N \geq \ell + 1$ . We will split into two cases:  $N = \ell + 1$ , and  $N > \ell + 1$ . There are two reasons for this case split. First, the behavior of the stationary points is different in these two cases. Second, Lemmas 3.9 and 3.11 require  $N \geq \ell + 2$ .

### 5.1. If $N = \ell + 1$ .

**Proposition 5.1.** *Let  $N'$  be an integer with  $N' \geq c.i.(\Gamma) + 1$ . Suppose that the global optimum for the Simplified Problem for  $N = N'$  occurs on a face  $I$  with  $c.i.(\Gamma) < \ell = N' - 1$ , where  $\ell = \max(I)$ . Then either the stationary point on face  $I$  is persistent, or the stationary point on face  $I' = I \cup \{\ell + 1\}$  is persistent.*

*Proof.* Consider the KKT matrix equation when  $N = N'$ . The last row corresponds to the equation  $2x_{N+1} = 4$ , so we have  $x_{N+1} = 2$ . If this solution also has  $x_N = 0$ , then it is persistent.

So suppose that  $x_N \neq 0$ . Let  $I' = I \cup \{\ell + 1\}$ , and consider the KKT matrix equation when  $N = N' + 1 = \ell + 2$ . The last two rows correspond to the equations  $2x_N + 2x_{N+1} = 4$  and  $2x_{N+1} = 4$ . Hence  $x_N = 0$  and  $x_{N+1} = 2$ , so we have a persistent solution.  $\square$

**Definition 5.2.** We say that a set of corners  $I$  heralds persistence if the stationary point on face  $I$  is not persistent, but the stationary point on face  $I' = I \cup \{\ell + 1\}$  is persistent, where  $\ell = \max I$ .

**5.2. If  $N > \ell + 1$ .** When there is at least one corner above the conductor, and  $N > \ell + 1$ , we will show that the polynomials  $\chi$  and  $\psi$  factor in a specific form, and their signs are determined by their leading coefficients. It is thus easier to relate optimality and persistence in this case than it is when all the corners are below the conductor.

**Lemma 5.3.** *Write  $\ell = \max I$ . Suppose  $\ell \geq c.i.(\Gamma)$ , and  $N > \ell + 1$ . Then*

$$\chi = -\frac{2}{3}(N - \ell - 1)\psi.$$

*Proof.* Sum the KKT equations starting in row  $\ell + 1$ , where  $\ell = \max I$ . For each  $j$  satisfying  $\ell + 2 \leq j \leq N - 2$ , the sum in column  $j$  is 0. Furthermore, since  $\ell \geq c.i.$ , in column  $\ell + 1$  we have the entries  $-1, 2, -1$  in rows  $\ell, \ell + 1, \ell + 2$ .

We obtain

$$x_{\ell+1} + \left( \sum_{i=0}^{N-\ell-1} 2i \right) x_N + 2(N - \ell)x_{N+1} = 4(N - \ell)$$

Rearranging and substituting  $-x_{N-1} = 4 - 2x_{N+1}$  yields

$$(5.1) \quad x_{\ell+1} = (N - \ell)(-x_{N-1} - (N - \ell - 1)x_N)$$

By Lemma 3.12 we also have

$$(5.2) \quad x_{\ell+1} = (N - \ell - 1)(N - \ell) \left( \frac{1}{2}x_{N-1} + \frac{1}{3}(N - \ell - 2)x_N \right)$$

Combining equations (5.1) and (5.2) yields

$$(N - \ell)(-x_{N-1} - (N - \ell - 1)x_N) = (N - \ell - 1)(N - \ell) \left( \frac{1}{2}x_{N-1} + \frac{1}{3}(N - \ell - 2)x_N \right)$$

Cancelling the factor  $(N - \ell)$  on both sides yields

$$-x_{N-1} - (N - \ell - 1)x_N = (N - \ell - 1) \left( \frac{1}{2}x_{N-1} + \frac{1}{3}(N - \ell - 2)x_N \right).$$

This rearranges to

$$x_{N-1} = -\frac{2}{3}(N - \ell - 1)x_N.$$

Cancelling the common denominator  $Q$  yields the desired result.  $\square$

**Lemma 5.4.** *A polynomial  $c_2(x - \ell - 1)^2 + c_1(x - \ell - 1) + c_0$  factors as  $c_2(x - \ell)(x - \ell + 1)$  if and only if  $c_0 = 2c_2$  and  $c_1 = 3c_2$ .*

*Proof.* Left to the reader.  $\square$

**Proposition 5.5.** *Write  $\ell = \max I$ . Suppose  $\ell \geq \text{c.i.}(\Gamma)$ , and  $N > \ell + 1$ . Then the polynomials  $\chi$  and  $\psi$  factor as follows.*

$$\begin{aligned}\chi &= (N - \ell - 1)(N - \ell)(N - \ell + 1)\chi_3 \\ \psi &= (N - \ell)(N - \ell + 1)\psi_2\end{aligned}$$

*Proof.* First, we prove the claim for the polynomial  $\psi$ .

Lemma 3.11 gives formulas for the Taylor expansion of  $\psi$  with center  $k = \ell + 1$ . We have:

$$\psi = \psi_2(N - \ell - 1)^2 + \psi_2(N - \ell - 1) + \psi_0$$

where

$$\begin{aligned}\psi_2 &= (-1)^{\ell+2} (\delta_{\ell+2}^{\ell+2} A'(\ell + 1, \ell + 1) + \delta_{\ell+1}^{\ell+2} A'(\ell + 1, \ell + 1)) \\ \psi_1 &= (-1)^{\ell+2} (3\delta_{\ell+2}^{\ell+2} A'(\ell + 1, \ell + 1) + \delta_{\ell+1}^{\ell+2} A'(\ell + 1, \ell + 1)) \psi_0 = (-1)^{\ell+2} A'(\ell + 1, \ell + 1).\end{aligned}$$

By Lemma 5.4, to achieve the desired result, it is enough to check that  $\psi_0 = 2\psi_2$  and  $\psi_1 = 3\psi_2$ .

The bottom row of the matrix  $A'(\ell + 1, \ell + 1)$  is 0, except in the last column, where it is 2. Thus expanding along the bottom row yields

$$\det A'(\ell + 1, \ell + 1) = \delta_{\ell+2}^{\ell+2} A'(\ell + 1, \ell + 1).$$

Also,  $\delta_{\ell+1}^{\ell+2} A'(\ell + 1, \ell + 1) = 0$  because this bottom row of this minor is 0.

We have  $\psi_0 = 2\psi_2$  and  $\psi_1 = 3\psi_2$ , and hence the polynomial  $\psi$  factors as claimed.

To prove the desired claim for  $\chi$ , first we apply Lemma 5.3 to get

$$\chi = -\frac{2}{3}(N - \ell - 1)\psi = -\frac{2}{3}\psi_2(N - \ell - 1)(N - \ell)(N - \ell + 1).$$

Next, by Lemma 3.13, we have  $\chi_3 = -\frac{2}{3}\psi_2$ . This yields the desired result.  $\square$

**Proposition 5.6.** *Write  $\ell = \max I$ . Suppose  $\ell \geq \text{c.i.}(\Gamma)$ . If the global optimum for the Simplified Problem lies on face  $I$  for at least one  $N$ , then either this solution is persistent, or it heralds persistence.*

*Proof.* If  $N = \ell + 1$ , then Proposition 5.1 gives the desired result.

So suppose  $N > \ell + 1$ , and suppose that the global optimum lies on face  $I$  for this  $N$ . We split into two cases according to whether  $\psi_2$  is 0.

*Case 1.* Suppose that  $\psi_2 \neq 0$ . We argue as we did in the proof of Proposition 4.3. Consider the last line segment  $L$  in the graph. It connects the points  $(\gamma_\ell, w_\ell)$  and  $(\gamma_N, w_N)$ .

We will show that the leading terms of  $2 - w_\ell$  and  $2 - w_N$  have opposite signs. Computing as we did in the proof of Proposition 4.3, we have

$$2 - w_N = -\frac{\chi}{2Q} = \frac{-\chi_3(N - \ell - 1)(N - \ell)(N - \ell + 1)}{2Q}$$

and

$$2 - w_\ell = -\frac{(\chi + 2(N - \ell)\psi)}{2Q}.$$

By Lemma

$$\begin{aligned}
\chi + 2(N - \ell)\psi &= -\frac{2}{3}(N - \ell - 1)\psi + 2(N - \ell)\psi \\
&= \frac{4}{3}(N - \ell)\psi + \frac{2}{3}\psi \\
&= \frac{4}{3}(N - \ell + \frac{1}{2})\psi \\
&= \frac{4}{3}\psi_2(N - \ell + \frac{1}{2})(N - \ell)(N - \ell + 1).
\end{aligned}$$

Thus

$$2 - w_\ell = \frac{-\frac{4}{3}\psi_2(N - \ell + \frac{1}{2})(N - \ell)(N - \ell + 1)}{2Q}$$

Since  $\chi_3 = -\frac{2}{3}\psi_2$ , this implies that  $2 - w_\ell$  and  $2 - w_N$  have opposite signs. By Lemma 3.14 it follows that the stationary point on face  $I$  will not be the global optimum for the Simplified Problem for  $N$ . This contradicts the hypothesis that  $I$  carries the global optimum for this  $N$ .

*Case 2.* Suppose that  $\psi_2 = 0$ . Then by Lemma 5.5 and Lemma 3.13, we have  $\psi \equiv 0$  and  $\chi \equiv 0$ , hence the solution is persistent.  $\square$

## 6. PERSISTENCE FOR THE SIMPLIFIED PROBLEM AND THE WORST 1-PS

### 6.1. Persistence for the Simplified Problem.

**Theorem 6.1** (Persistence of the global optimum for the Simplified Problem). *Let  $\Gamma$  be a numerical semigroup. There exists an integer  $N_0^{\text{simp}}$  and a set of corner indices  $I^{\text{simp}}$  (both depending on  $\Gamma$ ) such that for all  $N \geq N_0^{\text{simp}}$ , the global optimum for the Simplified Problem for  $\Gamma$  is the persistent solution for the face of  $W$  corresponding to  $I^{\text{simp}}$ .*

*Proof.* Let  $N_1 = \max_{I: \max(I) \leq c.i.(\Gamma)} \{\nu(\Gamma, I)\}$  as in Corollary 4.4. Consider the global optimum for the Simplified Problem when  $N = N_1 + 1$ . By the corollary, this solution has at least one corner above the conductor. By Proposition 5.6, it is either persistent, or it heralds a persistent solution.

Thus, the global optimum for the Simplified Problem when  $N = N_1 + 2$  must be persistent. Take  $N_0^{\text{simp}} = N_1 + 2$  to complete the proof.  $\square$

### 6.2. Persistence for the worst 1-PS.

**Lemma 6.2.** *Consider the set of points  $\{(0, 0), \dots, (N - \ell - 1, 0), (N - \ell, -1)\}$ . Then the least squares regression line for this set of points has slope  $m$  and  $y$ -intercept  $b'$  given by the following formulas.*

$$\begin{aligned}
m &= \frac{-6}{(N - \ell + 1)(N - \ell + 2)} \\
b' &= \frac{2(N - \ell - 1)}{(N - \ell + 1)(N - \ell + 2)}
\end{aligned}$$

**Theorem 6.3** (Persistence of the worst 1-PS). *Let  $\Gamma$  be a numerical semigroup.*

- (1) *There exists an integer  $N_0$  and a set of corner indices  $I$  (both depending on  $\Gamma$ ) such that for all  $N \geq N_0$ , the global optimum for the Unsimplified Problem for  $\Gamma$  lies on the face of  $W$  corresponding to  $I$ , and on no smaller face.*
- (2) *Write  $\ell = \max I$ . For all  $N \geq N_0$ , the coordinates  $w_0^*, \dots, w_{\ell-1}^*$  are constant with respect to  $N$ , and the coordinates  $w_\ell^*, \dots, w_N^*$  are given by the formula*

$$w_i^* = mi + b$$

where

$$\begin{aligned}
m &= \frac{-6}{(N - \ell + 1)(N - \ell + 2)} \\
b &= 2 + \frac{2(N + 2\ell - 1)}{(N - \ell + 1)(N - \ell + 2)}
\end{aligned}$$

*Proof.* We modify the solution to the Simplified Problem to obtain a solution to the Unsimplified Problem.

Choose  $N_0 \geq N_0^{\text{simp}}$ . Let  $I^{\text{simp}}$  be the set of corner indices giving the persistent solution to the Simplified Problem. Let  $\ell = \max(I^{\text{simp}})$ . We take  $I = I^{\text{simp}} \cup \{\ell - 1\}$ . (Note:  $\ell - 1$  may already be in  $I^{\text{simp}}$ , in which case  $I = I^{\text{simp}}$ .) By Corollary 4.4 we have  $\ell > \text{c.i.}(\Gamma)$ .

Let  $x$  be the solution to the KKT matrix equation for the Simplified Problem on face  $I$ . If  $I = I^{\text{simp}}$ , it is the global optimum to the Simplified Problem. If  $I \neq I^{\text{simp}}$ , the only difference is that we added an unused corner at index  $\ell - 1$ . So the solution  $x$  gives the global optimum to the Simplified Problem, but with the parameter  $x_{\ell-1} = 0$ .

We will define a vector  $\tilde{x}$  and then show that it is the solution to the KKT matrix equation for the Unsimplified Problem.

For the first  $\ell - 2$  coordinates of  $\tilde{x}$ , we use the solution  $x$ .

Set  $\tilde{x}_N$  and  $\tilde{x}_{N+1}$  to give the least squares regression line through the points  $\{(\gamma_\ell, 2), \dots, (\gamma_{N-1}, 2), (\gamma_N, 1)\}$ . (We can translate the least squares regression line described in Lemma 6.2 by  $(\gamma_\ell, 2)$  to obtain the desired line.) Recursively solve rows  $\ell + 1, \dots, N + 1$  of the KKT matrix for the Unsimplified Problem to obtain  $\tilde{x}_{\ell+1}, \dots, \tilde{x}_{N-1}$ .

Finally, we modify  $x_{\ell-1}$  and  $x_\ell$  to account for the changes in the slopes of the last two line segments.

Explicitly, we have

$$(6.1) \quad \tilde{x}_i = \begin{cases} x_i, & 1 \leq i \leq \ell - 2 \\ x_{\ell-1} + b', & i = \ell - 1 \\ x_\ell + m - b', & i = \ell \\ \frac{2(i-\ell+1)(i-\ell)(N-i)}{(N-\ell+1)(N-\ell+2)}, & \ell + 1 \leq i \leq N - 1 \\ -m, & i = N \\ m(N - \ell) + 2 + b', & i = N + 1 \end{cases}$$

Here  $m$  and  $b'$  are the quantities given in Lemma 6.2.

To finish the proof of the first statement, we need to show that for  $N$  sufficiently large, the following three conditions are satisfied.

- (i).  $\tilde{x}$  satisfies the KKT matrix equation for the Unsimplified Problem
- (ii).  $\tilde{x}_i > 0$  for  $i \in I$
- (iii).  $\tilde{x}_i \geq 0$  for  $i \notin I$  and  $i \leq N - 1$ .

*Proof of (i).*

The KKT matrix equations for the Simplified Problem and Unsimplified Problem are the same except for the last coordinate of the right hand side. Thus, when  $i \leq N$ , the equation  $A\tilde{x} = 2a$  in row  $i$  is equivalent to  $A(\tilde{x} - x) = 0$  in row  $i$ .

First, consider rows 1 through  $\ell$  of the KKT matrix equation. These rows are zero in columns  $\ell + 1, \dots, N - 1$ . Furthermore,  $(\tilde{x} - x)_i = 0$  in coordinates  $1 \leq i \leq \ell - 2$ . So it's enough to show that

$$A_{i,\ell-1}(\tilde{x}_{\ell-1} - x_{\ell-1}) + A_{i,\ell}(\tilde{x}_\ell - x_\ell) + A_{i,N}(\tilde{x}_N - x_N) + A_{i,N+1}(\tilde{x}_{N+1} - x_{N+1}) = 0.$$

When  $i \leq \ell - 1$ , substituting the formulas for  $A_{i,j}$  from Proposition 3.4 and the formulas for  $\tilde{x}$  from (6.1) yields

$$2(\gamma_{\ell-1} - \gamma_{i-1})(b') + 2(\gamma_\ell - \gamma_{i-1})(m - b') + 2(\gamma_N - \gamma_{i-1})(-m) + 2(m(N - \ell) + b') = 0.$$

This rearranges to

$$[\gamma_{\ell-1} - \gamma_\ell + 1](2b') + [(\gamma_\ell - \gamma_N + N - \ell)(2m) = 0.$$

But we have  $\gamma_\ell - \gamma_{\ell-1} = 1$  and  $\gamma_N - \gamma_\ell = N - \ell$  because  $\ell > \text{c.i.}(\Gamma)$ . This gives the desired result.

A similar calculation yields the desired result when  $i = \ell$ .

For rows  $\ell + 1$  through  $N + 1$ , we need to verify the following equations.

For row  $\ell + 1$ :

$$-\tilde{x}_{\ell+1} + 2(N - \ell)\tilde{x}_N + 2\tilde{x}_{N+1} = 4.$$

For row  $\ell + 2$ :

$$2\tilde{x}_{\ell+1} - \tilde{x}_{\ell+2} + 2(N - \ell - 1)\tilde{x}_N + 2\tilde{x}_{N+1} = 4.$$

For  $\ell + 3 \leq i \leq N - 1$ :

$$-\tilde{x}_{i-2} + 2\tilde{x}_{i-1} - \tilde{x}_i + 2(N - i + 1)\tilde{x}_N + 2\tilde{x}_{N+1} = 4.$$

For row  $N$ :

$$-\tilde{x}_{N-2} + 2\tilde{x}_{N-1} + 2\tilde{x}_N + 2\tilde{x}_{N+1} = 4.$$

For row  $N + 1$ :

$$-\tilde{x}_{N-1} + 2\tilde{x}_{N+1} = 2.$$

We substitute the formulas for  $\tilde{x}_i$  given in (6.1) and verify these identities of rational functions.

*Proof of (ii).* Let  $i \in I$ .

If  $i \leq \ell - 2$ , then we have  $\tilde{x}_i > 0$ , since  $\tilde{x}_i = x_i$  in this range, and  $x$  is the global optimum for the Simplified Problem.

If  $i = \ell - 1$ , we have  $\tilde{x}_{\ell-1} = x_{\ell-1} + b' > 0$ , since  $x_{\ell-1} \geq 0$  and  $b' > 0$ .

If  $i = \ell$ , we explain how to choose  $N_0$  sufficiently large to ensure that  $\tilde{x}_\ell > 0$ .

We have  $\tilde{x}_\ell = x_\ell + m - b'$ . We want to choose  $N$  sufficiently large that

$$x_\ell + m - b' > 0.$$

Substituting the formulas for  $m$  and  $b'$  yields

$$x_\ell + \frac{-6}{(N - \ell + 1)(N - \ell + 2)} - \frac{2(N - \ell - 1)}{(N - \ell + 1)(N - \ell + 2)} > 0.$$

Clearing denominators, we have

$$x_\ell(N - \ell + 1)(N - \ell + 2) - 6 - 2(N - \ell - 1) > 0.$$

This rearranges to

$$(6.2) \quad x_\ell(N - \ell)^2 + (3x_\ell - 2)(N - \ell) + 2x_\ell - 4 > 0.$$

The roots of this quadratic polynomial are

$$N - \ell = \frac{-(3x_\ell - 2) \pm \sqrt{(3x_\ell - 2)^2 - 4x_\ell(2x_\ell - 4)}}{2x_\ell}$$

The square root simplifies as follows.

$$\sqrt{(3x_\ell - 2)^2 - 4x_\ell(2x_\ell - 4)} = x_\ell + 2.$$

Thus, the roots of the quadratic polynomial in (6.2) are  $\frac{-2x_\ell + 4}{2x_\ell}$  and  $\frac{-4x_\ell}{2x_\ell}$ . Since  $x_\ell \geq 0$ , the larger root is given by  $-2x_\ell + 4$ .

Thus, to ensure  $\tilde{x}_\ell > 0$ , it suffices to take

$$N - \ell > \frac{-2x_\ell + 4}{2x_\ell}.$$

This rearranges to

$$N > \ell + \frac{2}{x_\ell} - 1.$$

*Proof of (iii).* Since  $x_i \geq 0$  for all  $i$ , it follows that  $\tilde{x} \geq 0$  for  $i = 1, \dots, \ell - 2$ . We already checked in Part (ii) that  $\tilde{x}_{\ell-1} \geq 0$  and  $\tilde{x}_\ell \geq 0$ . For  $\ell + 1 \leq i \leq N + 1$  it is clear from the formulas that  $\tilde{x} \geq 0$ .

To complete the proof of the first statement, take  $N_0 = \max\{N_0^{\text{simp}}, \lfloor \ell + \frac{2}{x_\ell} \rfloor\}$ .

For the second statement: by our parametrization of the cone  $W$  (see Definition 3.3), a solution  $\tilde{x}$  to the KKT matrix equation corresponds to the point  $w \in W$  given by

$$(6.3) \quad w = \left( \sum_{i \in I} \tilde{x}_i F_{\gamma_i}(\gamma) \right) + \tilde{x}_N L_{\gamma_N}(\gamma) + \tilde{x}_{N+1} L_1(\gamma).$$

If  $k \geq \ell$ , then the  $k^{\text{th}}$  coordinate of  $F_{\gamma_i}(\gamma)$  is 0 for all  $i \in I$ , so

$$\begin{aligned} w_k &= [\tilde{x}_N L_{\gamma_N}(\gamma) + \tilde{x}_{N+1} L_1(\gamma)]_k \\ &= \tilde{x}_N(\gamma_N - \gamma_k) + \tilde{x}_{N+1} \cdot 1 \\ &= -m(N - k) + m(N - \ell) + 2 + b' \\ &= mk + (-m\ell + 2 + b') \\ &= mk + b. \end{aligned}$$

If  $k = \ell - 1$ , then the  $k^{\text{th}}$  coordinate of  $F_{\gamma_i}(\gamma)$  is 0 for all  $i \in I$  except  $i = \ell$ , so

$$\begin{aligned} w_{\ell-1} &= [\tilde{x}_\ell L_{\gamma_\ell}(\gamma) + \tilde{x}_N L_{\gamma_N}(\gamma) + \tilde{x}_{N+1} L_1(\gamma)]_{\ell-1} \\ &= \tilde{x}_\ell(\gamma_\ell - \gamma_{\ell-1}) + m(\ell - 1) + (-m\ell + 2 + b') \\ &= x_\ell + 2. \end{aligned}$$

Then  $w_{\ell-1}$  is constant with respect to  $N$ , since  $x_\ell$  is constant with respect to  $N$ .

If  $k \leq \ell - 2$ , then

$$\begin{aligned} w_k &= \left[ \left( \sum_{i \in I} \tilde{x}_i F_{\gamma_i}(\gamma) \right) + \tilde{x}_N L_{\gamma_N}(\gamma) + \tilde{x}_{N+1} \right]_k \\ &= \left[ \left( \sum_{\substack{i \in I \\ i \leq \ell-2}} \tilde{x}_i F_{\gamma_i}(\gamma) \right) + \tilde{x}_{\ell-1} F_{\gamma_{\ell-1}}(\gamma) + \tilde{x}_\ell F_{\gamma_\ell}(\gamma) + \tilde{x}_N L_{\gamma_N}(\gamma) + \tilde{x}_{N+1} L_1(\gamma) \right]_k. \end{aligned}$$

The sum

$$\sum_{\substack{i \in I \\ i \leq \ell-2}} \tilde{x}_i F_{\gamma_i}(\gamma)$$

is constant with respect to  $N$ , since  $\tilde{x}_i = x_i$  in this range, and both  $x_i$  and  $F_{\gamma_i}(\gamma)$  are constant with respect to  $N$ .

It remains to show that the sum of the last four terms is also constant with respect to  $N$ . We have

$$\begin{aligned} &[\tilde{x}_{\ell-1} F_{\gamma_{\ell-1}}(\gamma) + \tilde{x}_\ell F_{\gamma_\ell}(\gamma) + \tilde{x}_N L_{\gamma_N}(\gamma) + \tilde{x}_{N+1} L_1(\gamma)]_k \\ &= (x_{\ell-1} + b')(\gamma_{\ell-1} - \gamma_k) + (x_\ell + m - b')(\gamma_\ell - \gamma_k) - m(\gamma_N - \gamma_k) + m(N - \ell) + 2 + b' \\ &= x_{\ell-1}(\gamma_{\ell-1} - \gamma_k) + x_\ell(\gamma_\ell - \gamma_k). \end{aligned}$$

Since  $\gamma_k$ ,  $\gamma_\ell$ ,  $x_{\ell-1}$ , and  $x_\ell$  are constant with respect to  $N$ , this expression is constant with respect to  $N$ . This completes the proof of the second statement.  $\square$

*Remark.* It is possible to give a more conceptual proof of the second statement in Theorem 6.3. When the corner set ends in two consecutive corners  $\{\ell - 1, \ell\}$ , to minimize  $\|w - a\|^2$ , we may minimize  $\sum_{i=0}^{\ell-1} (w_i - a_i)^2$  and  $\sum_{i=\ell}^N (w_i - a_i)^2$  separately. Then the first sum is independent of  $N$ , and is the same for the Simplified and Unsimplified Problems; and the least squares regression line minimizes the second sum.

## 7. EXAMPLES

In Table 1, for five different numerical semigroups, we describe the corner set  $I$  giving the worst 1-PS for each  $N$ . Here is how we generated this table.

- (1) For each  $N$  starting at  $\text{c.i.}(\Gamma) + 2$ , we compute the face  $I^{\text{simpl}}(N)$  giving the optimal solution to the Simplified Problem. We increase  $N$  by one until we find the persistent solution to the Simplified Problem.
- (2) We use Theorem 6.3 to compute the persistent solution to the Unsimplified Problem.
- (3) For each  $N$  in  $[\text{c.i.}(\Gamma) + 2, N_0]$ , we compute the face  $I(N)$  giving the optimal solution to the Unsimplified Problem.

TABLE 1. The worst 1-PS for five singularities

Simple cusp	$\langle 2, 3 \rangle$	$I$	$N$	
		$\emptyset$	$4 \leq N \leq 15$	
		$\{4\}$	$16 \leq N \leq 28$	
		$\{4, 5\}$	$29 \leq N \leq 74$	
		$\{5\}$	$75 \leq N \leq 145$	
		$\{5, 6\}$	$146 \leq N$	
Rhamphoid cusp	$\langle 2, 5 \rangle$	$I$	$N$	
		$\emptyset$	$4 \leq N \leq 11$	
		$\{5\}$	$12 \leq N \leq 18$	
		$\{5, 6\}$	$19 \leq N \leq 69$	
		$\{6\}$	$70 \leq N \leq 117$	
		$\{6, 7\}$	$118 \leq N$	
	$\langle 4, 9 \rangle$	$I$	$N$	
		$\emptyset$	$4 \leq N \leq 13$	
		$\{7\}$	$14 \leq N \leq 18$	
		$\{7, 8\}$	$N = 19$	
		$\{7, 8, 10\}$	$N = 20$	
		$\{7, 10\}$	$N = 21$	
		$\{7, 10, 14\}$	$22 \leq N \leq 23$	
		$\{7, 10, 14, 15\}$	$24 \leq N \leq 27$	
		$\{7, 10, 15\}$	$28 \leq N \leq 45$	
		$\{7, 8, 10, 15, 16\}$	$46 \leq N$	
		$\langle 5, 7 \rangle$	$I$	$N$
			$\{9\}$	$13 \leq N \leq 15$
	$\{6, 9\}$		$N = 16$	
	$\{9\}$		$17 \leq N \leq 23$	
	$\{9, 15\}$		$24 \leq N \leq 34$	
	$\{9, 15, 16\}$		$35 \leq N \leq 83$	
	$\{9, 16\}$		$84 \leq N \leq 127$	
	$\{9, 16, 17\}$		$128 \leq N$	
$\langle 8, 13 \rangle$	$I$	$N$		
	$\{6, 11, 19, 26\}$	$43 \leq N \leq 55$		
	$\{6, 11, 26\}$	$56 \leq N \leq 58$		
	$\{6, 11, 26, 37\}$	$59 \leq N \leq 60$		
	$\{6, 11, 26, 38\}$	$61 \leq N \leq 63$		
	$\{6, 11, 26, 38, 47\}$	$64 \leq N \leq 67$		
	$\{6, 11, 26, 38, 47, 48\}$	$68 \leq N \leq 70$		
	$\{6, 11, 26, 38, 48\}$	$71 \leq N \leq 85$		
	$\{6, 11, 26, 38, 48, 49\}$	$86 \leq N \leq 112$		
	$\{6, 11, 26, 38, 49\}$	$113 \leq N \leq 139$		
	$\{6, 11, 26, 38, 49, 50\}$	$140 \leq N$		

7.1. **The worst 1-PS for cusps.** We describe the persistent worst 1-PS for a cusp  $y^2 = x^{2r+1}$  of order  $r$ .

**Definition 7.1.** For any integer  $r \geq 1$ , we define the polynomial

$$(7.1) \quad f(r, x) := (4r - 2)x^3 + (6r - 6)x^2 - (12r^3 + 6r^2 + 4r + 4)x - (30r^3 + 18r^2).$$

**Lemma 7.2.** For any fixed value of  $r \geq 1$ , the polynomial  $f(r, x)$  has exactly one positive real root.

*Proof.* We can check this directly for  $r = 1$ . When  $r > 1$ , the first two coefficients are positive and the last two coefficients are negative, so by Descartes' Rule of Signs,  $f(r, x)$  has at most one positive real root. Since  $f(r, 0) < 0$  and  $\lim_{x \rightarrow \infty} f(r, x) > 0$ , the polynomial  $f(r, x)$  has exactly one positive real root.  $\square$

**Definition 7.3.** We define  $\alpha(r)$  to be the positive real root of  $f(r, x)$ .

**Proposition 7.4.** *Let  $r$  be a positive integer. Let  $j = \lceil \alpha(r) \rceil$ , and let*

$$N_0 = j + \frac{j^4 + 4j^3 + (6r^2 + 6r + 5)j^2 + (12r^2 + 12r + 2)j - 3r^4 - 6r^3 + 3r^2 + 6r}{(-2r + 1)j^3 + 3rj^2 + (6r^3 + 3r^2 + 2r - 1)j + 9r^3 + 6r^2 - 3r}$$

*Then for all  $N \geq N_0$ , the weights of the worst 1-PS for a cusp of order  $r$  lies on the face of  $W$  corresponding to  $I = \{j, j + 1\}$ , and on no smaller face.*

For a full proof, see Appendix B.

The values of  $j$  and  $N_0$  for some small values of  $r$  are given in the Table 2.

TABLE 2.  $j$  and  $N_0$  versus  $r$

$r$	$j$	$N_0$
1	5	146
2	6	118
3	7	30
4	9	53
5	11	174
6	12	37
7	14	55
8	16	107
9	18	8369
10	19	58
11	21	88
12	23	200
13	24	61
14	26	82
15	28	131

The value  $N_0 = 8369$  for  $r = 9$  appears surprisingly large compared to the other values in the table. For the semigroup  $\langle 2, 19 \rangle$ , the Simplified Problem has persistent solution  $I^{\text{simp}} = \{18, 19\}$  for all  $N \geq N_0^{\text{simp}} = 19$ . By the proof of Theorem 6.3,  $N_0$  is the smallest integer strictly larger than  $\ell + \frac{2}{x_\ell} - 1$ . We have  $x_{19} = \frac{1}{4175}$  when  $N = 19$ , and this small denominator leads to the large  $N_0$  observed in the table.

We can nearly give a closed formula for  $\lceil \alpha(r) \rceil$ , in the sense that we can identify this quantity as one of two consecutive integers.

**Proposition 7.5.** *For every positive integer  $r$ , we have*

$$\lceil \alpha(r) \rceil \in \left\{ \left\lceil \sqrt{3r} + \frac{\sqrt{3} + 1}{2} \right\rceil, \left\lceil \sqrt{3r} + \frac{\sqrt{3} + 1}{2} \right\rceil + 1 \right\}$$

Experimentally, it seems that  $\lceil \alpha(r) \rceil$  is almost always the smaller of these two possible values. This is true for all  $0 \leq r \leq 10^8$  except  $r = 0, 1, 9$ .

## 8. FUTURE WORK

We conclude by briefly discussing a few possibilities for future work. Perhaps the most natural question is whether one can prove similar persistence results for the worst 1-PS's of other curves: in particular those with more complicated singularities, and those of higher geometric genus. In the latter case, when the singularities are attached to an otherwise stable curve, it seems reasonable to expect that the worst 1-PS will depend only on the singularities, in some appropriate sense. One could also ask about toric surface singularities. This question has yet to be investigated properly, but the preliminary evidence of a few example computations suggests that surfaces will not exhibit persistence in the same way as curves.

Another direction is the interpretation of the results of this paper: what does the persistent worst 1-PS *mean*? We have explanations for why they are as they are, and for the time to persistence, but so far only from a constrained optimisation point of view. An interesting question is whether there is some explanation

purely in terms of algebraic geometry. In particular, is there some combinatorial algorithm that computes the persistent worst 1-PS purely from the singularity data?

Finally, as mentioned in the introduction, we intend in future work to combine our results here with tools from Non-reductive GIT, to construct new moduli spaces of singular curves.

## APPENDIX A. PROOF OF LEMMA 3.11

**A.1. Beginning the proof.** In this appendix we give a proof of the following result.

**Proposition A.1.** *Suppose  $N \geq \max\{\text{c.i.}(\Gamma) + 2, \max(I) + 2\}$ . Let  $x$  be the solution of the KKT matrix equation for face  $I$ . Then each coordinate of  $x$  is a rational function in  $N$ .*

We also obtain explicit formulas for the coefficients of two polynomials  $\chi$  and  $\psi$  that figure prominently in the proof. Together, these results prove Lemma 3.11.

Proposition A.1 follows from the following lemma.

**Lemma A.2.** *Suppose  $N \geq \max\{\text{c.i.}(\Gamma) + 2, \max(I) + 2\}$ .*

- (1) *The functions  $Q(N)$ ,  $\chi(N)$ ,  $\psi(N)$ , and  $\omega(N)$  are polynomials with the following degree bounds.*
  - (a)  $\deg Q = 4$
  - (b)  $\deg \chi \leq 3$
  - (c)  $\deg \psi \leq 2$
  - (d)  $\deg \omega = 4$ .
- (2) *If  $N \geq \max\{\text{c.i.}(\Gamma) + 2, \max(I) + 2\}$  and  $N \geq j + 1$ , the function  $P_j(N)$  is a polynomial of degree at most 4.*

A full proof is given in the following sections. However, we follow the same outline to prove each part, and we describe this outline now. As  $N$  grows, the matrices  $A(N)$  and  $A'(N, j)$  grow in a way that is easy to describe. We can write recurrences for the matrices, then show that the appropriate difference equations vanish to establish that each of these functions is a polynomial in  $N$  with the degree bounds claimed. Then, we use a second difference equation to obtain the leading coefficients.

**A.2.  $Q$  and  $P_j$  are polynomials.**

**Definition A.3.** For each  $n \geq 3$ , define

$$\Phi_n : \text{Mat}_{n \times n} \rightarrow \text{Mat}_{(n+1) \times (n+1)}$$

as follows.

- Columns 1 through  $(n - 2)$  in  $\Phi_n(M)$  are the same as in  $M$ , extended by 0 at the bottom.
- Column  $n - 1$  in  $\Phi_n(M)$  is 0 except the last three entries, which are  $-1, 2, -1$ .
- Column  $n$  in  $\Phi_n(M)$  is the sum of columns  $n - 1$  and  $n$  in  $M$ , extended by 0 at the bottom.
- Column  $n + 1$  in  $\Phi_n(M)$  is column  $n$  from  $M$ , extended by 2 at the bottom.

We write  $\Phi$  for the collection of maps  $\{\Phi_n\}$ . We refer to  $\Phi$  as a recurrence, and frequently omit the subscript.

The application to our problem is as follows.

**Lemma A.4.** *Suppose  $N \geq \max\{\text{c.i.}(\Gamma) + 2, \max(I) + 2\}$ .*

- (1) *The matrices  $A(N)$  satisfy the recurrence  $\Phi$ . That is,  $A(N + 1) = \Phi(A(N))$ .*
- (2) *Fix  $j$  and suppose  $N \geq j + 1$ . Then the matrices  $A'(N, j)$  satisfy the recurrence  $\Phi$ . That is,  $A'(N + 1, j) = \Phi(A'(N, j))$ .*

**Lemma A.5.** *Let  $\{M(N)\}$  be a sequence of  $N \times N$  matrices for  $N \geq N_0$  such that  $M(N + 1) = \Phi(M(N))$  for all  $N \geq N_0$ . Then  $h(N) = (-1)^{N+1} \det M(N + 1)$  is given by a polynomial of degree at most 4.*

We need a little more notation for the proof.

**Definition A.6.** Let  $I$  and  $J$  be subsets of the row and column indices of a matrix  $M$ , respectively.

We write  $D_I^J M$  for the matrix obtained by deleting the rows whose indices belong to  $I$ , and deleting the columns whose indices belong to  $J$ .

When  $D_I^J M$  is a square matrix, we write  $\delta_I^J M = \det D_I^J M$ .

**Definition A.7.** Let  $M(N)$  be an  $N \times N$  matrix.

We write  $\text{aug}(c, M(N))$  for the matrix obtained by adding  $c$  times column  $N$  to column  $N - 1$ .

We write  $\text{red } M(N)$  for the matrix obtained by subtracting column  $N$  from column  $N - 1$ .

*Proof of Lemma A.5.* We show that the following fifth-order difference equation vanishes for any integer  $k \geq N_0$ .

$$(A.1) \quad h(k+5) - 5h(k+4) + 10h(k+3) - 10h(k+2) + 5h(k+1) - h(k) = 0.$$

We define

$$\Delta_5 := h(k+5) - 5h(k+4) + 10h(k+3) - 10h(k+2) + 5h(k+1) - h(k)$$

Then our goal is to show that  $\Delta_5 = 0$ .

Substituting  $h(N) = (-1)^{N+1} \det M(N+1)$  yields the following expression.

$$(A.2) \quad (-1)^k \Delta_5 = \det M(k+6) + 5 \det M(k+5) + 10 \det M(k+4) + 10 \det M(k+3) + 5 \det M(k+2) + \det M(k+1)$$

Next, we find identities that will allow us to write each  $\det M(k+i)$  in terms of the determinants of  $M(k+6)$  and its minors.

We start by relating  $\det M(k+5)$  and  $\det M(k+6)$ . Expanding  $\det M(k+6)$  along its bottom row yields

$$\det M(k+6) = -\delta_{k+6}^{k+4} M(k+6) + 2\delta_{k+6}^{k+6} M(k+6).$$

But

$$D_{k+6}^{k+4} M(k+6) = \text{aug}(1, M(k+5))$$

so we obtain

$$\det M(k+6) = -\det M(k+5) + 2\delta_{k+6}^{k+6} M(k+6).$$

which rearranges to

$$(A.3) \quad \det M(k+5) = -\det M(k+6) + 2\delta_{k+6}^{k+6} M(k+6).$$

Next, we relate  $\det M(k+4)$  and  $\det M(k+6)$ . Expanding  $\text{aug}(1, M(k+5))$  along its bottom row yields

$$\det(\text{aug}(1, M(k+5))) = -\delta_{k+5}^{k+3} \text{aug}(1, M(k+5)) - 2\delta_{k+5}^{k+4} \text{aug}(1, M(k+5)) + 2\delta_{k+5}^{k+5} \text{aug}(1, M(k+5)).$$

But

$$\begin{aligned} D_{k+5}^{k+3} \text{aug}(1, M(k+5)) &= M(k+4) \\ D_{k+5}^{k+4} \text{aug}(1, M(k+5)) &= D_{k+5, k+6}^{k+4, k+5} M(k+6) \\ D_{k+5}^{k+5} \text{aug}(1, M(k+5)) &= D_{k+5, k+6}^{k+4, k+6} M(k+6) \end{aligned}$$

Combining these identities and rearranging yields

$$(A.4) \quad \det M(k+4) = -\det M(k+5) - 2\delta_{k+5, k+6}^{k+4, k+5} M(k+6) + 2\delta_{k+5, k+6}^{k+4, k+6} M(k+6).$$

In a similar fashion, we obtain the following identities.

$$(A.5) \quad \det M(k+3) = -\det M(k+4) - 4\delta_{k+4, k+5, k+6}^{k+3, k+4, k+5} M(k+6) + 2\delta_{k+4, k+5, k+6}^{k+3, k+4, k+6} M(k+6)$$

$$(A.6) \quad \det M(k+2) = -\det M(k+3) - 6\delta_{k+3, k+4, k+5, k+6}^{k+2, k+3, k+4, k+5} M(k+6) + 2\delta_{k+3, k+4, k+5, k+6}^{k+2, k+3, k+4, k+6} M(k+6)$$

$$(A.7) \quad \det M(k) = -\det M(k+2) - 8\delta_{k+2, k+3, k+4, k+5, k+6}^{k+1, k+2, k+3, k+4, k+5} M(k+6) + 2\delta_{k+2, k+3, k+4, k+5, k+6}^{k+1, k+2, k+3, k+4, k+6} M(k+6).$$

Substituting the identities (A.3), (A.4), (A.5), (A.6), and (A.7) into the expression (A.2) for  $(-1)^k \Delta_5$  yields

$$(A.8) \quad \begin{aligned} (-1)^k \Delta_5 &= -4\delta_{k+2, k+3, k+4, k+5, k+6}^{k+1, k+2, k+3, k+4, k+5} M(k+6) + \delta_{k+2, k+3, k+4, k+5, k+6}^{k+1, k+2, k+3, k+4, k+6} M(k+6) \\ &\quad - 12\delta_{k+3, k+4, k+5, k+6}^{k+2, k+3, k+4, k+5} M(k+6) + 4\delta_{k+3, k+4, k+5, k+6}^{k+2, k+3, k+4, k+6} M(k+6) - 12\delta_{k+4, k+5, k+6}^{k+3, k+4, k+5} M(k+6) \\ &\quad + 6\delta_{k+4, k+5, k+6}^{k+3, k+4, k+6} M(k+6) - 4\delta_{k+5, k+6}^{k+4, k+5} M(k+6) + 4\delta_{k+5, k+6}^{k+4, k+6} M(k+6) + \delta_{k+6}^{k+6} M(k+6) \end{aligned}$$

Note that this expression only involves the determinant of one matrix,  $M(k+6)$ , and its minors.

Next, we successively expand the terms with fewer than five deletions.

We begin with the term with one deletion. Expanding  $D_{k+6}^{k+6}M(k+6)$  along its bottom row yields

$$\delta_{k+6}^{k+6}M(k+6) = -\delta_{k+5,k+6}^{k+3,k+6}M(k+6) - 2\delta_{k+5,k+6}^{k+4,k+6}M(k+6) + 2\delta_{k+5,k+6}^{k+5,k+6}M(k+6).$$

We substitute this into the previous expression for  $(-1)^k\Delta_5$  and simplify to obtain

$$\begin{aligned} (-1)^k\Delta_5 &= -4\delta_{k+2,k+3,k+4,k+5,k+6}^{k+1,k+2,k+3,k+4,k+5}M(k+6) + \delta_{k+2,k+3,k+4,k+5,k+6}^{k+1,k+2,k+3,k+4,k+6}M(k+6) \\ &\quad - 12\delta_{k+3,k+4,k+5,k+6}^{k+2,k+3,k+4,k+5}M(k+6) + 4\delta_{k+3,k+4,k+5,k+6}^{k+2,k+3,k+4,k+6}M(k+6) - 12\delta_{k+4,k+5,k+6}^{k+3,k+4,k+5}M(k+6) \\ &\quad + 6\delta_{k+4,k+5,k+6}^{k+3,k+4,k+6}M(k+6) - 4\delta_{k+5,k+6}^{k+4,k+5}M(k+6) + 2\delta_{k+5,k+6}^{k+4,k+6}M(k+6) \\ (A.9) \quad &\quad - \delta_{k+5,k+6}^{k+3,k+6}M(k+6) + 2\delta_{k+5,k+6}^{k+5,k+6}M(k+6). \end{aligned}$$

Next, we expand the terms with two deletions along their bottom rows.

$$\begin{aligned} \delta_{k+5,k+6}^{k+4,k+5}M(k+6) &= -\delta_{k+4,k+5,k+6}^{k+2,k+4,k+5}M(k+6) - 2\delta_{k+4,k+5,k+6}^{k+3,k+4,k+5}M(k+6) + 2\delta_{k+4,k+5,k+6}^{k+4,k+5,k+6}M(k+6) \\ \delta_{k+5,k+6}^{k+4,k+6}M(k+6) &= -\delta_{k+4,k+5,k+6}^{k+2,k+4,k+6}M(k+6) - 2\delta_{k+4,k+5,k+6}^{k+3,k+4,k+6}M(k+6) + 4\delta_{k+4,k+5,k+6}^{k+4,k+5,k+6}M(k+6) \\ \delta_{k+5,k+6}^{k+3,k+6}M(k+6) &= -\delta_{k+4,k+5,k+6}^{k+2,k+3,k+6}M(k+6) + \delta_{k+4,k+5,k+6}^{k+3,k+4,k+6}M(k+6) + 4\delta_{k+4,k+5,k+6}^{k+3,k+5,k+6}M(k+6) \end{aligned}$$

We substitute these identities into the previous expression for  $\Delta_5$  and simplify to obtain

$$\begin{aligned} (-1)^k\Delta_5 &= -4\delta_{k+2,k+3,k+4,k+5,k+6}^{k+1,k+2,k+3,k+4,k+5}M(k+6) + \delta_{k+2,k+3,k+4,k+5,k+6}^{k+1,k+2,k+3,k+4,k+6}M(k+6) \\ &\quad - 12\delta_{k+3,k+4,k+5,k+6}^{k+2,k+3,k+4,k+5}M(k+6) + 4\delta_{k+3,k+4,k+5,k+6}^{k+2,k+3,k+4,k+6}M(k+6) - \delta_{k+4,k+5,k+6}^{k+2,k+3,k+6}M(k+6) \\ &\quad + 4\delta_{k+4,k+5,k+6}^{k+2,k+4,k+5}M(k+6) - 2\delta_{k+4,k+5,k+6}^{k+2,k+4,k+6}M(k+6) - 4\delta_{k+4,k+5,k+6}^{k+3,k+4,k+5}M(k+6) \\ (A.10) \quad &\quad + \delta_{k+4,k+5,k+6}^{k+3,k+4,k+6}M(k+6) - 4\delta_{k+4,k+5,k+6}^{k+3,k+5,k+6}M(k+6) - 2\delta_{k+4,k+5,k+6}^{k+4,k+5,k+6}M(k+6). \end{aligned}$$

Next, we expand the terms with three deletions along their bottom rows.

$$\begin{aligned} \delta_{k+4,k+5,k+6}^{k+2,k+3,k+6}M(k+6) &= -\delta_{k+3,k+4,k+5,k+6}^{k+1,k+2,k+3,k+6}M(k+6) + 6\delta_{k+3,k+4,k+5,k+6}^{k+2,k+3,k+5,k+6}M(k+6) \\ \delta_{k+4,k+5,k+6}^{k+2,k+4,k+5}M(k+6) &= -\delta_{k+3,k+4,k+5,k+6}^{k+1,k+2,k+4,k+5}M(k+6) + \delta_{k+3,k+4,k+5,k+6}^{k+2,k+3,k+4,k+5}M(k+6) \\ &\quad + 2\delta_{k+3,k+4,k+5,k+6}^{k+2,k+4,k+5,k+6}M(k+6) \\ \delta_{k+4,k+5,k+6}^{k+2,k+4,k+6}M(k+6) &= -\delta_{k+3,k+4,k+5,k+6}^{k+1,k+2,k+4,k+6}M(k+6) + \delta_{k+3,k+4,k+5,k+6}^{k+2,k+3,k+4,k+6}M(k+6) \\ \delta_{k+4,k+5,k+6}^{k+3,k+4,k+5}M(k+6) &= -\delta_{k+3,k+4,k+5,k+6}^{k+1,k+3,k+4,k+5}M(k+6) - 2\delta_{k+3,k+4,k+5,k+6}^{k+2,k+3,k+4,k+5}M(k+6) \\ &\quad + 2\delta_{k+3,k+4,k+5,k+6}^{k+3,k+4,k+5,k+6}M(k+6) \\ \delta_{k+4,k+5,k+6}^{k+3,k+4,k+6}M(k+6) &= -\delta_{k+3,k+4,k+5,k+6}^{k+1,k+3,k+4,k+6}M(k+6) - 2\delta_{k+3,k+4,k+5,k+6}^{k+2,k+3,k+4,k+6}M(k+6) \\ &\quad + 6\delta_{k+3,k+4,k+5,k+6}^{k+3,k+4,k+5,k+6}M(k+6) \\ \delta_{k+4,k+5,k+6}^{k+4,k+5,k+6}M(k+6) &= 2\delta_{k+3,k+4,k+5,k+6}^{k+3,k+4,k+5,k+6}M(k+6) \end{aligned}$$

Also, we have  $\delta_{k+4,k+5,k+6}^{k+3,k+5,k+6}M(k+6) = 0$  because this minor contains a column of zeroes.

We substitute these identities into the previous expression for  $(-1)^k\Delta_5$  and simplify to obtain

$$\begin{aligned} (-1)^k\Delta_5 &= -4\delta_{k+2,k+3,k+4,k+5,k+6}^{k+1,k+2,k+3,k+4,k+5}M(k+6) + \delta_{k+2,k+3,k+4,k+5,k+6}^{k+1,k+2,k+3,k+4,k+6}M(k+6) \\ &\quad - \delta_{k+3,k+4,k+5,k+6}^{k+1,k+2,k+3,k+6}M(k+6) - 4\delta_{k+3,k+4,k+5,k+6}^{k+1,k+2,k+4,k+5}M(k+6) + 2\delta_{k+3,k+4,k+5,k+6}^{k+1,k+2,k+4,k+6}M(k+6) \\ &\quad + 4\delta_{k+3,k+4,k+5,k+6}^{k+1,k+3,k+4,k+5}M(k+6) - \delta_{k+3,k+4,k+5,k+6}^{k+1,k+3,k+4,k+6}M(k+6) + 6\delta_{k+3,k+4,k+5,k+6}^{k+2,k+3,k+5,k+6}M(k+6) \\ (A.11) \quad &\quad + 8\delta_{k+3,k+4,k+5,k+6}^{k+2,k+4,k+5,k+6}M(k+6). \end{aligned}$$

Next, we analyze the terms with four deletions.

The following minors each have a column of zeroes, hence their determinants are zero.

$$\begin{aligned}\delta_{k+3,k+4,k+5,k+6}^{k+1,k+2,k+3,k+6} M(k+6) &= 0 \\ \delta_{k+3,k+4,k+5,k+6}^{k+1,k+2,k+4,k+5} M(k+6) &= 0 \\ \delta_{k+3,k+4,k+5,k+6}^{k+1,k+2,k+4,k+6} M(k+6) &= 0 \\ \delta_{k+3,k+4,k+5,k+6}^{k+2,k+3,k+5,k+6} M(k+6) &= 0 \\ \delta_{k+3,k+4,k+5,k+6}^{k+2,k+4,k+5,k+6} M(k+6) &= 0.\end{aligned}$$

We expand the following terms along their bottom rows.

$$\begin{aligned}\delta_{k+3,k+4,k+5,k+6}^{k+1,k+3,k+4,k+5} M(k+6) &= -\delta_{k+2,k+3,k+4,k+5,k+6}^{k,k+1,k+3,k+4,k+5} M(k+6) + \delta_{k+2,k+3,k+4,k+5,k+6}^{k+1,k+2,k+3,k+4,k+5} M(k+6) \\ &\quad + 2\delta_{k+2,k+3,k+4,k+5,k+6}^{k+1,k+3,k+4,k+5,k+6} M(k+6) \\ \delta_{k+3,k+4,k+5,k+6}^{k+1,k+3,k+4,k+6} M(k+6) &= -\delta_{k+2,k+3,k+4,k+5,k+6}^{k,k+1,k+3,k+4,k+6} M(k+6) + \delta_{k+2,k+3,k+4,k+5,k+6}^{k+1,k+2,k+3,k+4,k+6} M(k+6) \\ &\quad + 8\delta_{k+2,k+3,k+4,k+5,k+6}^{k+1,k+3,k+4,k+5,k+6} M(k+6)\end{aligned}$$

We substitute these identities into the previous expression for  $\Delta_5$  and simplify to obtain

$$(A.12) \quad (-1)^k \Delta_5 = -4\delta_{k+2,k+3,k+4,k+5,k+6}^{k,k+1,k+3,k+4,k+5} M(k+6) + \delta_{k+2,k+3,k+4,k+5,k+6}^{k,k+1,k+3,k+4,k+6} M(k+6).$$

Each of these minors on the right hand side of this equation has a column of zeroes, hence their determinants are zero. Thus,  $\Delta_5 = 0$ .  $\square$

**Lemma A.8.** *The coefficient of the degree four term is*

$$(A.13) \quad h_4 = \frac{1}{3}(-1)^k \delta_{k-1,k}^{k-1,k} M(k)$$

for any  $k \geq N_0$ .

*Proof.* When  $h$  is a polynomial of degree at most four, we have

$$h(l) - 4h(l+1) + 6h(l+2) - 4h(l+3) + h(l+4) = 24h_4$$

where  $h_4$  is the coefficient of the degree four term.

It is convenient to take  $l = k+1$ , as this allows us to use many of the identities established in the proof of the previous lemma.

$$h(k+1) - 4h(k+2) + 6h(k+3) - 4h(k+4) + h(k+5) = 24h_4$$

Substituting  $h(N) = (-1)^{N+1} M(N+1)$  yields

$$(A.14) \quad \det M(k+2) + 4 \det M(k+3) + 6 \det M(k+4) + 4 \det M(k+5) + \det M(k+6) = (-1)^k 24h_4$$

We apply the identities established in the proof of the previous lemma to the left hand side of (A.14) and obtain

$$\det M(k+2) + 4 \det M(k+3) + 6 \det M(k+4) + 4 \det M(k+5) + \det M(k+6) = 8\delta_{k+3,k+4,k+5,k+6}^{k+3,k+4,k+5,k+6} M(k+6).$$

Expanding along the last four columns shows that  $\delta_{k+3,k+4,k+5,k+6}^{k+3,k+4,k+5,k+6} M(k+6) = \delta_{k-1,k}^{k-1,k} M(k)$ .

We have

$$(-1)^k 24h_4 = 8\delta_{k-1,k}^{k-1,k} M(k),$$

which yields the desired result.  $\square$

**Lemma A.9.**  $\deg Q = 4$ .

*Proof.* By the previous lemma, the coefficient of the degree four term in  $Q$  is  $\frac{1}{3}(-1)^k \delta_{k-1,k}^{k-1,k} A(k-1)$ . The matrix  $D_{k-1,k}^{k-1,k} A(k-1)$  is the KKT matrix for finding the closest point on the cone  $W \cap \{x_N = x_{N+1} = 0\}$  to the vector  $a$  outside it. This problem has a unique solution, so this determinant cannot be zero.  $\square$

### A.3. $\chi$ is a polynomial.

**Definition A.10.** For each  $n \geq 4$ , define

$$X_n : \text{Mat}_{n \times n} \rightarrow \text{Mat}_{(n+1) \times (n+1)}$$

as follows.

- Columns 1 through  $(n-3)$  in  $X(M)$  are the same as in  $M$ , extended by 0 at the bottom.
- The last four entries in column  $n-2$  in  $X(M)$  are  $-1, 2, -1, 0$ , and this column is 0 otherwise.
- Column  $n-1$  in  $X(M)$  is column  $n-2$  in  $M$ , extended by 0 at the bottom.
- Column  $n$  in  $X(M)$  is the sum of columns  $n-1$  and  $n$  in  $M$ , extended by 0 at the bottom.
- Column  $n+1$  in  $X(M)$  is column  $n$  from  $M$ , extended by 2 at the bottom.

We write  $X$  for the collection of maps  $\{X_n\}$ . We refer to  $X$  as a recurrence, and frequently omit the subscript.

The application to our problem is as follows.

**Lemma A.11.** *Suppose  $N \geq \max\{c.i.(\Gamma) + 2, \max(I) + 2\}$ . Then the matrices  $A'(N, N-1)$  satisfy the recurrence  $X$ . That is,  $A'(N+1, N) = X(A'(N, N-1))$ .*

**Lemma A.12.** *Let  $\{M(N)\}$  be a sequence of  $N \times N$  matrices for  $N \geq N_0$  such that  $M(N+1) = X(M(N))$  for all  $N \geq N_0$ . Then  $\chi(N) = (-1)^{N+1} \det M(N+1)$  is given by a polynomial of degree at most 3 for all  $N \geq N_0 + 1$ .*

*Proof.* We show that the following fourth-order difference equation vanishes for any integer  $k \geq N_0$ .

$$(A.15) \quad \chi(k) - 4\chi(k+1) + 6\chi(k+2) - 4\chi(k+3) + \chi(k+4) = 0.$$

We define

$$\Delta_4 := \chi(k) - 4\chi(k+1) + 6\chi(k+2) - 4\chi(k+3) + \chi(k+4)$$

Thus, our goal is to prove that  $\Delta_4 = 0$ .

Substituting the definition of the function  $\chi(N) = (-1)^{N+1} \det M(N+1)$  yields the following expression.

$$(A.16) \quad (-1)^{k+1} \Delta_4 = \det M(k+1) + 4 \det M(k+2) + 6 \det M(k+3) + 4 \det M(k+4) + \det M(k+5).$$

Next, we find identities that will allow us to write each  $\det M(k+i)$  in terms of the determinants of  $M(k+5)$  and its minors.

To begin, expanding  $\det M(k+5)$  along its bottom row yields

$$(A.17) \quad \det M(k+5) = 2\delta_{k+5}^{k+5} M(k+5)$$

Next, we relate  $\det M(k+4)$  and determinants of minors of  $M(k+5)$ . Expanding  $\det \text{aug}(1, M(k+4))$  along its bottom row yields

$$\det \text{aug}(1, M(k+4)) = -2\delta_{k+4}^{k+3} \text{aug}(1, M(k+4)) + 2\delta_{k+4}^{k+4} \text{aug}(1, M(k+4)).$$

But

$$D_{k+4}^{k+3} \text{aug}(1, M(k+4)) = D_{k+4, k+5}^{k+2, k+4} M(k+5)$$

$$D_{k+4}^{k+4} \text{aug}(1, M(k+4)) = D_{k+4, k+5}^{k+2, k+5} M(k+5)$$

so we obtain

$$(A.18) \quad \det M(k+4) = -2\delta_{k+4, k+5}^{k+2, k+4} M(k+5) + 2\delta_{k+4, k+5}^{k+2, k+5} M(k+5)$$

In a similar fashion, we obtain the following identities.

$$(A.19) \quad \det M(k+3) = -4\delta_{k+3, k+4, k+5}^{k+1, k+2, k+4} M(k+5) + 2\delta_{k+3, k+4, k+5}^{k+1, k+2, k+5} M(k+5)$$

$$(A.20) \quad \det M(k+2) = -6\delta_{k+2, k+3, k+4, k+5}^{k, k+1, k+2, k+4} M(k+5) + 2\delta_{k+2, k+3, k+4, k+5}^{k, k+1, k+2, k+5} M(k+5)$$

$$(A.21) \quad \det M(k+1) = -8\delta_{k+1, k+2, k+3, k+4, k+5}^{k-1, k, k+1, k+2, k+4} M(k+5) + 2\delta_{k+1, k+2, k+3, k+4, k+5}^{k-1, k, k+1, k+2, k+5} M(k+5)$$

We substitute the identity (A.17) for  $\det M(k+5)$  into the expression (A.16) for  $(-1)^{k+1} \Delta_4$  to obtain the following.

$$(A.22) \quad (-1)^{k+1} \Delta_4 = \det M(k+1) + 4 \det M(k+2) + 6 \det M(k+3) + 4 \det M(k+4) + 2\delta_{k+5}^{k+5} M(k+5).$$

We expand the term with one deletion. Expanding  $D_{k+5}^{k+5}M(k+5)$  along its bottom row yields

$$\delta_{k+5}^{k+5}M(k+5) = -\delta_{k+4,k+5}^{k+2,k+5}M(k+5) + 2\delta_{k+4,k+5}^{k+4,k+5}M(k+5).$$

We substitute this into the expression (A.22) for  $(-1)^{k+1}\Delta_4$  and simplify.

$$(A.23) \quad \begin{aligned} (-1)^{k+1}\Delta_4 &= \det M(k+1) + 4\det M(k+2) + 6\det M(k+3) + 4\det M(k+4) \\ &\quad - 2\delta_{k+4,k+5}^{k+2,k+5}M(k+5) + 4\delta_{k+4,k+5}^{k+4,k+5}M(k+5). \end{aligned}$$

Next, we substitute the identity (A.18) for  $\det M(k+4)$  and simplify.

$$(A.24) \quad \begin{aligned} (-1)^{k+1}\Delta_4 &= \det M(k+1) + 4\det M(k+2) + 6\det M(k+3) \\ &\quad - 8\delta_{k+4,k+5}^{k+2,k+4} + 6\delta_{k+4,k+5}^{k+2,k+5}M(k+5) + 4\delta_{k+4,k+5}^{k+4,k+5}M(k+5). \end{aligned}$$

Next, we expand the terms with two deletions along their bottom rows.

$$\begin{aligned} \delta_{k+4,k+5}^{k+2,k+4}M(k+5) &= -\delta_{k+3,k+4,k+5}^{k+1,k+2,k+4}M(k+5) + 2\delta_{k+3,k+4,k+5}^{k+2,k+4,k+5}M(k+5) \\ \delta_{k+4,k+5}^{k+2,k+5}M(k+5) &= -\delta_{k+3,k+4,k+5}^{k+1,k+2,k+5}M(k+5) + 4\delta_{k+3,k+4,k+5}^{k+2,k+4,k+5}M(k+5) \\ \delta_{k+4,k+5}^{k+4,k+5}M(k+5) &= -\delta_{k+3,k+4,k+5}^{k+1,k+4,k+5}M(k+5) - 2\delta_{k+3,k+4,k+5}^{k+2,k+4,k+5}M(k+5) \end{aligned}$$

We substitute these identities into the expression (A.24) for  $(-1)^{k+1}\Delta_4$  and simplify.

$$(A.25) \quad \begin{aligned} (-1)^{k+1}\Delta_4 &= \det M(k+1) + 4\det M(k+2) + 6\det M(k+3) \\ &\quad - 8\delta_{k+3,k+4,k+5}^{k+1,k+2,k+4}M(k+5) - 6\delta_{k+3,k+4,k+5}^{k+1,k+2,k+5}M(k+5) - 4\delta_{k+3,k+4,k+5}^{k+1,k+4,k+5}M(k+5). \end{aligned}$$

We substitute the identity (A.19) for  $\det M(k+3)$  and simplify:

$$(A.26) \quad \begin{aligned} (-1)^{k+1}\Delta_4 &= \det M(k+1) + 4\det M(k+2) - 16\delta_{k+3,k+4,k+5}^{k+1,k+2,k+4}M(k+5) \\ &\quad + 6\delta_{k+3,k+4,k+5}^{k+1,k+2,k+5}M(k+5) - 4\delta_{k+3,k+4,k+5}^{k+1,k+4,k+5}M(k+5). \end{aligned}$$

Next, we expand the terms with three deletions along their bottom rows.

$$\begin{aligned} \delta_{k+3,k+4,k+5}^{k+1,k+2,k+4}M(k+5) &= -\delta_{k+2,k+3,k+4,k+5}^{k,k+1,k+2,k+4}M(k+5) + 2\delta_{k+2,k+3,k+4,k+5}^{k+1,k+2,k+4,k+5}M(k+5) \\ \delta_{k+3,k+4,k+5}^{k+1,k+2,k+5}M(k+5) &= -\delta_{k+2,k+3,k+4,k+5}^{k,k+1,k+2,k+5}M(k+5) + 6\delta_{k+2,k+3,k+4,k+5}^{k+1,k+2,k+4,k+5}M(k+5) \\ \delta_{k+3,k+4,k+5}^{k+1,k+4,k+5}M(k+5) &= -\delta_{k+2,k+3,k+4,k+5}^{k,k+1,k+4,k+5}M(k+5) + \delta_{k+2,k+3,k+4,k+5}^{k+1,k+2,k+4,k+5}M(k+5). \end{aligned}$$

We substitute these identities into the expression (A.26) for  $(-1)^{k+1}\Delta_4$  and simplify.

$$(A.27) \quad \begin{aligned} (-1)^{k+1}\Delta_4 &= \det M(k+1) + 4\det M(k+2) + 16\delta_{k+2,k+3,k+4,k+5}^{k,k+1,k+2,k+4}M(k+5) \\ &\quad - 6\delta_{k+2,k+3,k+4,k+5}^{k,k+1,k+2,k+5}M(k+5) + 4\delta_{k+2,k+3,k+4,k+5}^{k,k+1,k+4,k+5}M(k+5). \end{aligned}$$

We substitute the identity (A.20) for  $\det M(k+2)$  and simplify.

$$(A.28) \quad \begin{aligned} (-1)^{k+1}\Delta_4 &= \det M(k+1) - 8\delta_{k+2,k+3,k+4,k+5}^{k,k+1,k+2,k+4}M(k+5) \\ &\quad + 2\delta_{k+2,k+3,k+4,k+5}^{k,k+1,k+2,k+5}M(k+5) + 4\delta_{k+2,k+3,k+4,k+5}^{k,k+1,k+4,k+5}M(k+5). \end{aligned}$$

We expand the terms with four deletions along their bottom rows.

$$\begin{aligned} \delta_{k+2,k+3,k+4,k+5}^{k,k+1,k+2,k+4}M(k+5) &= (-1)\delta_{k+1,k+2,k+3,k+4,k+5}^{k-1,k,k+1,k+2,k+4}M(k+5) + 2\delta_{k+1,k+2,k+3,k+4,k+5}^{k,k+1,k+2,k+4,k+5}M(k+5) \\ \delta_{k+2,k+3,k+4,k+5}^{k,k+1,k+2,k+5}M(k+5) &= -\delta_{k+1,k+2,k+3,k+4,k+5}^{k-1,k,k+1,k+2,k+5}M(k+5) + 8\delta_{k+1,k+2,k+3,k+4,k+5}^{k,k+1,k+2,k+4,k+5}M(k+5) \end{aligned}$$

Note that  $D_{k+2,k+3,k+4,k+5}^{k,k+1,k+4,k+5}M(k+5)$  has a column of zeroes, hence this minor has determinant 0.

We substitute these identities into the expression (A.28) for  $(-1)^{k+1}\Delta_4$  and simplify.

$$(A.29) \quad \begin{aligned} (-1)^{k+1}\Delta_4 &= \det M(k+1) + 8\delta_{k+1,k+2,k+3,k+4,k+5}^{k-1,k,k+1,k+2,k+4}M(k+5) \\ &\quad - 2\delta_{k+1,k+2,k+3,k+4,k+5}^{k-1,k,k+1,k+2,k+5}M(k+5). \end{aligned}$$

Substituting the identity (A.18) for  $\det M(k+1)$  yields  $\Delta_4 = 0$ . □

**Lemma A.13.** *The coefficient of the degree three term is*

$$(A.30) \quad \chi_3 = \frac{2}{3}(-1)^{k+1} \left( \delta_{k,k+1}^{k,k+1} M(k+1) + \delta_{k-1,k+1}^{k,k+1} M(k+1) \right)$$

for any  $k \geq N_0$ .

*Proof.* When  $\chi$  is a polynomial of degree at most three, we have

$$(A.31) \quad \chi(l) - 3\chi(l+1) + 3\chi(l+2) - \chi(l+3) = -6\chi_3$$

where  $\chi_3$  is the coefficient of the degree three term.

It is convenient to take  $l = k+1$ , as this allows us to use many of the identities established in the proof of the previous lemma.

$$(A.32) \quad \chi(k+1) - 3\chi(k+2) + 3\chi(k+3) - \chi(k+4) = -6\chi_3$$

Substituting  $\chi(N) = (-1)^{N+1}M(N+1)$  yields

$$(A.33) \quad \det M(k+2) + 3\det M(k+3) + 3\det M(k+4) + \det M(k+5) = (-1)^{k+1}6\chi_3$$

We apply the identities established in the proof of the previous lemma to the left hand side of (A.33) and obtain

$$(A.34) \quad \begin{aligned} & \det M(k+2) + 3\det M(k+3) + 3\det M(k+4) + \det M(k+5) \\ &= -4\delta_{k+1,k+2,k+3,k+4,k+5}^{k-1,k+1,k+2,k+4,k+5} M(k+5) - 4\delta_{k+1,k+2,k+3,k+4,k+5}^{k,k+1,k+2,k+4,k+5} M(k+5) \end{aligned}$$

We examine each of the terms on the right hand side of (A.34).

Expanding  $D_{k+1,k+2,k+3,k+4,k+5}^{k-1,k+1,k+2,k+4,k+5} M(k+5)$  along column  $k$  yields

$$\delta_{k+1,k+2,k+3,k+4,k+5}^{k-1,k+1,k+2,k+4,k+5} M(k+5) = \delta_{k,k+1,k+2,k+3,k+4,k+5}^{k-1,k,k+1,k+2,k+4,k+5} M(k+5)$$

and

$$D_{k,k+1,k+2,k+3,k+4,k+5}^{k-1,k,k+1,k+2,k+4,k+5} M(k+5) = D_{k,k+1}^{k,k+1} M(k+1).$$

Thus we have

$$(A.35) \quad \delta_{k+1,k+2,k+3,k+4,k+5}^{k-1,k+1,k+2,k+4,k+5} M(k+5) = \text{del}(k, k+1, k, k+1, M(k+1)).$$

Expanding  $D_{k+1,k+2,k+3,k+4,k+5}^{k,k+1,k+2,k+4,k+5} M(k+5)$  along column  $k-1$  yields

$$\delta_{k+1,k+2,k+3,k+4,k+5}^{k,k+1,k+2,k+4,k+5} M(k+5) = -\delta_{k-1,k+1,k+2,k+3,k+4,k+5}^{k-1,k,k+1,k+2,k+4,k+5} M(k+5) - 2\delta_{k,k+1,k+2,k+3,k+4,k+5}^{k-1,k,k+1,k+2,k+4,k+5} M(k+5)$$

But

$$\begin{aligned} D_{k-1,k+1,k+2,k+3,k+4,k+5}^{k-1,k,k+1,k+2,k+4,k+5} M(k+5) &= D_{k-1,k+1}^{k,k+1} M(k+1) \\ D_{k,k+1,k+2,k+3,k+4,k+5}^{k-1,k,k+1,k+2,k+4,k+5} M(k+5) &= D_{k,k+1}^{k,k+1} M(k+1) \end{aligned}$$

Thus we have

$$(A.36) \quad \delta_{k+1,k+2,k+3,k+4,k+5}^{k,k+1,k+2,k+4,k+5} M(k+5) = -\delta_{k-1,k+1}^{k,k+1} M(k+1) - 2\delta_{k,k+1}^{k,k+1} M(k+1).$$

We combine equations (A.33), (A.34), (A.35), and (A.36).

$$\begin{aligned} (-1)^{k+1}6\chi_3 &= \det M(k+2) + 3\det M(k+3) + 3\det M(k+4) + \det M(k+5) \\ &= -4\delta_{k+1,k+2,k+3,k+4,k+5}^{k-1,k+1,k+2,k+4,k+5} M(k+5) - 4\delta_{k+1,k+2,k+3,k+4,k+5}^{k,k+1,k+2,k+4,k+5} M(k+5) \\ &= 4\delta_{k-1,k+1}^{k,k+1} M(k+1) + 4\delta_{k,k+1}^{k,k+1} M(k+1). \end{aligned}$$

This gives the desired result:

$$\chi_3 = \frac{2}{3}(-1)^{k+1} \left( \delta_{k-1,k+1}^{k,k+1} M(k+1) + \delta_{k,k+1}^{k,k+1} M(k+1) \right).$$

□

Fix a positive integer  $k$ . We use the following notation for the Taylor expansions of this polynomial centered at  $k$ .

$$\chi(N) = \chi_3(N-k)^3 + \chi_2(N-k)^2 + \chi_1(N-k) + \chi_0$$

**Lemma A.14.** *The coefficient of the degree two term in this Taylor expansion is*

$$\chi_2 = (-1)^{k+1} \left( 2\delta_{k-1,k+1}^{k,k+1} M(k+1) + 4\delta_{k,k+1}^{k,k+1} M(k+1) \right).$$

*Proof.* Evaluating the Taylor expansion for  $N = k$ ,  $N = k + 1$ , and  $N = k + 2$  yields

$$\begin{aligned} \chi(k) &= \chi_0 \\ \chi(k+1) &= \chi_3 + \chi_2 + \chi_1 + \chi_0 \\ \chi(k+2) &= 8\chi_3 + 4\chi_2 + 2\chi_1 + \chi_0. \end{aligned}$$

Solving this system for  $\chi_2$  yields

$$2\chi_2 = \chi(k+2) - 2\chi(k+1) + \chi(k) - 6\chi_3.$$

Substituting  $\chi(N) = (-1)^{N+1} M(N+1)$  yields

$$(A.37) \quad 2\chi_2 = (-1)^{k+1} (\det M(k+3) + 2 \det M(k+2) + M(k+1)) - 6\chi_3.$$

We use the identities established in the two previous proofs as well as one additional identity.

$$\delta_{k+2,k+3,k+4,k+5}^{k+1,k+2,k+4,k+5} M(k+5) = -\delta_{k+1,k+2,k+3,k+4,k+5}^{k-1,k+1,k+2,k+4,k+5} M(k+5) - 2\delta_{k+1,k+2,k+3,k+4,k+5}^{k,k+1,k+2,k+4,k+5} M(k+5).$$

This allows us to write

$$(A.38) \quad \det M(k+3) + 2 \det M(k+2) + M(k+1) = 8\delta_{k-1,k+1}^{k,k+1} M(k+1) + 12\delta_{k,k+1}^{k,k+1} M(k+1).$$

Substituting (A.38) and the expression (A.30) previously found for  $\chi_3$  into (A.37) yields

$$\begin{aligned} 2\chi_2 &= (-1)^{k+1} (\det M(k+3) + 2 \det M(k+2) + M(k+1)) - 6\chi_3 \\ &= (-1)^{k+1} (8\delta_{k-1,k+1}^{k,k+1} M(k+1) + 12\delta_{k,k+1}^{k,k+1} M(k+1)) \\ &\quad - 6\frac{2}{3}(-1)^{k+1} \left( \delta_{k-1,k+1}^{k,k+1} M(k+1) + \delta_{k,k+1}^{k,k+1} M(k+1) \right) \\ &= (-1)^{k+1} (4\delta_{k-1,k+1}^{k,k+1} M(k+1) + 8\delta_{k,k+1}^{k,k+1} M(k+1)). \end{aligned}$$

□

**Lemma A.15.** *The coefficient of the degree one term in this Taylor expansion is*

$$\chi_1 = \frac{1}{3}(-1)^{k+2} \left( 8\delta_{k-1,k+1}^{k,k+1} M(k+1) + 14\delta_{k,k+1}^{k,k+1} M(k+1) + 6\delta_{k+2}^{k+1} M(k+2) \right).$$

*Proof.* Evaluating the Taylor expansion for  $N = k + 1$  yields

$$\chi(k+1) = \chi_3 + \chi_2 + \chi_1 + \chi_0$$

Hence,

$$(A.39) \quad \begin{aligned} \chi_1 &= \chi(k+1) - \chi_0 - \chi_3 - \chi_2 \\ &= (-1)^{k+2} \det M(k+2) - (-1)^{k+1} \det M(k+1) - \chi_3 - \chi_2 \\ &= (-1)^{k+2} (\det M(k+2) + \det M(k+1)) - \chi_3 - \chi_2. \end{aligned}$$

We prove the following identity:

$$(A.40) \quad \det M(k+2) + \det M(k+1) = \delta_{k+2}^{k+1} M(k+2).$$

We obtain this as follows. Start by expanding  $\text{red } M(k+2)$  along its bottom row.

$$(A.41) \quad \det \text{red}(M(k+2)) = 2\delta_{k+2}^{k+1} \text{red } M(k+2) + 2\delta_{k+2}^{k+2} \text{red } M(k+2)$$

We have

$$(A.42) \quad \det \text{red } M(k+2) = \det M(k+2)$$

$$(A.43) \quad \delta_{k+2}^{k+1} \text{red } M(k+2) = \delta_{k+2}^{k+1} M(k+2)$$

Expanding  $D_{k+2}^{k+2} \text{red } M(k+2)$  along its bottom row yields

$$\delta_{k+2}^{k+2} \text{red } M(k+2) = -\delta_{k+1,k+2}^{k-1,k+2} \text{red } M(k+2).$$

Expanding  $\det M(k+1)$  along its bottom row yields

$$\det M(k+1) = 2\delta_{k+1}^{k+1} M(k+1),$$

and we have

$$D_{k+1, k+2}^{k-1, k+2} \text{red } M(k+2) = D_{k+1}^{k+1} M(k+1).$$

Thus

$$(A.44) \quad 2\delta_{k+2}^{k+2} \text{red } M(k+2) = -\det M(k+1).$$

Substituting (A.42), (A.43), and (A.44) into (A.41) yields (A.40).

Substituting (A.40) and the formulas for  $\chi_3$  and  $\chi_2$  from the previous lemmas yields the result.  $\square$

We summarize the results of this section with the following lemma.

**Lemma A.16.** *Suppose that  $k \geq \max\{\text{c. i.}(\Gamma) + 2, \max(I) + 2\}$ . Then*

$$\begin{aligned} \chi_3 &= \frac{1}{3}(-1)^{k+1} \left( 2\delta_{k, k+1}^{k, k+1} A'(k, k-1) + 2\delta_{k-1, k+1}^{k, k+1} A'(k, k-1) \right) \\ \chi_2 &= (-1)^{k+1} \left( 4\delta_{k, k+1}^{k, k+1} A'(k, k-1) + 2\delta_{k-1, k+1}^{k, k+1} A'(k, k-1) \right) \\ \chi_1 &= \frac{1}{3}(-1)^{k+1} \left( -14\delta_{k, k+1}^{k, k+1} A'(k, k-1) - 8\delta_{k-1, k+1}^{k, k+1} A'(k, k-1) - 6\delta_{k+2}^{k+1} A'(k+1, k) \right) \\ \chi_0 &= (-1)^{k+1} \det A'(k, k-1). \end{aligned}$$

*Proof.* We apply Lemmas A.13, A.14, and A.15 with  $M(k+1) = A'(k, k-1)$ .  $\square$

**Lemma A.17.**  $\deg \omega = 4$ .

*Proof.* The last row of the KKT matrix equation says  $-x_{N-1} + 2x_{N+1} = 4$ . Clearing denominators yields  $-\chi + 2\omega = 4Q$ . Since  $\deg \omega = 4$  and  $\deg \chi \leq 3$ , the result follows.  $\square$

A.4.  $\psi$  is a polynomial.

**Definition A.18.** For each  $n \geq 3$ , define

$$\Psi_n : \text{Mat}_{n \times n} \rightarrow \text{Mat}_{(n+1) \times (n+1)}$$

as follows.

- Columns 1 through  $(n-2)$  in  $\Psi(M)$  are the same as in  $M$ , extended by 0 at the bottom.
- The last three entries in column  $n-1$  in  $\Psi(M)$  are  $-1, 2, -1$ , and this column is 0 otherwise.
- Column  $n$  in  $\Psi(M)$  is column  $n-1$  in  $M$ , extended by 0 at the bottom.
- Column  $n+1$  in  $\Psi(M)$  is column  $n$  from  $M$ , extended by 2 at the bottom.

We write  $\Psi$  for the collection of maps  $\{\Psi_n\}$ . We refer to  $\Psi$  as a recurrence, and frequently omit the subscript.

The application to our problem is as follows.

**Lemma A.19.** *Suppose  $N \geq \max\{\text{c. i.}(\Gamma) + 2, \max(I) + 2\}$ . Then the matrices  $A'(N, N)$  satisfy the recurrence  $\Psi$ . That is,  $A'(N+1, N+1) = \Psi(A'(N, N))$ .*

**Lemma A.20.** *Let  $\{M(N)\}$  be a sequence of  $N \times N$  matrices for  $N \geq N_0$  such that  $M(N+1) = \Psi(M(N))$  for all  $N \geq N_0$ . Then  $\psi(N) = (-1)^{N+1} \det M(N+1)$  is given by a polynomial of degree at most two for all  $N \geq N_0$ .*

*Proof.* We show that the following third-order difference equation vanishes for any integer  $k \geq N_0 + 1$ .

$$(A.45) \quad \psi(k) - 3\psi(k+1) + 3\psi(k+2) - \psi(k+3) = 0.$$

We define

$$\Delta_3 := \psi(k) - 3\psi(k+1) + 3\psi(k+2) - \psi(k+3).$$

Thus, our goal is to prove that  $\Delta_3 = 0$ .

Substituting the definition of the function  $\psi(N) = (-1)^{N+1} \det M(N+1)$  yields the following expression for  $\Delta_3$ .

$$(A.46) \quad (-1)^{k+1} \Delta_3 = \det M(k+1) + 3 \det M(k+2) + 3 \det M(k+3) + \det M(k+4).$$

Next, we find identities that will allow us to write each  $\det M(k+i)$  in terms of the determinants of  $M(k+4)$  and its minors.

To begin, expanding  $\det M(k+4)$  along its bottom row yields

$$\det M(k+4) = -\delta_{k+4}^{k+2} M(k+4) + 2\delta_{k+4}^{k+4} M(k+4).$$

But

$$D_{k+4}^{k+2} M(k+4) = M(k+3)$$

so we obtain

$$(A.47) \quad \det M(k+3) = -\det M(k+4) + 2\delta_{k+4}^{k+4} M(k+4).$$

In a similar fashion, we obtain the following identities.

$$(A.48) \quad \det M(k+2) = -\det M(k+3) + 2\delta_{k+3, k+4}^{k+2, k+4} M(k+4).$$

$$(A.49) \quad \det M(k+1) = -\det M(k+2) + 2\delta_{k+2, k+3, k+4}^{k+1, k+2, k+4} M(k+4)$$

We substitute the identities (A.47), (A.48), and (A.49) into the expression (A.46) for  $(-1)^{k+1} \Delta_3$  to obtain the following.

$$(A.50) \quad (-1)^{k+1} \Delta_3 = 2\delta_{k+2, k+3, k+4}^{k+1, k+2, k+4} M(k+4) + 4\delta_{k+3, k+4}^{k+2, k+4} M(k+4) + 2\delta_{k+4}^{k+4} M(k+4).$$

We expand the term with one deletion. Expanding  $D_{k+4}^{k+4} M(k+4)$  along its bottom row yields

$$\delta_{k+4}^{k+4} M(k+4) = -\delta_{k+3, k+4}^{k+1, k+4} M(k+4) - 2\delta_{k+3, k+4}^{k+2, k+4} M(k+4).$$

We substitute this into the expression (A.50) for  $(-1)^{k+1} \Delta_3$  and simplify.

$$(A.51) \quad (-1)^{k+1} \Delta_3 = \delta_{k+2, k+3, k+4}^{k+1, k+2, k+4} M(k+4) - \delta_{k+3, k+4}^{k+1, k+4} M(k+4).$$

The minor  $D_{k+3, k+4}^{k+1, k+4} M(k+4)$  has only one nonzero entry in column  $k+2$ . Expanding along this column yields

$$\delta_{k+3, k+4}^{k+1, k+4} M(k+4) = \delta_{k+2, k+3, k+4}^{k+1, k+2, k+4} M(k+4).$$

Thus,  $(-1)^{k+1} \Delta_3 = 0$ . This completes the proof that  $\psi(N)$  is a polynomial of degree at most two.  $\square$

**Lemma A.21.** *The coefficient of the degree two term is*

$$\psi_2 = (-1)^{k+1} (\delta_{k+1}^{k+1} M(k+1) + \delta_k^{k+1} M(k+1))$$

for any  $k \geq N_0$ .

*Proof.* When  $\psi$  is a polynomial of degree at most two, we have

$$(A.52) \quad \psi(k+1) - 2\psi(k+2) + \psi(k+3) = 2\psi_2$$

where  $\psi_2$  is the coefficient of the degree two term.

Substituting  $\psi(N) = (-1)^{N+1} \det M(N+1)$  yields

$$(A.53) \quad \det M(k+1) + 2 \det M(k+2) + \det M(k+3) = (-1)^{k+1} 2\psi_2$$

We can argue as we did in the previous proof to obtain the following identities.

$$\det M(k+2) = -\det M(k+3) + 2\delta_{k+3}^{k+3} M(k+3)$$

$$\det M(k+1) = -\det M(k+2) + 2\delta_{k+2, k+3}^{k+1, k+3} M(k+3)$$

$$\delta_{k+3}^{k+3} M(k+3) = -\delta_{k+2, k+3}^{k, k+3} M(k+3) - 2\delta_{k+2, k+3}^{k+1, k+3} M(k+3).$$

We use these identities to evaluate the left hand side of (A.53) and obtain

$$(A.54) \quad \det M(k+1) + 2 \det M(k+2) + \det M(k+3) = -2\delta_{k+2, k+3}^{k+1, k+3} M(k+3) - 2\delta_{k+2, k+3}^{k, k+3} M(k+3)$$

Expanding the first term on the right in (A.54) along its bottom row yields

$$(A.55) \quad \delta_{k+2,k+3}^{k+1,k+3} M(k+3) = -\delta_{k+1,k+2,k+3}^{k-1,k+1,k+3} M(k+3) - 2\delta_{k+1,k+2,k+3}^{k,k+1,k+3} M(k+3)$$

Expanding the first term on the right in (A.55) along its bottom row yields

$$\delta_{k+1,k+2,k+3}^{k-1,k+1,k+3} M(k+3) = \delta_{k,k+1,k+2,k+3}^{k-1,k,k+1,k+3} M(k+3)$$

and

$$D_{k,k+1,k+2,k+3}^{k-1,k,k+1,k+3} M(k+3) = \delta_{k,k+1}^{k-1,k+1} M(k+1).$$

For the second term on the right in (A.55), we have

$$D_{k+1,k+2,k+3}^{k,k+1,k+3} M(k+3) = D_{k+1}^{k+1} M(k+1).$$

Hence, we obtain

$$(A.56) \quad \delta_{k+2,k+3}^{k+1,k+3} M(k+3) = -\delta_{k,k+1}^{k-1,k+1} M(k+1) - 2\delta_{k+1}^{k+1} M(k+1)$$

Expanding the second term on the right in (A.54) along its bottom row yields

$$(A.57) \quad \delta_{k+2,k+3}^{k,k+3} M(k+3) = -\delta_{k+1,k+2,k+3}^{k-1,k,k+3} M(k+3) + \delta_{k+1,k+2,k+3}^{k,k+1,k+3} M(k+3).$$

The first term on the right in (A.57) has a column of zeroes, hence its determinant is zero. For the second term on the right in (A.57), we have

$$D_{k+1,k+2,k+3}^{k,k+1,k+3} M(k+3) = D_{k+1}^{k+1} M(k+1).$$

Hence, we obtain

$$(A.58) \quad \delta_{k+2,k+3}^{k,k+3} M(k+3) = \delta_{k+1}^{k+1} M(k+1).$$

Substituting (A.56) and (A.58) into (A.54) yields

$$\det M(k+1) + 2 \det M(k+2) + \det M(k+3) = 2\delta_{k,k+1}^{k-1,k+1} M(k+1) + 2\delta_{k+1}^{k+1} M(k+1).$$

Finally, expanding along the bottom row yields

$$\delta_k^{k+1} M(k+1) = \delta_{k,k+1}^{k-1,k+1} M(k+1).$$

Thus, we have

$$\begin{aligned} (-1)^{k+1} 2\psi_2 &= \det M(k+1) + 2 \det M(k+2) + \det M(k+3) \\ &= 2\delta_{k,k+1}^{k-1,k+1} M(k+1) + 2\delta_{k+1}^{k+1} M(k+1). \\ &= 2\delta_k^{k+1} M(k+1) + 2\delta_{k+1}^{k+1} M(k+1). \end{aligned}$$

This gives the desired result:

$$\psi_2 = (-1)^{k+1} (\delta_k^{k+1} M(k+1) + \delta_{k+1}^{k+1} M(k+1)).$$

□

Fix a positive integer  $k$ . We use the following notation for the Taylor expansions of this polynomial centered at  $k$ .

$$\psi(N) = \psi_2(N-k)^2 + \psi_1(N-k) + \psi_0$$

**Lemma A.22.** *The coefficient of the degree 1 term in this Taylor expansion is*

$$\psi_1 = (-1)^{k+1} (3\delta_{k+1}^{k+1} M(k+1) + \delta_k^{k+1} M(k+1)).$$

*Proof.* We have

$$\psi = \psi_2(N-k)^2 + \psi_1(N-k) + \psi_0.$$

When  $N = k$ , we have  $\psi(k) = \psi_0$ , and when  $N = k+1$  we have

$$\psi(k+1) = \psi_2 + \psi_1 + \psi_0,$$

so

$$\begin{aligned}
\psi_1 &= \psi(k+1) - \psi_2 - \psi_0 \\
&= (-1)^{k+2} \det M(k+2) - \psi_2 - (-1)^{k+1} \det M(k+1) \\
\text{(A.59)} \quad &= (-1)^{k+2} (\det M(k+2) + \det M(k+1)) - \psi_2.
\end{aligned}$$

Expanding  $\det M(k+2)$  along its bottom row yields

$$\det M(k+2) = -\delta_{k+2}^k M(k+2) + 2\delta_{k+2}^{k+2} M(k+2).$$

But

$$D_{k+2}^k M(k+2) = M(k+1)$$

so we obtain

$$\text{(A.60)} \quad \det M(k+2) + \det M(k+1) = 2\delta_{k+2}^{k+2} M(k+2).$$

Substituting this and the expression for  $\psi_2$  yields

$$\text{(A.61)} \quad \psi_1 = (-1)^{k+2} 2\delta_{k+2}^{k+2} M(k+2) - (-1)^{k+1} (\delta_k^{k+1} M(k+1)) + \delta_{k+1}^{k+1} M(k+1).$$

We study the first term on the right in (A.61). Expanding along column  $k$  yields

$$\delta_{k+2}^{k+2} M(k+2) = -\delta_{k,k+2}^{k,k+2} M(k+2) - 2\delta_{k+1,k+2}^{k,k+2} M(k+2).$$

But

$$\begin{aligned}
D_{k,k+2}^{k,k+2} M(k+2) &= D_k^{k+1} M(k+1) \\
D_{k+1,k+2}^{k,k+2} M(k+2) &= D_{k+1}^{k+1} M(k+1)
\end{aligned}$$

so

$$\delta_{k+2}^{k+2} M(k+2) = -\delta_k^{k+1} M(k+1) - 2\delta_{k+1}^{k+1} M(k+1).$$

Substituting this into (A.61) yields

$$\begin{aligned}
\psi_1 &= (-1)^{k+2} (-2\delta_k^{k+1} M(k+1) - 4\delta_{k+1}^{k+1} M(k+1) - (-1)^{k+1} (\delta_k^{k+1} M(k+1)) + \delta_{k+1}^{k+1} M(k+1)) \\
&= (-1)^{k+1} (3\delta_{k+1}^{k+1} M(k+1) + \delta_k^{k+1} M(k+1)),
\end{aligned}$$

as desired. □

We summarize the results of this section with the following lemma.

**Lemma A.23.** *Suppose that  $k \geq \max\{c.i.(\Gamma) + 2, \max(I) + 2\}$ . Then*

$$\begin{aligned}
\psi_2 &= (-1)^{k+1} (\delta_{k+1}^{k+1} A'(k, k) + \delta_k^{k+1} A'(k, k)) \\
\psi_1 &= (-1)^{k+1} (3\delta_{k+1}^{k+1} A'(k, k) + \delta_k^{k+1} A'(k, k)) \\
\psi_0 &= (-1)^{k+1} \det A'(k, k).
\end{aligned}$$

*Proof.* We apply Lemmas A.21 and A.22 with  $M(k+1) = A'(k, k)$ . □

## APPENDIX B. PROOF OF PROPOSITION 7.4

Here we give a proof of Proposition 7.4, concerning worst 1-PS's for higher order cusps.

### B.1. Statement of the main result for cusps.

**Definition B.1.** For any integer  $r \geq 1$ , we define the polynomial

$$\text{(B.1)} \quad f(r, x) := (4r-2)x^3 + (6r-6)x^2 - (12r^3 + 6r^2 + 4r + 4)x - (30r^3 + 18r^2).$$

**Lemma B.2.** *For any fixed value of  $r \geq 1$ , the polynomial  $f(r, x)$  has exactly one positive real root.*

*Proof.* We can check this directly for  $r = 1$ . When  $r > 1$ , the first two coefficients are positive and the last two coefficients are negative, so by Descartes' Rule of Signs,  $f(r, x)$  has at most one positive real root. Since  $f(r, 0) < 0$  and  $\lim_{x \rightarrow \infty} f(r, x) > 0$ , the polynomial  $f(r, x)$  has exactly one positive real root. □

**Definition B.3.** We define  $\alpha(r)$  to be the positive real root of  $f(r, x)$ .

**Proposition B.4.** *Let  $r$  be a positive integer. Let  $j = \lceil \alpha(r) \rceil$ . Then for all  $N \geq j + 2$ , the persistent corner set for a cusp of order  $r$  and the simplified problem is  $I^{\text{simp}} = \{j, j + 1\}$ .*

In the proof, we will exhibit explicit formulas for a nonnegative solution  $x$  to the KKT matrix equation for this face. This will prove the claim.

First, we give a hint about how we obtained these quantities.

**B.2. How we obtained some formulas appearing in the proof.** When  $I = \{j, j + 1\}$ , the piecewise linear graph through the points  $\{(\gamma_i, w_i)\}$  consists of three line segments. Let  $y = m_k x + b_k$  be the equations of these three line segments for  $k = 1, 2, 3$ .

The KKT matrix equation variables  $x_j$  and  $x_{j+1}$  represent parameters of  $\text{face}_{\gamma(I)}$ , and we have

$$(B.2) \quad x_j = -m_1 + m_2$$

$$(B.3) \quad x_{j+1} = -m_2 + m_3.$$

For the Simplified Problem, we have  $m_3 = 0$  and  $b_3 = 2$ .

The middle line segment joins  $(\gamma_j, w_j)$  and  $(\gamma_{j+1}, w_{j+1})$ . Since  $j > \text{cond}(\Gamma)$ , we have  $\gamma_{j+1} - \gamma_j = 1$ , and hence

$$m_2 = (w_{j+1} - w_j) / (\gamma_{j+1} - \gamma_j) = (2 - w_j) / 1 = 2 - w_j.$$

We can use the equation for the first line segment to compute  $w_j$ . This yields  $w_j = m_1(j + r) + b_1$ , and hence  $m_2 = 2 - m_1(j + r) - b_1$ .

Substituting these expressions for  $m_2$  and  $m_3$  into (B.2) and B.3) yields

$$(B.4) \quad x_j = 2 - m_1(j + r + 1) - b_1$$

$$(B.5) \quad x_{j+1} = -2 + m_1(j + r) + b_1.$$

The simple structure of the corner sets  $I = \{j, j + 1\}$  helps us in another way. Whenever we have two consecutive corners, the optimisation problem breaks up: we can optimize  $\sum_{i=0}^j (w_i - a_i)^2$  and  $\sum_{i=j+1}^N (w_i - a_i)^2$  separately. When there are exactly two corners, as in  $I = \{j, j + 1\}$ , the solutions to each of these problems is given by least squares regression. We thus obtain formulas for  $m_1$  and  $b_1$  as follows.

**Lemma B.5.** *Let  $\Gamma = \langle 2, r \rangle$ .*

(1) *Applying least squares linear regression to  $(\gamma_0, a_0), \dots, (\gamma_j, a_j)$  yields the following formulas.*

$$\begin{aligned} n &= j + 1 \\ \sum \gamma_i w_i &= (r + j)(r + j + 1) \\ \sum \gamma_i &= \frac{1}{2}(j^2 + 2jr + j - r^2 + r) \\ \sum w_i &= 2j + 2r + 1 \\ \sum \gamma_i^2 &= -r^3 + r^2j + rj^2 + \frac{1}{3}j^3 + \frac{1}{2}r^2 + rj + \frac{1}{2}j^2 + \frac{1}{2}r + \frac{1}{6}j \end{aligned}$$

(2) *Recall that*

$$\begin{aligned} m_1 &= \frac{n(\sum \gamma_i w_i) - (\sum \gamma_i)(\sum w_i)}{n(\sum \gamma_i^2) - (\sum \gamma_i)^2} \\ b_1 &= \frac{1}{n} \left( \sum w_i \right) - m_1 \frac{1}{n} \left( \sum \gamma_i \right) \end{aligned}$$

(3) *The denominator of  $m_1$  is*

$$n \left( \sum \gamma_i^2 \right) - \left( \sum \gamma_i \right)^2 = \frac{1}{12} (j^4 + 4j^3 + (6r^2 + 6r + 5)j^2 + (12r^2 + 12r + 2)j - 3r^4 - 6r^3 + 3r^2 + 6r)$$

*We scale this polynomial by 12 so that the  $j^4$  term is monic, and call this polynomial  $h$ . Then*

$$\begin{aligned} m_1 &= \frac{6((-2r + 1)j^2 + j + 2r^3 + r^2 + r)}{h} \\ b_1 &= \frac{2(j^4 + (4r + 2)j^3 + (12r^2 + 6r + 2)j^2 + (12r^2 + 8r + 1)j - 9r^4 + 6r^2 + 3r)}{h} \end{aligned}$$

Substituting the formulas for  $m_1$  and  $b_1$  from Lemma B.5 into the expression (B.4) yields

$$(B.6) \quad x_j = \frac{f}{h},$$

where

$$\begin{aligned} f(r, j) &= (4r - 2)j^3 + (6r - 6)j^2 + (-12r^3 - 6r^2 - 4r - 4)j - 30r^3 - 18r^2 \\ h(r, j) &= j^4 + 4j^3 + (6r^2 + 6r + 5)j^2 + (12r^2 + 12r + 2)j - 3r^4 - 6r^3 + 3r^2 + 6r \end{aligned}$$

We also find that

$$x_{j+1} = -\frac{f(r, j-1)}{h}.$$

### B.3. Proof of the main result for cusps.

*Proof of Proposition B.4.* We exhibit explicit formulas for a nonnegative solution  $x$  to the KKT matrix equation for this face. This proves the claim.

We begin by defining the following quantities. See Section B.2 for a discussion of how we obtained these formulas.

$$(B.7) \quad h(r, j) = j^4 + 4j^3 + (6r^2 + 6r + 5)j^2 + (12r^2 + 12r + 2)j - 3r^4 - 6r^3 + 3r^2 + 6r$$

$$(B.8) \quad x_j = \frac{f}{h}$$

$$(B.9) \quad x_{j+1} = -\frac{f(r, j-1)}{h}$$

$$(B.10) \quad m_1 = -x_j - x_{j+1}$$

$$(B.11) \quad b_1 = (j+r)(x_j + x_{j+1}) + 2 + x_{j+1}$$

We then define  $x_k$  for  $1 \leq k \leq j-1$  in terms of the quantities above.

$$(B.12) \quad x_k = \begin{cases} \frac{1}{3}(k-1)k(k+1)m_1 + \frac{1}{2}k(k+1)b_1 - 2k^2 & \text{if } 1 \leq k \leq r-1 \\ \frac{1}{6}(j-r)(j-r+1)((j-r-1)x_j + (j-r+2)x_{j+1}) & \text{if } k = r \\ \frac{1}{3}(j-k)(j-k+1)((j-k-1)x_j + (j-k+2)x_{j+1}) & \text{if } r+1 \leq k \leq j-1 \\ 0 & \text{if } j+2 \leq k \leq N \\ 2 & \text{if } k = N+1 \end{cases}$$

To finish the proof, we need to show that the following three conditions are satisfied.

- (i).  $x$  satisfies the KKT matrix equation for the Simplified Problem
- (ii).  $x_i > 0$  for  $i \in I$
- (iii).  $x_i \geq 0$  for  $i \notin I$  and  $i \leq N-1$ .

To establish (i), we need to verify several identities of rational functions.

Recall the formulas for the KKT matrix equation:

$$A_{i,j} = \begin{cases} \gamma_j - \gamma_{j+1} & \text{if } j \leq N-1, j \notin I, \text{ and } i = j \\ \gamma_{j+1} - \gamma_{j-1} & \text{if } j \leq N-1, j \notin I, \text{ and } i = j+1 \\ \gamma_{j-1} - \gamma_j & \text{if } j \leq N-1, j \notin I, \text{ and } i = j+2 \\ 2(\gamma_j - \gamma_{i-1}) & \text{if } j \leq N-1, j \in I, \text{ and } i \leq j \\ 2(\gamma_N - \gamma_{i-1}) & \text{if } j = N \\ 2 & \text{if } j = N+1 \\ 0 & \text{otherwise} \end{cases}$$

The vector  $a$  on the right hand side of the KKT matrix equation is given by the following formula.

$$a_i = \begin{cases} \gamma_1 & \text{if } i = 1 \\ \gamma_i - \gamma_{i-2} & \text{if } 2 \leq i \leq N \\ 2 & \text{if } i = N+1 \text{ (Simplified Problem)} \end{cases}$$

For the semigroup  $\Gamma = \langle 2, 2r + 1 \rangle$ , we have

$$\gamma_i = \begin{cases} 2i & \text{if } 0 \leq i \leq r \\ i + r & \text{if } r + 1 \leq i \leq N \end{cases}$$

Thus

$$a_i = \begin{cases} 2 & \text{if } i = 1 \\ 4 & \text{if } 2 \leq i \leq r \\ 3 & \text{if } i = r + 1 \\ 2 & \text{if } r + 2 \leq i \end{cases}$$

Now we check each row of the KKT matrix equation.

In row 1, we have  $A_{1,k} = 0$  when  $k \notin \{1, j, j + 1, N, N + 1\}$ , and  $x_N = 0$ . Then

$$\begin{aligned} & A_{1,1}x_1 + A_{1,j}x_j + A_{1,j+1}x_{j+1} + A_{1,N+1}x_{N+1} \\ &= (-2)(b_1 - 2) + (2(j + r))x_j + (2(j + r + 1))x_{j+1} + (2)(2) \\ &= 4 \\ &= 2a_1. \end{aligned}$$

In row 2, we have  $A_{2,k} = 0$  when  $k \notin \{1, 2, j, j + 1, N, N + 1\}$ , and  $x_N = 0$ . Then

$$\begin{aligned} & A_{2,1}x_1 + A_{2,2}x_2 + A_{2,j}x_j + A_{2,j+1}x_{j+1} + A_{2,N+1}x_{N+1} \\ &= (4)(b_1 - 2) + (-2)(2m_1 + 3b_1 - 8) + (2(j + r - 2))x_j + (2(j + r - 1))x_{j+1} + (2)(2) \\ &= 8 \\ &= 2a_2. \end{aligned}$$

Let  $3 \leq i \leq r - 1$ . In row  $i$ , we have  $A_{i,k} = 0$  when  $k \notin \{i - 2, i - 1, i, j, j + 1, N, N + 1\}$ , and  $x_N = 0$ . Then

$$\begin{aligned} & A_{i,i-2}x_{i-2} + A_{i,i-1}x_{i-1} + A_{i,i}x_i + A_{i,j}x_j + A_{i,j+1}x_{j+1} + A_{i,N+1}x_{N+1} \\ &= (-2)\left(\frac{1}{3}(i - 3)(i - 2)(i - 1)m_1 + \frac{1}{2}(i - 2)(i - 1)b_1 - 2(i - 2)^2\right) \\ &\quad + (4)\left(\frac{1}{3}(i - 2)(i - 1)im_1 + \frac{1}{2}(i - 1)(i)b_1 - 2(i - 1)^2\right) \\ &\quad + (-2)\left(\frac{1}{3}(i - 1)i(i + 1)m_1 + \frac{1}{2}(i)(i + 1)b_1 - 2i^2\right) \\ &\quad + (2(j + r - 2(i - 1)))x_j + (2(j + r + 1 - 2(i - 1)))x_{j+1} + (2)(2) \\ &= 8 \\ &= 2a_i. \end{aligned}$$

In row  $r$ , we have  $A_{r,k} = 0$  when  $k \notin \{r - 2, r - 1, r, j, j + 1, N, N + 1\}$ , and  $x_N = 0$ . Then

$$\begin{aligned} & A_{r,r-2}x_{r-2} + A_{r,r-1}x_{r-1} + A_{r,r}x_r + A_{r,j}x_j + A_{r,j+1}x_{j+1} + A_{r,N+1}x_{N+1} \\ &= (-2)\left(\frac{1}{3}(r - 3)(r - 2)(r - 1)m_1 + \frac{1}{2}(r - 2)(r - 1)b_1 - 2(r - 2)^2\right) \\ &\quad + (4)\left(\frac{1}{3}(r - 2)(r - 1)rm_1 + \frac{1}{2}(r - 1)(r)b_1 - 2(r - 1)^2\right) \\ &\quad + (-1)\left(\frac{1}{6}(j - r)(j - r + 1)((j - r - 1)x_j + (j - r + 2)x_{j+1})\right) \\ &\quad + (2(j - r + 2))x_j + (j - r + 3)x_{j+1} + (2)(2) \\ &= 8 \\ &= 2a_r. \end{aligned}$$

In row  $r + 1$ , we have  $A_{r+1,k} = 0$  when  $k \notin \{r - 1, r, r + 1, j, j + 1, N, N + 1\}$ , and  $x_N = 0$ . Then

$$\begin{aligned}
& A_{r+1,r-1}x_{r-1} + A_{r+1,r}x_r + A_{r+1,r+1}x_{r+1} + A_{r+1,j}x_j + A_{r+1,j+1}x_{j+1} + A_{r+1,N+1}x_{N+1} \\
&= (-2)\left(\frac{1}{3}(r-2)(r-1)rm_1 + \frac{1}{2}(r-1)(r)b_1 - 2(r-1)^2\right) \\
&\quad + (3)\frac{1}{6}(j-r)(j-r+1)((j-r-1)x_j + (j-r+2)x_{j+1}) \\
&\quad + (-1)\left(\frac{1}{3}(j-(r+1))(j-(r+1)+1)((j-(r+1)-1)x_j + (j-(r+1)+2)x_{j+1})\right) \\
&\quad + (2(j-r))x_j + (2(j-r+1))x_{j+1} + (2)(2) \\
&= 6 \\
&= 2a_{r+1}.
\end{aligned}$$

In row  $r + 2$ , we have  $A_{r+2,k} = 0$  when  $k \notin \{r, r + 1, r + 2, j, j + 1, N, N + 1\}$ , and  $x_N = 0$ . Then

$$\begin{aligned}
& A_{r+2,r}x_r + A_{r+2,r+1}x_{r+1} + A_{r+2,r+2}x_{r+2} + A_{r+2,j}x_j + A_{r+2,j+1}x_{j+1} + A_{r+2,N+1}x_{N+1} \\
&= (-2)\frac{1}{6}(j-r)(j-r+1)((j-r-1)x_j + (j-r+2)x_{j+1}) \\
&\quad + (2)\left(\frac{1}{3}(j-(r+1))(j-(r+1)+1)((j-(r+1)-1)x_j + (j-(r+1)+2)x_{j+1})\right) \\
&\quad + (-1)\left(\frac{1}{3}(j-(r+2))(j-(r+2)+1)((j-(r+2)-1)x_j + (j-(r+2)+2)x_{j+1})\right) \\
&\quad + (2(j-r-1))x_j + (2(j-r))x_{j+1} + (2)(2) \\
&= 4 \\
&= 2a_{r+2}.
\end{aligned}$$

Let  $r + 3 \leq i \leq j - 1$ . In row  $i$ , we have  $A_{i,k} = 0$  when  $k \notin \{i - 2, i - 1, i, j, j + 1, N, N + 1\}$ , and  $x_N = 0$ . Then

$$\begin{aligned}
& A_{i,i-2}x_{i-2} + A_{i,i-1}x_{i-1} + A_{i,i}x_i + A_{i,j}x_j + A_{i,j+1}x_{j+1} + A_{i,N+1}x_{N+1} \\
&= (-1)\left(\frac{1}{3}(j-(i-2))(j-(i-2)+1)((j-(i-2)-1)x_j + (j-(i-2)+2)x_{j+1})\right) \\
&\quad + (2)\left(\frac{1}{3}(j-(i-1))(j-(i-1)+1)((j-(i-1)-1)x_j + (j-(i-1)+2)x_{j+1})\right) \\
&\quad + (-1)\left(\frac{1}{3}(j-k)(j-i+1)((j-i-1)x_j + (j-i+2)x_{j+1})\right) \\
&\quad + (2(j-i+1))x_j + (2(j-i+2))x_{j+1} + (2)(2) \\
&= 4 \\
&= 2a_i.
\end{aligned}$$

In row  $j$ , we have  $A_{j,k} = 0$  when  $k \notin \{j - 2, j - 1, j, j + 1, N, N + 1\}$ , and  $x_N = 0$ . Then

$$\begin{aligned}
& A_{j,j-2}x_{j-2} + A_{j,j-1}x_{j-1} + A_{j,j}x_j + A_{j,j+1}x_{j+1} + A_{j,N+1}x_{N+1} \\
&= (-1)(2x_j + 4x_{j+1}) + (2)(2x_{j+1}) + (2)x_j + (4)x_{j+1} + (2)(2) \\
&= 4 \\
&= 2a_j.
\end{aligned}$$

In row  $j + 1$ , we have  $A_{j+1,k} = 0$  when  $k \notin \{j - 1, j + 1, N, N + 1\}$ , and  $x_N = 0$ . Then

$$\begin{aligned}
& A_{j+1,j-1}x_{j-1} + A_{j+1,j+1}x_{j+1} + A_{j+1,N+1}x_{N+1} \\
&= (-1)(2x_{j+1}) + (2)(x_{j+1}) + (2)(2) \\
&= 4 \\
&= 2a_{j+1}.
\end{aligned}$$

Let  $j + 2 \leq i \leq N + 1$ . We have  $A_{i,k} = 0$  when  $k \notin \{i - 2, i - 1, i, N, N + 1\}$ . But  $x_{i-2} = x_{i-1} = x_i = x_N = 0$ , so this row of the KKT matrix equation encodes the equation  $A_{i,N+1}x_{N+1} = 2 \cdot 2 = 2a_i$ .

Thus,  $x$  is a solution to the KKT matrix equation for the Simplified Problem for this face.

Next, we establish (ii). We have  $I = \{j, j + 1\}$ , so we need to analyze  $x_j = f(r, j)/h$  and  $x_{j+1} = -f(r, j - 1)/h$ . First, we argue that the denominator is positive. For this, fix  $r > 0$ , and consider  $h(r, j)$  as a polynomial in  $j$ . We have  $\frac{dh}{dj} > 0$  and  $h(r, r) > 0$ , so  $h(r, j) > 0$  for all  $j > r$ .

Now consider the numerators. The hypothesis that  $j = \lceil \alpha(r) \rceil$  is equivalent to the statement that  $j$  is the smallest integer value of  $x$  for which  $f(r, x)$  is positive. Equivalently,  $j$  is the unique positive integer such that  $f(r, j - 1) < 0$  and  $f(r, j) > 0$ . It follows that  $x_j > 0$  and  $x_{j+1} > 0$ .

Next, we establish (iii). For  $r \leq k \leq j - 1$ , the expressions used to define  $x_k$  are nonnegative linear combinations of  $x_j$  and  $x_{j+1}$ , so  $x_k \geq 0$  when  $k$  is in this range.

To establish the result when  $k \leq r - 1$ , we define an auxiliary sequence  $(\bar{x})$  as follows.

Define

$$\bar{x}_k := \frac{x_k}{\frac{k(k+1)}{2}}$$

for  $k \leq r - 1$ .

Combining these definitions with the formulas in (B.12) yields

$$\bar{x}_k = \begin{cases} \frac{2(k-1)}{3}m_1 + b_1 - \frac{4k}{k+1} & \text{if } 1 \leq k \leq r - 1 \\ \frac{2(r-1)}{3}m_1 + b_1 - \frac{4r}{r+1} & \text{if } k = r \end{cases}$$

As  $k$  increases, the fractions  $\frac{2(k-1)}{3}$  and  $\frac{4k}{k+1}$  increase. But these terms appear in the formula for  $\bar{x}_k$  with negative coefficients (recall that  $m_1 < 0$ ). So the sequence  $\bar{x}_k$  decreases as  $k$  increases. The denominators used to define  $\bar{x}_k$  from  $x_k$  are positive. Since  $x_r$  is positive,  $\bar{x}_r$  is positive, so  $\bar{x}_1, \dots, \bar{x}_{r-1}$  are positive, and hence  $x_1, \dots, x_{r-1}$  are positive. □

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