MATRIX GENERATORS OF IRREDUCIBLE REPRESENTATIONS OF FINITE GROUPS

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Given a finite group G and an irreducible character χ of G, modern software such as `GAP` and `Magma` contain commands for producing matrix generators of a representation V of G with character χ. Finding fast algorithms to produce "nice" matrix generators is a subject of ongoing research. It seems that these computer algebra systems implement several different algorithms that cover many special cases.

I do not know a reference for a general algorithm. Hence, I briefly present an algorithm that was suggested to me by Valery Alexeev and James McKernan. This algorithm is not expected to perform efficiently; it is included merely to establish that this can be performed algorithmically.

Algorithm 1.
INPUTS: *a finite group* G *with generators* $g_1, \ldots, g_r$*; an irreducible character* $\chi : G \to \mathbb{C}$ *of degree* n*.*
OUTPUTS: *matrices* $M_1, \ldots, M_r \in \mathrm{GL}(n, \mathbb{C})$ *such that the homomorphism* $g_i \mapsto M_i$ *is a representation with character* χ

(1) *Compute matrix generators for the regular representation* V *of* G*. The entries of these matrices are in* $\{0, 1\}$*.*
(2) *Use the projection formula to compute matrix generators* $\rho_W(g)$ *for a representation* W *with character* $n\chi$*. Let* K *be the smallest field containing* $\{\chi(g) : g \in G\}$*. Note that* $\mathbb{Q} \subseteq K \subseteq \mathbb{Q}[\zeta_{\#G}]$*. Then the matrix generators* $\rho_W(g)$ *lie in* $\mathrm{GL}(n^2, K)$*.*
(3) *Let* $x_1, \ldots, x_{n^2}$ *be indeterminates. Let* M *be the* $\#G \times n^2$ *matrix over* K *whose rows are given by the vectors* $\rho_W(g).(x_1, \ldots, x_{n^2})$*. Let* $X \subset \mathbb{P}_K^{n^2-1}$ *be the determinantal variety* $\operatorname{rank} M \leq n$*. Since representations of finite groups are complete reducible in characteristic zero, the representation* W *is isomorphic over* K *to the direct sum* $V_\chi^{\oplus n}$*, and therefore* $X(K)$ *is non-empty.*
(4) *Intersect* X *with generic hyperplanes over* K *to obtain a zero-dimensional variety* Y*.*

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(5) *If necessary, pass to a finite field extension L of K to obtain a reduced closed point $y \in Y(L)$.*
(6) *The point y (thought of as a vector in $W \otimes L$) generates the desired representation.*

1. Example

We use this algorithm to produce matrix generators for the degree two irreducible representation of the symmetric group S_3.

The character table for S_3 is

Class	Id	$(1,2)$	$(1,2,3)$
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	$-1.$

Matrix generators for the regular representation of S_3 with respect to the ordered basis $\{e_{\mathrm{Id}}, e_{(1,2)}, e_{(1,3)}, e_{(2,3)}, e_{(1,2,3)}, e_{(1,3,2)}\}$. are given below. (Here, the linear algebra convention we follow is that matrices of linear transformation act on the left of column vectors.)

$$\rho_{\mathrm{reg}}((1,2)) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad \rho_{\mathrm{reg}}((1,2,3)) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The projection formula (see for instance Fulton and Harris formula (2.31))

$$\pi_i = \frac{\dim(V_i)}{\#G} \sum_{g \in G} \overline{\chi_i(g)} g.$$

yields the following matrix for the projection π_3 onto the four-dimensional isotypical subspace $W \cong V_3^{\oplus 2}$.

$$\pi_3 = \frac{2}{6}\left(2\begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix} - \begin{bmatrix} 0&0&0&0&0&1\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&1&0 \end{bmatrix} - \begin{bmatrix} 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} \frac{2}{3}&0&0&0&-\frac{1}{3}&-\frac{1}{3}\\ 0&\frac{2}{3}&-\frac{1}{3}&-\frac{1}{3}&0&0\\ 0&-\frac{1}{3}&\frac{2}{3}&-\frac{1}{3}&0&0\\ 0&-\frac{1}{3}&-\frac{1}{3}&\frac{2}{3}&0&0\\ -\frac{1}{3}&0&0&0&\frac{2}{3}&-\frac{1}{3}\\ -\frac{1}{3}&0&0&0&-\frac{1}{3}&\frac{2}{3} \end{bmatrix}.$$

A basis for Image(π_3) is given by the first, second, third, and fifth columns. This yields the following matrix generators for the representation W:

$$\rho_W((1,2)) = \begin{bmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{bmatrix}, \quad \rho_W((1,2,3)) = \begin{bmatrix} 0&0&0&-1\\ 0&-1&1&0\\ 0&-1&0&0\\ 1&0&0&-1 \end{bmatrix}.$$

Let x_1, x_2, x_3, x_4 be indeterminates. The image of (x_1, x_2, x_3, x_4) under the ρ_W images of $\{\mathrm{Id}, (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\}$ are given in the rows of the matrix M below:

$$M = \begin{bmatrix} x_1 & x_2 & x_3 & x_4\\ x_2 & x_1 & x_4 & x_3\\ -x_2+x_3 & -x_4 & x_1-x_4 & -x_2\\ -x_3 & -x_1+x_4 & -x_1 & x_2-x_3\\ -x_4 & -x_2+x_3 & -x_2 & x_1-x_4\\ -x_1+x_4 & -x_3 & x_2-x_3 & -x_1 \end{bmatrix}.$$

I used `Macaulay2` to produce the ideal generated by the 3×3 minors of this matrix; it has 72 generators. Let $X \subset \mathbb{P}^3$ be the determinantal subvariety defined by these equations. We know $X(\mathbb{Q}) \neq 0$ since $W \cong V_3^{\oplus 2}$ over $\mathbb{Q}$. I intersected X with the hyperplanes $x_1 = 1$, $x_2 = 0$, $x_3 = 0$ to obtain a zero-dimensional variety Y defined by the equations $x_1 - 1, x_2, x_3, x_4^2 - x_4 + 1$. By passing to the field $L = \mathbb{Q}[\zeta_3]$ we can split $x_4^2 - x_4 + 1$ to obtain the solutions $x_4 = \zeta_3 + 1, -\zeta_3$. Thus we obtain a vector $v = (1, 0, 0, -\zeta_3)$ whose orbit under $\rho_W(G)$ generates a representation isomorphic to V_3.

We have

$$\rho_W((1,2))(v) = (0,1,-\zeta_3,0)$$
$$\rho_W((1,2,3))(v) = (\zeta_3,0,0,\zeta_3+1)$$
$$\rho_W((1,2,3))(0,1,-\zeta_3,0) = (0,-\zeta_3-1,-1,0)$$

Hence, with respect to the ordered basis $\{v, \rho_W((1,2))(v)\} = \{(1,0,0,-\zeta_3),(0,1,-\zeta_3,0)\}$, we obtain matrix generators for a two-dimensional representation V_3 with character χ_3.

$$\rho_{V_3}((1,2)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \rho_{V_3}((1,2,3)) = \begin{bmatrix} \zeta_3 & 0 \\ 0 & -\zeta_3 - 1 \end{bmatrix}.$$

Remark. If instead we had intersected X with the hyperplanes $x_1 = 0$, $x_2 = 1$, $x_3 = 0$, we would have obtained the points $x_4 = \pm 1$. This would have led to rational matrix generators

$$\rho_{V_3}((1,2)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \rho_{V_3}((1,2,3)) = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$

The field extension L of K in the algorithm arises because although $X(K)$ is nonempty, there is no guarantee that when we intersect X with generic hyperplanes that we will capture a point in $X(K)$.

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