This note was originally written in October 2025. At that time, the formula shown in Conjecture 1.1 below was only a conjecture. In April 2026, we proved it. See
https://faculty.fordham.edu/dswinarski/InvariantPolynomialsAndMukaiModels/InvariantPolynomialFormula.pdf
for a proof.

A CONJECTURAL COMBINATORIAL FORMULA FOR SOME SL_n-INVARIANT POLYNOMIALS

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ABSTRACT. Let $V(\lambda)$ be an irreducible representation of SL_n with highest weight λ. We give a conjectural combinatorial formula for the SL_n-invariant polynomial in $V(\lambda) \otimes V(\lambda^*)$, in terms of the Gelfand-Tsetlin bases of $V(\lambda)$ and $V(\lambda^*)$.

1. INTRODUCTION

Let G be a simple algebraic group over $\mathbb{C}$. Let $V(\lambda)$ be an irreducible representation with highest weight λ. Then the space of invariants of the following form is one-dimensional.

$$\dim(V(\lambda) \otimes V(\lambda^*))^G = 1$$

Many bases of the irreducible representations $V(\lambda)$ have been introduced in the literature, particularly when $G = \mathrm{SL}_n$. In this note we work with the Gelfand-Tsetlin basis of SL_n. See [2, Section 2] for an introduction.

Here is the main conjecture. The notation will be explained below in Section 2.

Conjecture 1.1. *Let*

$$F = \sum_{P \in \mathrm{GT}(\lambda)} \frac{(-1)^{\ell(P)}}{\|P\| \|P^*\|} B_P B_{P^*}.$$

Then F is an SL_n-invariant polynomial in $V(\lambda) \otimes V(\lambda^*)$.

Algebraic combinatorics and invariant theory are research fields with a long history and a large literature; it would not surprise the author if this conjecture is already known. Please contact the author with relevant references.

2. NOTATION

2.1. The pattern P.

We follow the definitions of [2, Section 2].

Let λ be a partition with n parts, written in nonincreasing order. A *Gelfand-Tsetlin pattern of shape* λ is a triangular array of the following form.

$$\begin{array}{ccccccccc} x_{n,1} & & x_{n,2} & & x_{n,3} & & \cdots & & x_{n,n} \\ & x_{n-1,1} & & x_{n-1,2} & & \cdots & & x_{n-1,n-1} & \\ & & \ddots & & & & & & \\ & & & x_{2,1} & & x_{2,2} & & & \\ & & & & x_{1,1} & & & & \end{array}$$

Each entry $x_{i,j}$ is a nonnegative integer, the top row $x_{n,i}$ corresponds to λ, and the entries satisfy the inequalities $x_{k,i} \geq x_{k-1,i} \geq x_{k,i+1}$.

We write $\mathrm{GT}(\lambda)$ for the set of all Gelfand-Tsetlin patterns of shape λ. This set indexes a basis of $V(\lambda)$. We write B_P for the basis element corresponding to the pattern $P \in \mathrm{GT}(\lambda)$.

2.2. The pattern P^*.

Let $P \in \mathrm{GT}(\lambda)$. We define the *dual pattern* P^* by the formula

$$y_{i,j} = n - x_{n-i,j}.$$

2.3. **The numerator $(-1)^{\ell(P)}$.** The *level* $\ell(P)$ is defined as the level of the weight $\mathrm{wt}(P)$. We briefly recall these definitions.

Let $P \in \mathrm{GT}(\lambda)$. Then we define the *content* of P as $(c_1, \dots, c_n)$, where

$$c_i = \sum_{j=1}^{i+1} x_{i+1,j} - \sum_{j=1}^{i} x_{i,j}.$$

The *weight* of P (written in the basis of fundamental dominant weights ω_i for $\mathfrak{sl}_n$) is

$$\begin{aligned}\mathrm{wt}(P) &= \sum_{i=1}^{n-1} (c_i - c_{i+1})\omega_i \\ &= \sum_{i=1}^{n-1} \left(-\sum_{j=1}^{i-1} x_{i-1,j} + 2\sum_{j=1}^{i} x_{i,j} - \sum_{j=1}^{i+1} x_{i+1,j} \right) \omega_i.\end{aligned}$$

Let μ be a weight occuring in the representation $V(\lambda)$. Then we may write $\mu = \lambda - \sum_{i=1}^{n-1} k_i\alpha_i$, where the α_i are the simple roots for $\mathfrak{sl}_n$. Then we define the *level* of μ as $\ell(\mu) = \sum_{i=1}^{n-1} k_i$. See for instance [1, Section 4].

2.4. **The norm $\|P\|$.** Write $\tilde{x}_{i,j} = x_{i,j} - j + 1$.

We define $\|P\|$ by

$$\|P\|^2 = \prod_{k=2}^{n} \prod_{1 \le i \le j \le k} \frac{(\tilde{x}_{k,i} - \tilde{x}_{k-1,j})!}{(\tilde{x}_{k-1,i} - \tilde{x}_{k-1,j})!} \prod_{1 \le i < j \le k} \frac{(\tilde{x}_{k,i} - \tilde{x}_{k,j} - 1)!}{(\tilde{x}_{k-1,i} - \tilde{x}_{k,j} - 1)!}$$

See [2, Section 2].

References

[1] W. A. de Graaf, *Constructing representations of split semisimple Lie algebras*, J. Pure Appl. Algebra **164** (2001), no. 1-2, 87–107, DOI 10.1016/S0022-4049(00)00150-X. Effective methods in algebraic geometry (Bath, 2000). MR1854331 ↑2

[2] A. I. Molev, *Gelfand-Tsetlin bases for classical Lie algebras*, Handbook of algebra. Vol. 4, Handb. Algebr., vol. 4, Elsevier/North-Holland, Amsterdam, 2006, pp. 109–170, DOI 10.1016/S1570-7954(06)80006-9. MR2523420 ↑1, 2

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