Solution #1: Since \( \sin \) is periodic with period \( 2\pi \), it suffices to prove that \( \sin(0) = \sin(\pi) = 0 \), that \( \sin(x) > 0 \) for \( x \in (0, \pi) \), and that \( \sin(x) < 0 \) for \( x \in (\pi, 2\pi) \). Now \( \sin(0) = 0 \) by definition, and \( \sin(x + \pi) = -\sin(x) \) from the proof of Theorem 5.8, so that \( \sin(\pi) = -\sin(0) = 0 \). Again from the proof of Theorem 5.8, \( \sin(x) > 0 \) for \( x \in (0, \frac{\pi}{2}) \). Also, \( \sin(y + \frac{\pi}{2}) = \cos(y) \) for all \( y \in \mathbb{R} \), so if \( x \in [\frac{\pi}{2}, \pi] \) then put \( y = x - \frac{\pi}{2} \) to find that \( \sin(x) = \cos(y) > 0 \) because \( y \in [0, \frac{\pi}{2}] \) and we already know that \( \cos \) is strictly positive on \([0, \frac{\pi}{2}]\). Thus, \( \sin(x) > 0 \) for all \( x \in (0, \pi) \). Finally, we use once again that \( \sin(y + \pi) = -\sin(y) \) for all \( y \in \mathbb{R} \) and plug in \( y = x - \pi \) to learn that if \( x \in (\pi, 2\pi) \) then \( \sin(x) = -\sin(y) < 0 \) since \( y \in (0, \pi) \) and we just saw that \( \sin(y) > 0 \).

Solution #2: Let \( P_n = \{x_0, x_1, \ldots, x_n\} \) be the partition of \([0, b]\) defined by \( x_i = \frac{ib}{n} \). Then

\[
L(f, P_n) = \sum_{i=1}^{n} \left(\frac{(i-1)b}{n}\right)^3 \frac{b}{n} = b^4 \left(1 - \frac{1}{n}\right)^2, \quad U(f, P_n) = \sum_{i=1}^{n} \left(\frac{ib}{n}\right)^3 \frac{b}{n} = b^4 \left(1 + \frac{1}{n}\right)^2,
\]

following the computations in class. Thus,

\[
\frac{b^4}{4} = \sup\{L(f, P_n) : n \in \mathbb{N}\} \leq \int_0^b f \leq \inf\{U(f, P_n) : n \in \mathbb{N}\} = \frac{b^4}{4},
\]

and hence \( f \) is integrable with \( \int_0^b f = \frac{1}{4}b^4 \) as desired.

Solution #3: Define \( x_i = \frac{b}{2(n-1)} \) for \( i \in \{0, 1, \ldots, n-1\} \) and \( x_n = 1 \). Then clearly

\[
L(f, P_n) = L(f, P'_n) + \frac{1}{16}, \quad U(f, P_n) = U(f, P'_n) + \frac{1}{2},
\]

where \( P'_n = \{x_0, x_1, \ldots, x_{n-1}\} \) is a uniform partition of \([0, \frac{1}{2}]\) into \( n - 1 \) intervals of equal length. Using the work in Solution #2 with \( b = \frac{1}{2} \), it follows that

\[
\sup\{L(f, P_n) : n \in \mathbb{N}\} = \frac{1}{4 \cdot 2^4} + \frac{1}{16} < \frac{1}{4 \cdot 2^4} + \frac{1}{2} = \inf\{U(f, P_n) : n \in \mathbb{N}\}.
\]

Notice that the left-hand [right-hand] side here is strictly less [greater] than \( \int_0^1 x^3 \, dx = \frac{1}{4} \).

Solution #4: Write \( P = \{x_0, x_1, \ldots, x_n\} \), \( m_i = \inf \{f(x_i-1, x_i)\} \), \( M_i = \sup \{f(x_i-1, x_i)\} \) as usual. Notice that \( m_i \leq M_i \). I claim that if \( L(f, P) = U(f, P) \), then \( m_i = M_i \) for all \( i \in \{1, 2, \ldots, n\} \). Indeed, if not, then there would exist some \( i_0 \in \{1, 2, \ldots, n\} \) with \( m_{i_0} < M_{i_0} \). But then

\[
L(f, P) = m_{i_0}(x_{i_0} - x_{i_0-1}) + \sum_{i \neq i_0} m_i(x_i - x_{i-1}) < M_{i_0}(x_{i_0} - x_{i_0-1}) + \sum_{i \neq i_0} M_i(x_i - x_{i-1}) \leq M_{i_0}(x_{i_0} - x_{i_0-1}) + \sum_{i \neq i_0} M_i(x_i - x_{i-1}) = U(f, P),
\]

contradicting our assumption that \( L(f, P) = U(f, P) \). Thus, we have proved that \( m_i = M_i \) for all \( i \). But then, for any \( x \in [x_{i-1}, x_i] \), \( m_i \leq f(x) \leq M_i = m_i \), proving that \( f(x) = m_i = M_i \) for all such \( x \), so \( f \) is constant on \([x_{i-1}, x_i]\). Finally, if \( x, y \in [a, b] \), \( x \leq y \), then \( x \in [x_{i-1}, x_i] \), \( y \in [x_{j-1}, x_j] \) for some \( i \leq j \), and if \( i < j \), then \( f(x) = f(x_i) = f(x_{i+1}) = \cdots = f(x_{j-1}) = f(y) \).