Solution #1: (a) False. The function \( f(x) = x^3 \) is strictly increasing but \( f'(0) = 0 \).
(b) True. Consider that \( \frac{f(x)-f(x_0)}{x-x_0} \geq 0 \) if \( x > x_0 \) because \( f(x) \geq f(x_0) \). Likewise for \( x < x_0 \).
(c) True. See Lemma 4.16, or the proof of Rolle’s Theorem given in class.
(d) False. The function \( f(x) = x \) trivially satisfies \( f(x) \leq f(1) \) for \( x \leq 1 \), but \( f'(1) = 1 \).

Solution #2: We argue by contradiction. If the equation \( f(x) = 0 \) had at least \( n+1 \) solutions, there would exist real numbers \( x_0 < x_1 < \cdots < x_n \) such that \( f(x_i) = 0 \) for \( i \in \{0,1,\ldots,n\} \).
Applying Rolle’s Theorem to \( f \) on the interval \([x_{i-1},x_i]\) for each \( i \in \{1,2,\ldots,n\} \). This gives us points \( c_i \in (x_{i-1},x_i) \) for each \( i \in \{1,2,\ldots,n\} \) such that \( f'(c_i) = 0 \). Notice that \( c_i < x_i < c_{i+1} \) for \( i \in \{1,2,\ldots,n-1\} \), so no two of the \( n \) numbers \( c_1,c_2,\ldots,c_n \) are equal, so the equation \( f'(c_i) = 0 \) has at least \( n \) solutions, contradicting our assumption that it has at most \( n-1 \).

Solution #3: Let \( f(x) = x^4 + 2x^2 - 6x + 2 \). Then \( f'(x) = 4x^3 + 4x - 6 \) and \( f''(x) = 12x^2 + 4 \).
Clearly \( f''(x) > 0 \) for all \( x \in \mathbb{R} \), so \( f''(x) = 0 \) has no solution, so, by Problem 2, \( f'(x) = 0 \) has at most one solution, so again by Problem 2, \( f(x) = 0 \) has at most two solutions. However, \( f(0) = 2 \), \( f(1) = -1 \), and \( f(2) = 14 \), so the Intermediate Value Theorem tells us that there exists at least one solution to \( f(x) = 0 \) in each of the two intervals \((0, 1) \) and \((1, 2) \). Thus, there are at most two solutions but also at least two, hence exactly two.

Solution #4: By definition, \( \log : \mathbb{R}^+ \to \mathbb{R} \) is the unique differentiable function satisfying the two conditions \( \log'(x) = \frac{1}{x} \) for all \( x \in \mathbb{R}^+ \) and \( \log(1) = 0 \). Thus, since \( \lim_{n \to \infty}(1 + \frac{1}{n}) = 1 \),
\[
1 = \log'(1) = \lim_{x \to 1} \frac{\log(x) - \log(1)}{x - 1} = \lim_{n \to \infty} \frac{\log(1 + \frac{1}{n}) - 0}{(1 + \frac{1}{n}) - 1} = \lim_{n \to \infty} \left[n \log(1 + \frac{1}{n})\right].
\]
Now we use that \( \exp : \mathbb{R} \to \mathbb{R}^+ \) has been defined as the inverse function to \( \log \). In particular, \( \exp \) is continuous, so we may conclude from the above that
\[
\exp(1) = \exp \left( \lim_{n \to \infty} \left[n \log(1 + \frac{1}{n})\right]\right) = \lim_{n \to \infty} \exp(n \log(1 + \frac{1}{n})) = \lim_{n \to \infty} (1 + \frac{1}{n})^n.
\]

Bonus Solution: The function \( g : \mathbb{R} \to \mathbb{R} \) defined by \( g(x) := f''(x) \) is twice differentiable and satisfies \( g'(x_0) = 0 \) but \( g''(x_0) > 0 \). Thus, the Extreme Value Criterion tells us that \( x_0 \) is a local minimizer of \( g \). Slightly more precisely (this is what the proof of the Extreme Value Criterion really tells us), \( x_0 \) is a strict local minimizer of \( g \), meaning that there exists some \( \delta > 0 \) such that \( g(x) > g(x_0) \) for \( x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\} \). Since by assumption \( g(x_0) = 0 \), it follows that \( g \) is strictly positive on the two open intervals \((x_0 - \delta, x_0) \) and \((x_0, x_0 + \delta) \). But \( g \) is the derivative of \( f' \), so \( f' \) must be strictly increasing on both of these intervals by the Monotonicity Criterion. Since \( f' \) is continuous at \( x_0 \) with \( f'(x_0) = 0 \), it follows that \( f'(x) < 0 \) for \( x \in (x_0 - \delta, x_0) \) and \( f'(x) > 0 \) for \( x \in (x_0, x_0 + \delta) \) (more precisely, if \( x \in (x_0, x_0 + \delta) \), then it holds that
\[
0 = f'(x_0) = \lim_{n \to \infty} f'(x_0 + \frac{1}{n + 1}(x - x_0)) \leq f'(x_0 + \frac{1}{2}(x - x_0)) < f'(x), \text{ etc.}
\]
Applying the Monotonicity Criterion again, we learn that \( f \) is strictly decreasing on \((x_0 - \delta, x_0) \) and strictly increasing on \((x_0, x_0 + \delta) \). Since \( f \) is continuous at \( x_0 \), it follows that \( f(x) > f(x_0) \) whenever \( 0 < |x - x_0| < \delta \), so, by definition, \( x_0 \) is a (strict) local minimizer of \( f \).