Solution #1: If \( f : D \rightarrow \mathbb{R} \) is continuous at \( x_0 \), then for all \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that whenever \( x \in D \) and \( |x - x_0| < \delta \), then \( |f(x) - f(x_0)| < \epsilon \). Since \( A, B \) are subsets of \( D \), this holds in particular whenever \( x \in A \) or \( x \in B \) (and \( |x - x_0| < \delta \)), proving that \( f : A \rightarrow \mathbb{R} \) and \( f : B \rightarrow \mathbb{R} \) are continuous at \( x_0 \). Vice versa, if \( f : A \rightarrow \mathbb{R} \) and \( f : B \rightarrow \mathbb{R} \) are continuous at \( x_0 \), then for all \( \epsilon > 0 \) there exist numbers \( \delta_A, \delta_B > 0 \) such that if either \( x \in A \), \( |x - x_0| < \delta_A \), or \( x \in B \), \( |x - x_0| < \delta_B \), then \( |f(x) - f(x_0)| < \epsilon \). Now define \( \delta := \min\{\delta_A, \delta_B\} > 0 \). If \( x \in D \) and \( |x - x_0| < \delta \), then it follows that either \( x \in A \), \( |x - x_0| < \delta \leq \delta_A \), or \( x \in B \), \( |x - x_0| < \delta \leq \delta_B \), and hence \( |f(x) - f(x_0)| < \epsilon \) in both cases, as required for the continuity of \( f \) at \( x_0 \).

Solution #2: Recall that a function \( f : D \rightarrow \mathbb{R} \) is called \( \alpha \)-Hölder if there exists a constant \( C \) such that \( |f(x) - f(y)| \leq C|x - y|^{\alpha} \) for all \( x, y \in D \). Using the \( \varepsilon\)-\( \delta \)-characterization, it is very easy to see that \( f \) is uniformly continuous: given \( \varepsilon > 0 \), we need to find some \( \delta > 0 \) depending only on \( \epsilon \) such that whenever \( x, y \in D \) and \( |x - y| < \delta \), it follows that \( |f(x) - f(y)| < \epsilon \); using the above, it then clearly suffices to take \( \delta = (\frac{\varepsilon}{C})^{\frac{1}{\alpha}} \). Alternatively, we can work with the definition of uniform continuity based on sequences: given any two sequences \( \{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \subset D \) with \( |x_n - y_n| \rightarrow 0 \), we need to show that \( |f(x_n) - f(y_n)| \rightarrow 0 \); so it suffices to show that if \( \{t_n\}_{n=1}^{\infty} \) is a sequence with \( t_n \geq 0 \) for all \( n \in \mathbb{N} \) and \( t_n \rightarrow 0 \), then \( t_n^{\alpha} \rightarrow 0 \); but this is almost immediate from the definition of a limit: given \( \varepsilon > 0 \), we need to find \( N \in \mathbb{N} \) such that \( t_n^{\alpha} < \varepsilon \) for \( n > N \); for this, simply use the assumption that \( t_n \rightarrow 0 \) to find \( N \in \mathbb{N} \) such that \( t_n < \varepsilon^{\frac{1}{\alpha}} \) for \( n > N \).

Note: Both solutions crucially rely on the fact that the function \( x \mapsto x^\alpha \) is increasing. To prove this fact, notice that, strictly speaking, we have defined \( x^\alpha \) only for \( \alpha \in \mathbb{Q} \); in this case, write \( \alpha = \frac{m}{n} \) with \( m, n \in \mathbb{N} \) and use that the functions \( y = \sqrt[n]{x} \) and \( z = y^m \) are increasing (which we already know), so that \( x_1 < x_2 \) implies \( y_1 < y_2 \) and hence \( z_1 < z_2 \), i.e. \( x_1^\alpha < x_2^\alpha \) as desired.

Solution #3: If \( f \) was not constant, then there would exist \( a, b \in [0, 1] \) with \( f(a) \neq f(b) \). Then \( a \neq b \), so we can assume without loss of generality that \( a < b \). Since the irrationals are dense (Corollary 1.10), there exists an irrational \( c \) between \( f(a) \) and \( f(b) \). It follows from the IVT that there exists an \( x_0 \in (a, b) \) such that \( f(x_0) = c \), contradicting the fact that \( f(x_0) \in \mathbb{Q} \).

Solution #4: Given \( x, \varepsilon \), define \( S := \{\delta > 0 : f(x + \delta) < f(x) + \varepsilon\} \) and define \( \delta_0 := \varepsilon^2 + 2\varepsilon\sqrt{x} \). Then we need to prove two things: (1) If \( \delta < \delta_0 \) then \( \delta \in S \) (this tells us that no number less than \( \delta_0 \) can be an upper bound of \( S \)). (2) If \( \delta > \delta_0 \) then \( \delta \notin S \) (this shows that \( \delta_0 \) is an upper bound of \( S \)). Given (1) and (2), it then follows that \( \delta_0 = \sup S \), the least upper bound of \( S \), as desired. The proofs of (1) and (2) are very similar, using the fact that \( f \) is strictly increasing: (1) If \( \delta < \delta_0 \) then \( f(x + \delta) < f(x + \delta_0) = \sqrt{x + 2\varepsilon\sqrt{x} + \varepsilon^2} = \sqrt{x} + \varepsilon = f(x) + \varepsilon \), so that \( \delta \in S \). (2) If \( \delta > \delta_0 \) then \( f(x + \delta) > f(x + \delta_0) = f(x) + \varepsilon \), so that \( \delta \notin S \).

Bonus Solution: Let \( x, y \in D \) and define \( m := \frac{x+y}{2} \). Notice that \( |x - m| = |y - m| = \frac{|x-y|}{2} \). By the triangle inequality and the definition of \( \alpha \)-Hölder,

\[
|f(x) - f(y)| \leq |f(x) - f(m)| + |f(m) - f(y)| \leq C \left( \frac{|x-y|}{2} \right)^\alpha + C \left( \frac{|x-y|}{2} \right)^\alpha = qC|x-y|^\alpha,
\]

where \( q := 2^{1-\alpha} \). Repeating this argument \( n \) times, we find that \( |f(x) - f(y)| \leq q^n C|x-y|^\alpha \). Now notice that \( q < 1 \) because \( \alpha > 1 \). Letting \( n \rightarrow \infty \), we deduce that \( |f(x) - f(y)| = 0 \).