Solution #1: To reduce to the special case $\ell = 1$, notice that if $|q| < 1$, then $Q := \sqrt{|q|} < \sqrt{1} = 1$ (proved in class), so if we can solve the problem for $\ell = 1$, then we will know that $nQ^n \to 0$. On the other hand, $|n!q^n| = n!|q|^n = (nQ^n)^\ell$, which (being a product of $\ell$ sequences that we just proved converge to zero) converges to zero. This solves the problem in general.

Now for the case $\ell = 1$, let $a := \frac{1}{|q|} - 1$. Then $a > 0$ because $|q| < 1$. As noted in class,

$$\frac{1}{|q|^n} = (1 + a)^n = 1 + \binom{n}{1}a + \binom{n}{2}a^2 + \cdots + \binom{n}{n}a^n \ (*)$$

by the binomial formula. In particular, $\frac{1}{|q|^n} > na$ because $a > 0$, so that $|q^n| = |q|^n < \frac{1}{na} \to 0$, proving that $q^n \to 0$. Now this is not quite enough: it only tells us that $\{nq^n\}_{n=1}^\infty$ is bounded. To fix this, we can just pick up the $a^2$ term in $(*)$ rather than the $a$ term (very similar to the proof in class that $\sqrt{n} \to 1$). Then $\frac{1}{|q|^n} > \frac{n(n-1)}{2}a^2$, so $|nq^n| = n|q|^n < \frac{2}{n^2a^2}$ (at least for all $n \geq 2$), which converges to zero, proving that $nq^n$ converges to zero as desired. (Alternatively one could pick up the $a^\ell+1$ term in $(*)$, assuming that $n \geq \ell + 1$, which solves the problem right away. Or prove by induction that $|n!q^n| \leq \tilde{q}^n$ for some $\tilde{q} \in (|q|, 1)$ and all large enough $n$.)

Solution #2: By the Monotone Convergence Theorem, it suffices to prove that the sequence $\{s_n\}_{n=1}^\infty$ with $s_n := \sum_{i=1}^n \frac{1}{i^2}$ is bounded from above. Since we already know that $\sum_{n=1}^\infty \frac{1}{n(n+1)}$ converges, it is clear that the sequence $\{\hat{s}_n\}_{n=1}^\infty$ with $\hat{s}_n := \sum_{i=1}^n \frac{1}{i(i+1)}$ is bounded; in fact, we even know from class that $\hat{s}_n = 1 - \frac{1}{n+1} < 1$ for all $n \in \mathbb{N}$. So it remains to find an inequality that bounds $s_n$ from above in terms of $\hat{s}_n$. Now notice that, for all $n \geq 2$,

$$s_n = \sum_{i=1}^n \frac{1}{i^2} = 1 + \sum_{i=2}^n \frac{1}{i^2} \leq 1 + \sum_{i=2}^n \frac{1}{(i-1)i} = 1 + \sum_{j=1}^{n-1} \frac{1}{j(j+1)} = 1 + \hat{s}_{n-1} < 2,$$

where we have set $j := i - 1$. Since $s_1 = 1 < 2$, it follows that $s_n < 2$ for all $n \in \mathbb{N}$. This is the desired upper bound. (Recall that $s_n \to \frac{\pi^2}{6}$, so we have just learned that $\pi \leq \sqrt{12}$. Also, you should compare this solution to the proof of the Condensation Test, and to Example 2.26.)

Solution #3: Define $\{a_n\}_{n=1}^\infty$ by $a_n := 2 - \frac{1}{n}$ if $n$ is odd and $a_n := 1$ if $n$ is even. Then no index $n \in \mathbb{N}$ is a peak index: indeed, if $n$ is odd, then $a_{n+2} = 2 - \frac{1}{n+2} > 2 - \frac{1}{n} = a_n$, whereas if $n$ is even, then $a_{n+1} = 2 - \frac{1}{n+1} \geq 2 - \frac{1}{3} > 1 = a_n$. On the other hand, $\{a_{2k-1}\}_{k=1}^\infty = \{1, 1, 1, 1, \ldots\}$ is clearly a monotone decreasing subsequence of $\{a_n\}_{n=1}^\infty$. (Observe that $\{a_{2k-1}\}_{k=1}^\infty$ is a strictly increasing subsequence, and we knew from class that such a subsequence had to exist. So one natural approach to solving this problem would be to start with some strictly increasing and some (not necessarily strictly) decreasing sequence and then blend the two together.)

Solution #4: Let $a, b \in \mathbb{R}$ with $a < b$. Then the function $f : (a, b) \to \mathbb{R}$, $f(x) := \frac{1}{x-a}$, is clearly continuous (by Proposition 1(i) from class), but it is not bounded above because $f(a + \frac{1}{n}) = n$ for all $n \in \mathbb{N}$ with $a + \frac{1}{n} < b$, so that $f((a, b)) \supset \{n \in \mathbb{N} : n > \frac{1}{b-a}\}$, which we already know is an unbounded set. On the other hand, the function $g : (a, b) \to \mathbb{R}$, $g(x) := x$, is bounded from above (by $b$) because $g((a, b)) = (a, b)$, but then $\sup g((a, b)) = \sup \{a, b \in (a, b) \}$ so that $g$ does not have a maximizer (more precisely, $g$ does not attain its supremum). (Both of these examples are slight modifications of examples from class.)