Solution #1: I overlooked that this is basically Theorem 6.19 in the book.

Solution #2: Let \( f(x) = \sqrt{1+x} \), so that \( f'(x) = \frac{1}{2\sqrt{1+x}} \) and \( f''(x) = -\frac{1}{4\sqrt{1+x^3}} \). Thus,
\[
1 + \frac{x}{2} - \frac{x^2}{8} \leq 1 + \int_0^x \frac{1}{2} - \int_0^t \frac{ds}{4\sqrt{1+s^3}} dt = \sqrt{1+x} = 1 + \int_0^x \frac{dt}{2\sqrt{1+t}} \leq 1 + \frac{x}{2}.
\]
To get the inequalities, I have used that \( \sqrt{1+t} \geq 1 \) and \( \sqrt{1+s^3} \geq 1 \) because \( t, s \geq 0 \).

Solution #3: Under the assumptions of this exercise, the Lagrange remainder estimate tells us that for all \( x \neq x_0 \) there exists some \( x_1 \) strictly between \( x \) and \( x_0 \) such that
\[
f(x) - f(x_0) = \frac{1}{(n+1)!} f^{(n+1)}(x_1)(x-x_0)^{n+1},
\]
simply because \( (T^n_{x_0}f)(x) = f(x_0) \). In (a) [(b)], the right-hand side is positive [negative] for all \( x_1 \neq x_0 \), so then \( x_0 \) is clearly a (strict) local minimizer [maximizer] of \( f \). In (c), \( (x_1 - x_0)^{n+1} \) is positive [negative] for \( x_1 > x_0 \) \( [x_1 < x_0] \), so \( f(x) - f(x_0) \) changes sign as \( x \) crosses \( x_0 \) from the left, so \( x_0 \) is not a local extremizer of \( f \) (but is a point of inflection).

Solution #4: It is easy to prove by induction that \( f^{(j)}(x) = (-1)^j j! x^{-j-1} \) for all \( j \in \mathbb{N}_0 \). Thus, in particular, \( f^{(j)}(x_0) = (-1)^j j! \) for all \( j \in \mathbb{N}_0 \). Hence, by definition,
\[
(T^n_{x_0}f)(x) = \sum_{j=0}^n \frac{f^{(j)}(x_0)}{j!} (x-x_0)^j = \sum_{j=0}^n (1-x)^j = 1 + (1-x) + (1-x)^2 + \cdots + (1-x)^n.
\]
This means that the Taylor series of \( f \) is simply a geometric series, \( (T^n_{x_0}f)(x) = \sum_{n=0}^\infty q^n \) with \( q = 1-x \), which we know converges if and only if \( |q| < 1 \), or in other words, \( x \in (0,2) \). And if this geometric series converges, then its limit is given by \( \frac{1}{1-q} = \frac{1}{x} = f(x) \).

Bonus Solution: That \( F \) is differentiable (with \( F'(0) = 0 \)) follows as in HW7, #3. That \( F' \) is integrable follows from #1 above. Now Theorem 6.22 tells us that \( F(x) = \int_0^x F(t) \, dt \) because, in this theorem, \( F' \) is allowed to be discontinuous at one point (I overlooked that too). In fact, the proof of Theorem 6.22 still works even if we assume only that \( F' \) is integrable.

For an alternative solution, given \( x > 0 \), take any \( \epsilon \in (0,x) \) and write
\[
0 \leq \left| \left( \int_0^x f \right) - (F(x) - F(\epsilon)) \right| \overset{(1)}{=} \left| \int_0^x f - \int_\epsilon^x f \right| = \left| \int_0^\epsilon f \right| \overset{(2)}{\leq} \int_0^\epsilon |f| \overset{(3)}{\leq} (2\epsilon + 1)\epsilon.
\]
(In 1) I have used that the version of the FTC given in class applies to \( F \) on the interval \( [\epsilon,x] \) because \( f = F' \) is continuous there. (2) is the triangle inequality for integrals. (3) follows from the fact that \( |f(t)| = |2t\sin(\frac{1}{t}) - \cos(\frac{1}{t})| \leq 2\epsilon + 1 \) for all \( t \in (0,\epsilon) \).) Now let \( \epsilon \to 0 \) and observe that \( F(\epsilon) \to F(0) = 0 \) because \( F \), being differentiable, is in particular continuous.

A different way would be to generalize Theorem 6.29 to our setting, but this is much harder because the continuity of \( f \) (including at the boundary points of the interval) plays an essential role in the proof of that theorem. One would have to show that \( \frac{1}{x} \int_0^x \cos \frac{1}{t} \, dt \to 0 \) as \( x \to 0 \). To see roughly why this is true, approximate \( \cos \frac{1}{t} \) by a sawtooth function. Then
\[
\lim_{x \to 0} \frac{1}{x} \int_0^x \cos \frac{1}{t} \, dt = \lim_{n \to \infty} \sum_{i=n}^\infty \left[ \left( \frac{1}{2i} - \frac{1}{2i+1} \right) - \left( \frac{1}{2i+1} - \frac{1}{2i+2} \right) \right] = \lim_{n \to \infty} \sum_{i=n}^\infty \frac{1}{i^3} = 0.
\]