For extra credit only!

Let \( f(x) = x^k \) for \( k \in \mathbb{N} \). The partition method for computing \( \int_0^1 f \) discussed in class requires information about the sum \( \sum_{i=1}^n i^k \). I showed you formulas for this sum when \( k = 1, 2, 3 \).

Problem 1: Prove that \((i + 1)^5 - i^5 = 5i^4 + 10i^3 + 10i^2 + 5i + 1\). Then sum both sides of this equation from \( i = 1 \) to \( n \). From this, derive an explicit formula for \( \sum_{i=1}^n i^4 \).

This ingenious idea (due to Pascal) can be viewed as a discretized version of the Fundamental Theorem of Calculus. The following problem shows that no such ingenuity is required if one is willing to work with more complicated partitions than the standard \( \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\} \).

Problem 2: Given \( q > 1 \) and \( n \in \mathbb{N} \), define a partition \( P_{n,q} = \{x_0, x_1, \ldots, x_n\} \) of \([0,1]\) by \( x_0 = 0 \) and \( x_i = q^{i-n} \) for \( i \in \{1, \ldots, n\} \). Using the geometric series formula, show that

\[
L(f, P_{n,q}) = \sum_{i=2}^{n} (q^{i-n} - q^{i-1-n})(q^{i-1-n})^k = \frac{q - 1}{q^{k+1} - 1}(1 - q^{(1-n)(k+1)}),
\]

\[
U(f, P_{n,q}) = q^{-n}q^{(1-n)k} + \sum_{i=1}^{n} (q^{i-n} - q^{i-1-n})(q^{i-n})^k = q^{k-(k+1)n} + \frac{q - 1}{q^{k+1} - 1}q^k(1 - q^{-n(k+1)}).
\]

Fix \( q > 1 \) and let \( n \) tend to infinity to conclude that

\[
\frac{q - 1}{q^{k+1} - 1} \leq \int_0^1 f \leq \frac{q - 1}{q^{k+1} - 1}q^k.
\]

Finally, let \( q \) tend to 1 to conclude that \( f \) is integrable with \( \int_0^1 f = \frac{1}{k+1} \).