RESEARCH SUMMARY

CRIS POOR

1. Introduction

My current research concerns the theory and computation of Siegel modular forms. Complex analytic techniques are the main tool although, besides being a part of several complex variables and transcendental algebraic geometry, the field of automorphic functions increasingly falls under the domain of number theory. I also publish results in the geometry of numbers, although I do not stray far from applications to Siegel modular forms.

In the past I have published in the fields of mixed Hodge theory, Riemann surfaces, theta functions and particle coagulation. I may return to these fields and write up unpublished results but currently I am completely occupied with questions about Siegel modular forms.

Let Γ be a group commensurable with $\mathrm{Sp}_g(\mathbb{Z})$. Siegel modular forms of weight k for Γ are holomorphic functions on the Siegel upper half space that are bounded at the cusps and invariant under an action of Γ that depends upon the weight k. They give mappings of $\mathcal{A}_g(\Gamma)$, the moduli space of principally polarized abelian varieties with level structure Γ, into projective spaces. The moduli space $\mathcal{A}_g$ has many important subvarieties such as modular curves, the hyperelliptic locus and the Jacobian locus. One important topic is the behavior of Siegel modular forms on these subvarieties; for example, the Schottky problem asks for the ideal of Siegel forms vanishing on the Jacobian locus. Articles 1, 2, 3, 5 and 15 concern these subvarieties.

The L-functions of Siegel modular Hecke eigenforms are a second important topic. As part of the Langlands program, we should look for generalizations of A. Wiles' Modularity Theorem to higher genera. Articles 20 and 22 are directly concerned with L-functions of Siegel modular Hecke eigenforms.

Every Siegel modular form has a Fourier expansion.

$$f(Z) = \sum_T a(T) e\left(\langle Z, T \rangle\right).$$

Date: October 7, 2008.

The Fourier coefficients $a(T)$ encode geometric, number theoretic and combinatorial information and there are surprising identities among modular forms arising from quite different sources. For proofs of these identities, the dimension of a space of Siegel modular cusp forms is crucial information; in concrete examples, determining sets of Fourier coefficients will sometimes suffice. For genus $g \leq 3$, many generating functions giving these dimensions are known but none are known for $g \geq 4$. Prof. D. Yuen and I have been developing general methods to compute dimensions of Siegel forms in specific cases. From a computational point of view, Siegel cusp form are huge objects. Each stage of progress in the computation of spaces of Siegel modular cusp forms has been due to a new theoretical insight, not merely to increased computing power. First, the Vanishing Theorems giving estimations for determining sets of Fourier coefficients were improved in articles 8 and 11. This goal required extensions to the geometry of numbers that were given in articles 11 and 16; these extensions have proven useful in their own right, resolving one conjecture of J. Martinet, see article 17. Second, the Restriction Technique, see articles 14, 18, 19 and 20, was introduced to find linear relations among Fourier coefficients. Third, in article 22, a method of computing integral closures is introduced for paramodular forms of genus two. The next innovation, using the pullback of Eisenstein series, will appear in article 26. Our current project, articles 22 and 24, is building evidence for higher modularity theorems according to the Paramodular Conjecture as explained to us by my colleague A. Brumer.

2. Thesis

1. *Cross-Ratio Identities for Theta Functions on Jacobi Varieties,* Princeton University, (1-24), 1988.

My thesis derived the *cross-ratio identities* for Jacobian theta function in general and for hyperelliptic theta functions in particular. For the hyperelliptic case, this work was directly continued in article 3. The cyclic-covering case was further developed in [7] by H. Farkas.

3. Published Refereed Articles

2. *Fay's Trisecant Formula and Cross-Ratios,* Proc. AMS, vol. 114. no. 3, (667-671), 1992.

This proof of Fay's trisecant formula is based on the Riemann surface function theory developed by my thesis advisor, R. C. Gunning. I regret not explicitly including a similar proof of the multisecant identity. One can find it hidden on page 848 of article 3. An exposition of R. Gunning's cross-ratio function is in the final section added to the second edition of *Riemann Surfaces* by H. Farkas and I. Kra, the excellent text book whose first edition I used as a graduate student.

3. *The Hyperelliptic Locus,*
Duke Math. J., vol. 76. no. 3, (809-884), 1994.

Here it is proved that irreducible hyperelliptic Jacobians are characterized among all principally polarized abelian varieties by their vanishing nullwerte, an unresolved problem for over a century. In 1984, D. Mumford [22] proved that hyperelliptic Jacobians are characterized among all principally polarized abelian varieties by the vanishing and nonvanishing of their nullwerte. Neither result seems to be a corollary of the other. This article is a continuation of article 1. H. J. Weber [28] used this result to locate hyperelliptic Jacobians related to modular curves. I have unpublished results that hyperelliptic Jacobians are characterized by their vanishing nullwerte alone in genera four and five.

4. *Relations on the Period Mapping giving Extensions of Mixed Hodge Structures on Compact Riemann Surfaces.*
Geometriae Dedicata, vol. 59, (243-291), 1996. (with D. Yuen)

R. Gunning [10] gave a novel proof of the symmetry of the Riemann period matrix by studying the periods of iterated integrals of holomorphic differentials, work continued by his student E. Jablow [18]. R. Hain [11] and his student M. Pulte [26] studied the mixed Hodge structure on compact Riemann surfaces via periods of homotopy functionals consisting of twice iterated integrals of holomorphic and anti-holomorphic differentials. We attempted to unify these two approaches. We proved that the holomorphic periods alone generically determine the mixed Hodge structure. We found all higher order symmetries of the period map arising from homotopy functionals among triple iterated integrals. An intrinsic formulation of these symmetries is that that period map from Teichmüller space factors through a third exterior power bundle over $\mathcal{A}_g$. This subject as been furthered [19] in the thesis of R. Kaenders.

5. *Schottky's Form and the Hyperelliptic Locus*
Proc. AMS, vol. 124, no. 7, (1987-1991), 1996.

J. I. Igusa proved [17] that Schottky's form in genus four was given by the difference of the theta series for the two classes of even unimodular lattices in dimension sixteen. Schottky's form vanishes on the Jacobian locus in genus four. Here it is proven that this difference of theta series vanishes on the hyperelliptic locus in every genus. S. Grushevsky and R. Salvati Manni recently proved [9] that this difference does not vanish on the Jacobian locus in genus five but rather cuts out the trigonal divisor on $\mathcal{M}_5$.

6. *Dimensions of spaces of Siegel modular forms of low weight in degree four.*
Bull. Austral. Math, Soc., vol. 54, (309-315), 1996. (with D. Yuen)
We found the dimensions of genus four Siegel cusp forms in weights 6, 8 and 12. The initial classification of the theta series of the Niemeier lattices [5][6] by V. A. Erokhin was the main tool. The cases of weight 6 and 8 are corollaries of a theorem [27] by R. Salvati Manni but we were unaware of this at the time. This note was immediately responded to by W. Duke and O. Imamoglu in an article [3] using explicit formulae. Our results were also used by E. Freitag and M. Oura in [8].

7. *Scaling theory and solutions for steady-state coagulation and settling of fractal aggregates in aquatic systems.*
Colloids and Surfaces A, vol. 107, (155-174), 1996. (S. Grant, S. Relle)
The Smoluchowski coagulation equation, in integro-differential form, is used to model steady-state coagulation in combination with gravitational settling. Our model fits the particle volume concentrations of Lake Zurich, Switzerland pretty well except for the large aggregates which are likely being removed from the water column by other means.

8. *Estimates for Dimensions of Spaces of Siegel Modular Cusp Forms*
Abh. Math. Sem. Univ. Hamburg, vol. 66, (337-354), 1996. (D. Yuen)
This paper grew from a desire to explicitly compute with theta series in higher genera. Siegel gave a constructive proof of the finite dimensionality of vector spaces of Siegel modular forms. He showed that the vector spaces were determined by Fourier coefficients with indices of bounded trace. M. Eichler has a sharper vanishing theorem [4] that uses the Minimum function $m(v) = \min_{x \in \mathbb{Z}^g \setminus \{0\}} x'vx$ and relies on Hermite's constant from the geometry of numbers. We interpolated the proofs of Siegel and Eichler to get a reasonably sharp estimation theorem for any convex function. As an application, we reproved Witt's conjecture [29] by computing just one Fourier coefficient. In this sense, this article is a sequel to [29], where Witt lamented the *ungeheuere Rechnungen*. This work is continued in article 11.

9. *Particle coagulation and the memory of initial conditions.*
J. Phys. A., vol. 31, (9241-9254), 1998. (A. Boehm, S. Grant)
The Smoluchowski coagulation equations are a countably infinite set of coupled nonlinear ordinary differential equations. When there is uniform attraction among particles of different size, the large particle

normalized distributions converge to a fixed function of the scaling parameter, independent of the initial conditions. The scaling parameter is a ratio of particle size to time. It had been widely assumed that the long time particle distributions were independent of the initial conditions as well but we showed that this is not the case. The small particle distributions retain a memory of the initial conditions indefinitely. The magnitude of the effect depends upon the proximity to the unit circle of the the complex roots of the generating function given by the initial conditions.

10. *Dimensions of Spaces of Siegel Modular Forms and Theta-Series with Pluri-harmonics.*
Far East J. Math. Sci., vol. 1. no. 6, (849-863), 1999. (D. Yuen)

We classified spaces of Siegel modular forms spanned by theta series with pluriharmonic coefficients in some cases in genus four. The main value of the article lies in the computational techniques. In particular, we symmetrized the pluriharmonic polynomials under the automorphism group of the lattice and used the automorphisms to reduce the complexity of the computations.

11. *Linear dependence among Siegel modular forms.*
Math. Ann., vol. 318. (205-234), 2000. (with D. Yuen)

The intrinsic "vanishing order" of a Fourier series f is a convex set $\nu(f)$: the convex semihull of the support of f. We prove the Semihull Theorem: For Siegel modular cusp forms f of weight k, if $Y^{k/2}|f(Z)|$ attains its maximum at $X_0 + iY_0$, then $\frac{k}{4\pi} Y_0^{-1} \in \nu(f)$. From the Semihull Theorem, we may recover the estimates for convex functions first proven in article 8. The theorems proven here are used in articles 13, 14, 15, 16, 17, 18, 19 and 22.

We introduce an important convex function, the dyadic trace, and use it to give improved determining sets of Fourier coefficients. For example, in weight 12 and genus 4, a determining set consists of the 23 classes whose dyadic trace is less than or equal to 4; whereas the trace estimate of Siegel requires over $100,000$ classes. The dyadic trace has been defined and applied in the case of Hermitian modular forms in the thesis [20] of M. Klein.

12. *Kinetic Theories for the Coagulation and Sedimentation of Particles.* (S. Grant, J. Kim)
J. Colloid and Interface Science, vol. 238. (238-250), 2001.

The Smoluchowski coagulation equations are modified to include sedimentation. We assume that there is uniform attraction among particles of different size and that sedimentation increases linearly with particle size. A solution is obtained for the corresponding nonlinear partial differential equation and we compare the solutions with the predictions of two common approximations: the similarity theory and the quasi-steady-state hypothesis. The latter does not fare well but the similarity theory predicts critical exponents that are within about 20% of those of our solution. The boundary of the coagulation dominated zone is found by analyzing a nonlinear ordinary differential equation.

13. *The Dyadic Trace and Odd Weight Computations For Siegel Modular Cusp Forms.*
Bull. Austral. Math, Soc., vol. 63, (269-271), 2001. (with D. Yuen)

We illustrate the results of article 11 in the case of odd weights. A Fourier coefficient is no longer a class function but its vanishing is.

14. *Restriction of Siegel modular forms to modular curves.*
Bull. Austral. Math, Soc., vol. 65, (239-252), 2002. (with D. Yuen)

The Restriction Technique is introduced to produce linear relations among the Fourier coefficients of Siegel modular forms. Siegel forms are restricted to modular curves and the known linear relations among elliptic modular forms are pulled back. After article 11 provided tractable determining sets of Fourier coefficients, a method was needed to find the linear relations among these Fourier coefficients. This is an old problem, equivalent to finding the linear relations among certain Poincare series. As an application, we computed the dimension of S_4^{10}, a space just beyond the reach of the explicit formula method [3]. If you wish to learn the Restriction Technique, this is the article to read. Further applications of the Restriction Technique are given in articles 18, 19 and 20.

15. *Slopes of Integral lattices.*
J. of Number Theory, vol. 100. (363-380), 2003. (with D. Yuen)

J. I. Igusa found critical slopes for the hyperelliptic locus [16]. J. Harris and I. Morrison found critical slopes for the trigonal locus and gave an elegant conjecture for the Jacobian locus [12]. We use the dyadic trace to find critical slopes for the modular curves from article 14. Our modular curves arise from lattices and the slope of a lattice is a new integral invariant. The dyadic traces and slopes of all root lattices are computed. The main result is that a Siegel modular cusp form vanishes on a modular curve if the slope of the cusp form is less than the slope of the lattice.

16. *The Extreme Core.*
Abh. Math. Sem. Univ. Hamburg, vol. 75, (51-75), 2005. (D. Yuen)

This article is a continuation of article 11 and advances the geometry of numbers for its applications to Siegel modular forms. The topics include kernels, cores and noble forms. We show the existence of the extreme core in each genus. This is a core $\mathcal{C}_{\text{ext}}$ such that $k\,\mathcal{C}_{\text{ext}} \subseteq \nu(f)$ for every nontrivial cusp form f of weight k. Thus the existence of the extreme core is a generalization of the Valence Inequality in genus one. We estimate the extreme core in all genera and almost specify it in genus two, leaving the true value of $w_0 = \sup_f \inf\langle A_2, \frac{1}{k}\nu(f)\rangle$ as an open problem. The constants in the Vanishing theorems are improved and the improvements are used in articles 19 and 22 and in [20].

17. *The Bergé-Martinet constant and slopes of Siegel Cusp Forms.* Bull. London Math. Soc., vol. 38. (913-924), 2006. (with D. Yuen)

This is a topic in the geometry of numbers. We use the dyadic trace to determine the value of the Bergé-Martinet constant for degrees $5, 6$ and 7. We prove Conjecture 6.4.16 in J. Martinet's recent book [21] by finding all dual-critical pairs. As an application to Siegel modular forms, we use these newly found values of the Bergé-Martinet constant to improve upon M. Eichler's lower bound [4] for the optimal slope of a cusp form in degrees $5, 6$ and 7.

18. *Computations of spaces of Siegel modular cusp forms.*
J. Math. Soc. Japan, vol. 59. no. 1, (185-222), 2007. (with D. Yuen)

This is a systematic exposition of the Restriction Technique written with Professor David S. Yuen, a preliminary version was published in [24]. Advances are made in both theory and computation. Determining sets of Fourier coefficients from article 11 and the Restriction Technique from article 14 are combined into a systematic method for computing individual spaces of Siegel modular cusp forms. For $g > 3$, most known cases are recovered and some new cases are computed in genera 4, 5 and 6. We compute Hecke eigenforms in the nontrivial cases.

It is natural to ask whether our method of computing spaces of Siegel modular cusp forms always works. We were able to prove a partial result: the linear relations generated by the Restriction Technique characterize the Fourier expansions of Siegel modular cusp forms from among all *convergent* Fourier series. Whether or not the linear relations generated by the Restriction Technique characterize the Fourier expansions of Siegel modular cusp forms from among all formal Fourier series is an open problem.

A similar open problem will appear in article 25. The natural application of our method to congruence subgroups is in article 19. The L-functions of the new Hecke eigenforms will appear in article 21.

19. *Dimensions of Cusp Forms for $\Gamma_0(p)$ in Degree Two and Small Weights.*
Abh. Math. Sem. Univ. Hamburg, vol. 77, (59-80), 2007. (D. Yuen)

We explain how to use the method of article 18 and the constants of article 16 for spaces of Siegel modular cusp forms of finite index in the Siegel modular group. We applied our method to the weight one spaces $S_2^1(\Gamma_0(p))$ and found that they were trivial for primes $p \leq 97$. This was unexpected and brought attention to the weight one case. T. Ibukiyama and N. Skoruppa proved that $S_2^1(\Gamma_0(N))$ always vanishes [14] and their article appears in the same volume of the *Abhandlungen.* In weight two, we examined the cases $p \leq 41$ and found only Saito-Kurokawa and Yoshida lifts; unlike the weight one case, this pattern cannot continue indefinitely. An enumeration of weight two forms will require connections with rational abelian surfaces, see article 22 below.

In weights three and four, we verified conjectures of K. Hashimoto [13] in some cases. T. Ibukiyama has since proven [15] these conjectures. The obstruction to the weight three case was the weight one case, which vanishes as mentioned above.

20. *Toward the Siegel Ring in Genus Four.*
Inter. J. Number Th. vol. 4. no. 4, (563-586), 2008. (M. Oura, D. Yuen)

In [8], E. Freitag and M. Oura found the first second-order theta relation in genus four. This article is a direct continuation of that work but the computations require the Restriction Technique from articles 14 and 18. We classify all relations through degree 32, finding six new relations. A sequel to this paper, showing that the Th_2 map is not surjective in genus four, has already been written by M. Oura and R. Salvati Manni [23].

4. Research accepted for publication

21. *Lifting Puzzles in Degree Four.*
(authors: C. Poor, N, Ryan, D. Yuen)
(accepted to: The Bulletin of the Australian Mathematical Society)

Dr. N. Ryan wrote his thesis on the computation of Satake parameters. We use his work and the Hecke eigenforms found in article 18 to compute Euler factors of L-functions in genus four. These are the first examples of their type in genus four. Not all the Satake parameters in our examples are unimodular; therefore, if the Generalized Ramanujan-Petersson conjecture can be properly reformulated at all, there must be two new types of lifts yet to be discovered.

References

[1] Ash, A. and Gunnells, P. and McConnell, M.: *Cohomology of Congruence Subgroups of* $\mathrm{SL}(4,\mathbb{Z})$ *II.* (preprint)

[2] Böcherer, S.: *Siegel Modular Forms and Theta Series.* Proc. of Symposia in Pure Math., Part 2, (1989), 3-17.
[3] Duke, W. and Imamoḡlu, O.: *Siegel modular forms of small weight.* Math. Ann. **310**, *(no. 1),* (1998), 73-82.
[4] Eichler, M.: *Über die Anzahl der linear unabhängigen Siegelschen Modulformen von gegebenem Gewicht.* Math. Ann. **213**, (1975), 281-291.
[5] Erokhin, V. A.: *Theta series of even unimodular 24-dimensional lattices.* LOMI. **86**, (1979), 82-93.
[6] Erokhin, V. A.: *Theta series of even unimodular lattices.* LOMI. **199**, (1981), 59-70.
[7] Farkas, H. M.: *Generalizations of the λ-function.* Israel. Math. Conf. Proc. **9**, (1996), 231-239.
[8] Freitag, E. and Oura, M.: *A theta relation in genus* 4. Nagoya Math. J. **161**, (2001), 69-83.
[9] Grushevsky, S. and Salvati Manni, R.: *On the Cosmological constant for the chiral superstring measure.* arXiv: 0809 1391v1, September 8, 2008.
[10] Gunning, R. C.: *Quadratic periods of hyperelliptic abelian integrals,* Problems in Analysis, Princeton Univ. Press: Princeton, (1970), 239-247.
[11] Hain, R.: *The geometry of the mixed Hodge structure on the fundamental group.* Proc. Sympos. Pure Math. **46**, *(II),* (1987), 247-282.
[12] Harris, J. and Morrison, I.: *Slopes of effective divisors on the moduli space of stable curves.* Invent. Math. **99**, (1990), 321-355.
[13] Hashimoto, K.: *The dimension of spaces of cusp forms on Siegel upper half-plane of degree two (I).* J. Fac. Sci. Univ. Tokyo Sect. IA Math.**30**, (1983), 403-488.
[14] Ibukiyama, T. and Skoruppa, N.: *A vanishing theorem for Siegel modular forms of weight one.* Abh. Math. Sem. Univ. Hamburg **77**, (2007), 229-235.
[15] Ibukiyama, T.: *Dimension Formulas of Siegel Modular Forms of Weight 3 and Supersingular Abelian Surfaces* Proceedings of the 4-th Spring Conference: Siegel Modular Forms and Abelian Varieties, Hamana Lake, Japan (2007), 39-60.
[16] Igusa, J.-I.: *Modular forms and projective invariants.* Amer. J. Math. **89**, (1967), 817-855.
[17] Igusa, J.-I.: *Schottky's invariant and quadratic forms* E. B. Christoffel Int. Symp., Aachen (1981), 352-362 .
[18] Jablow, E.: *Quadratic vector classes on Riemann surfaces.* Duke Math. J. **53**, (1986), 221-232.
[19] Kaenders, R. J.: *On De Rham Homotopy Theory for Plane Algebraic Curves and their Singularities.* Proefschrift, Katholieke Universiteit Nijmegen (1997), 1-167.
[20] Klein, M.: *Verschwindungssaätze für Hermitesche Modulforme sowie Siegelsche Modulformen zu den Kongruenzuntergruppen $\Gamma_0^{(n)}(N)$ und $\Gamma^{(n)}(N)$.* Dissertation der Universität des Saarlandes, Saarbrüken (2004), 1-170.
[21] Martinet, J.: *Perfect lattices in Euclidean spaces.*. Grundlehren der mathematischen Wissenschaften, Band **327**. Springer-Verlag, New York-Heidelberg, 2003.
[22] Mumford, D.: *Tata Lectures on Theta II.* Progress in Math., Vol. **43**, Birkhäuser: Boston, Basel, Stuttgart. (1984).
[23] Oura, M. and R. Salvati Manni: *On the image of code polynomials under theta map.* arXiv:0803.4389

[24] Poor, C.: *Computations of Spaces of Siegel Modular Cusp Forms* RIMS Proceedings: Automorphic forms and their Dirichlet series (editor: H. Katsurada), Kyoto, Japan (2002).

[25] Poor, C. and Yuen, D.: *Paramodular Cusp Forms* Proceedings of the 4-th Spring Conference: Siegel Modular Forms and Abelian Varieties, Hamana Lake, Japan (2007), 198-215.

[26] Pulte, M.: *The fundamental group of a Riemann surface: mixed Hodge structures and algebraic cycles.* Duke Math. J. **57**, (1988), 721-760.

[27] Salvati Manni, Riccardo.: *Modular forms of the fourth degree.* (Remark on a paper of Harris and Morrison) Lect. Notes Math. **1515**, (Classification of Irregular Varieties), (1992), 106-111.

[28] Weber, H.-J.: *Hyperelliptic simple factors of* $J_0(N)$ *with dimension at least* 3. Exper. Math. **6**, *(no. 4)*, (1997), 273-287.

[29] Witt, E.: *Eine Identität zwischen Modulformen zweiten Grades.* Abh. Math. Sem. Hans. Univ. **14**, (1941), 323-337.

Department of Mathematics, Fordham University, Bronx, NY 10458-5165

E-mail address: poor@fordham.edu