THE SYNTAX AND SEMANTICS OF STANDARD QUANTIFICATION THEORY WITH
IDENTITY (SQT)

The construction of this theory begins with the description of the syntax of the formal language of the
theory, by first enumerating the primitive symbols and then giving a recursive definition of the set of
formulae that can be built up from these.

So the language of the theory is defined as the following set theoretical structure:

\[ L := \langle C, P, V, T, F \rangle \]

where \( C \), the set of logical constants is defined as follows: \( C := \{ \neg, &, \forall, =, (,) \} \). \( P \), the set of parameters is
the union of the set of individual parameters, \( P_{\text{ind}} \), intuitively corresponding to proper nouns, and the set of
predicate parameters, \( P_{\text{pred}} \), corresponding to predicates of natural languages. Elements of \( P_{\text{ind}} \) are the small
Latin letters \( a, b, c \) and \( d \), while elements of \( P_{\text{pred}} \) are the capital letters, \( F, G \) and \( H \). \( V \) is the set of
individual variables, intuitively corresponding to pronouns of natural languages, containing the letters \( x, y \) and \( z \). Occasionally, if we need more distinct symbols in these categories, we may use subscripts to the
above letters.

\( T \), the set of terms is the union of \( P_{\text{ind}} \) and \( V \), i.e., \( T := P_{\text{ind}} \cup V \).

\( F \), the set of formulae of \( L \), contains the complex expressions of \( L \), corresponding to sentences of natural
languages. \( F \) is defined by the following recursive definition:

1. If \( t_1, \ldots, t_n \in T \) and \( P^n \in P_{\text{pred}} \) then \( 'P^n(t_1)\ldots(t_n)' \), \( 't_1=t_2' \in F \)
2. If \( A, B \in F \), then \( '(A&B)' \), \( '~(A)' \in F \)
3. If \( v \in V \), \( A \in F \), then \( '(\forall v)(A)' \in F \)
4. No other strings of symbols are elements of \( F \).

By these clauses, for example, \( '(\forall x)((\forall y)((F(x)(y))\&\neg(G(x)(a)(z))))' \) is a formula, while \( '=a((Pxy)' \) is not. The
number of parentheses in the well-formed formulae may be reduced by quite obvious conventions which I
shall not enumerate here. Further conventions introduce the other usual connectives, like
\( 'A\rightarrow B'=_{df.}'\neg(A\&\neg(B))' \), \( '(\exists x)(A)'=_{df.}'(\forall x)\neg(A)' \) etc., but these are needed only for practical reasons, they are
always analyzable in terms of their definientia.

A semantics for this language is constructed by defining a model that interprets its symbols by assigning
semantic values to them. The basic requirement that such a definition should meet is the rule of
compositionality, according to which the semantic values of the complex expressions of the language are
to be determined as functions of the semantic values of their components. The semantic values of
primitive symbols of the language, however, are determined by free-choice functions.

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1 Note here that the recursiveness of such a definition consists in the fact that some of its clauses may be applied to the result of its application. The single quotes in this definition (and in the semantic definitions below) should be interpreted as quasi-quotes, indicating the concatenation of the values of metavariables and logical constants enclosed by them in the order they appear.

2 The heuristic rationale for this rule is that in this way the formal semantics will reflect how a natural language user will construct the meaning of a complex expression, e.g. a sentence, that he or she never heard before, on the basis of an understanding of its components.

3 Which reflects the fact that the meanings of primitive expressions are conventional. That the idea of compositionality in semantics was not alien to mediaeval logicians either is shown by the following interesting remark by G. Nuchelmans: "... the signification of the whole complex was commonly held to be of a compositional nature and to be determined by the signification of its parts. As Pardo put it, only incomplex expressions have been given conventional meanings in a primary and immediate way; a propositional complex, such as Homo est animal, on the other hand, has been destined to signify its meaning only in a mediate, consequential and secondary manner, since its signification can be derived from the
So a model for L is defined as the following set-theoretical structure:

\[ M := \langle W, R \rangle \]

where \( W \) is a nonempty set, called the domain of discourse, and \( R \) is a function assigning semantic values to the parameters of L, defined by the following clauses:

1. If \( a \in \mathcal{P}_{\text{ind}} \), then \( R(a) \in W \)
2. If \( P_n \in \mathcal{P}_{\text{pred}} \), then \( R(P_n) \subseteq W^n \), i.e., it is a subset of the set of ordered n-tuples of elements of \( W \).

Given a model, we can define a value assignment, \( f \), a function that assigns individuals to terms, and the values 1 or 0 to formulae. (Intuitively, this function represents the interpretation of pronouns and the truth or falsity of propositions depending on this interpretation, like 'He is listening to Paul' is true when the pronoun is interpreted as standing for Peter, while false for Paul. Following Tarski, we also say that an assignment of variables satisfies a formula, to express the same informal idea.)

(1) If \( a \in \mathcal{P}_{\text{ind}} \), then \( f(a) = R(a) \)
(2) If \( v \in V \), then \( f(v) \in W \)
(3) \[ f(P^a(t_1), \ldots, t_n) = 1 \iff \langle f(t_1), \ldots, f(t_n) \rangle \in R(P) \]
(4) \[ f(t_1 = t_2) = 1 \iff f(t_1) = f(t_2) \]
(5) If \( v \in V \), \( A, B \in F \), then \( f('\neg A') = 1 \iff f(A) = 0 \), \( f('A \& B') = 1 \iff f(A) = f(B) = 1 \) and \( f('\forall v)(A)') = 1 \iff \) for every \( u \in W \), \( f[v:u][A] = 1 \), where \( f[v:u](w) = u \), if \( w = v \) (that is to say, \( f[v:u](v) = u \)), otherwise \( f[v:u](w) = f(w) \), that is to say, \( f[v:u] \) (read: \( f \) changed in \( v \) to \( u \)) is the same as \( f \) except, perhaps, that it assigns \( u \) to \( v \).

Finally, the definition of truth in this system is as follows:

\[ M \Rightarrow A \text{ iff for every } f \text{ in } M, f(A) = 1 \]

that is to say, \( A \) is true in a model \( M \), if it comes out as true in, or is satisfied by, every assignment in that model.

With this definition of truth at hand we can easily define the central logical notion of consequence, or logical implication as validity of the corresponding material implication, validity of a formula being defined as its truth in every model:

\[ A \Rightarrow B \iff \Rightarrow 'A \rightarrow B' \]

where

\[ \Rightarrow C \text{ iff for every } M, M \Rightarrow C \text{ and } A, B, C \in F \]


Incidentally, I think I should mention here an alternative, but equivalent possible formulation of the semantic function of predicates, which does not conceive of predicates as denoting sets of n-tuples, but rather as denoting functions from n-tuples to truth-values. Indeed, this is what we can call the Fregean theory of predication, as opposed to the Tarskian theory presented above. We can formulate Frege's original conception as follows: \( R(P)(<u_1, \ldots, u_n>) \in \{T, F\} \), where \( u_1, \ldots, u_n \in W \), and \( T \) and \( F \) are the True and the False, respectively. As can be seen, \( R(P) \) is definable as \( \{<v_1, \ldots, v_n>: R'(P)(<u_1, \ldots, u_n>) = v_T\} \), that is, the Tarskian extension of an n-place predicate is the set of n-tuples for which the Fregean function gives the True as its value. As a matter of fact, we could also define this function somewhat differently, not as one from n-tuples to truth-values, but as one which for an individual gives another function that can take another individual in its argument-place and gives as its value a further function which, again, can take in its argument-place another individual, and so on, up to \( n \) embedded functions, of which only the last one takes the truth-values as its possible values, like this: \( R'(P)(u_1 \ldots, u_n) \in \{T, F\} \). But we shall see the significance of these formal alternatives later, when we shall examine different mediaeval theories of predication.
In order to give a brief illustration of how these definitions are supposed to work let me prove the validity of the following formula:

$$(\forall x)((Fx \& \neg Fx) \to Gx)$$

This formula could be false in a model, only if there were an assignment $f$ in that model such that $f(\forall x)((Fx \& \neg Fx) \to Gx) = 0$. But this is so iff for some $u \in W$, $f[x:u][(Fx \& \neg Fx) \to Gx] = 0$, which is so iff $f[x:u](Fx) = 1$ and $f[x:u](\neg Fx) = 1$ and $f[x:u](Gx) = 0$. However, this could be so only if both $f[x:u](x) = u \in R(F)$ and its denial would hold, which is impossible. So our formula can be false in no model, whence it is true in every model, that is, it is valid. (Note that here, for the sake of simplicity, I have omitted scarce-quotes around the formulae of our object-language, i.e., I have used these formulae autonomously in our metalanguage.) Q.e.d.

By using the analytic tableau method, the proof is even simpler if we take the negation of our formula:

$$(\exists x)(Fx \& \neg Fx \& \neg Gx)$$

$Fa \& \neg Fa \& Ga$

$Ga$

$\neg Fa$

$Fa$

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**MODAL (INTENSIONAL) SEMANTICS**

Modal semantics may best be understood as a generalization of classical, extensional semantics. The basic idea is that the modal operators, 'necessary' and 'possible', may be regarded as a universal and a particular (or existential) quantifier over classical models, respectively. A modal theory, therefore, will have to be an extension of SQT both in its syntax and its semantics. In syntax it will have to incorporate primitive symbols for the modal operators and formation rules governing their use in building up complex expressions; while in semantics it will have to incorporate some machinery interpreting these expressions in line with this intuitive idea.

A modal language can be obtained from the language of SQT above by the following simple additions:

$L := \langle C, P, V, T, F \rangle$

where $C := C \cup \{\Diamond\}$ and $F$ is obtained by appending to the clauses /1/-/4/ defining $F$ above the clause:

/5/ If $A \in F$, then '$\Diamond(A)$' $\in F$

and replacing in the resulting list all occurrences of $F$ by $F$.

The semantics for the resulting language may be obtained as a generalization of the above definition of a model, if we define a modal interpretation $I$, as containing a multitude of classical models in the following manner:

$I := \langle M, W, R \rangle$

where $M$ is a set of classical models ("possible worlds"); $W$ is a function such that $W(M) = W_M$, where $W_M$ is some set, the domain of the model $M \in M$, and $R$ is a function such that $R(M) = R_M$, where $R_M$ is a function assigning semantic values to the parameters of $L$ according to the following clauses:

[1] If $a \in P_M$, then $R_M(a) \in W_M$
If $P^u \in P\text{pred}$, then $R_M(P^u) \subseteq W_M^u$, i.e., it is a subset of the set of ordered n-tuples of elements of $W_M$.

The definition of a value-assignment in I for an $M$, $f_M$, is going to be entirely parallel to the above definition of $f$ in an $M$, except that we shall have a further clause taking care of the evaluation of modal formulae (not surprisingly, one that will do justice to our intuition concerning modal operators quantifying over classical models representing the fanciful idea of "possible worlds"):

1. If $a \in P\text{std}$, then $f_M(a) = R_M(a)$
2. If $v \in V$, then $f_M(v) \in W_M$
3. $f_M('t_1,...,t_n') = 1$ iff $<f_M(t_1),...,f_M(t_n)> \in R_M(P)$
4. $f_M('t_1=t_2') = 1$ iff $f_M(t_1) = f_M(t_2)$
5. If $v \in V$, $A,B \in F$, then $f_M('\neg(A)') = 1$ iff $f_M(A) = 0$, $f_M('A\&B') = 1$ iff $f_M(A) = f_M(B) = 1$ and $f_M('\forall v)(A)') = 1$ iff for every $u \in W_M$, $f_M[v:u](A) = 1$, where $f_M[v:u](w) = u$ if $w = v$ (that is to say, $f_M[v:u](v) = u$), otherwise $f_M[v:u](w) = f_M(w)$, that is to say, $f_M[v:u]$ (read: $f_M$ changed in $v$ to $u$) is the same as $f_M$ except, perhaps, that it assigns $u$ to $v$.
6. If $\Box(A) \in F$, then $f_M(\Box(A)) = 1$ iff for some $M \in M$, $f_M(A) = 1$

Truth in an interpretation I, then, is defined as follows:
$I \Rightarrow A$ iff for every $f_M$ in $I$, $f_M(A) = 1$

Finally, validity is defined as truth in every interpretation:
$\Rightarrow A$ iff for every I, $I \Rightarrow A$

Since something is necessary if and only its contradictory is impossible, the necessity operator $\Box$ can be introduced by a contextual definition as an abbreviation in the following manner:

$'\Box(A)':='\neg(\neg A)'$ (where the analogy with '$\forall x(A)':='\exists x(\neg(A)' should be clear)

Now this basic idea of a modal interpretation, coupled with considerations concerning the notions of existence, actuality, tenses, "rigidity" of reference, essentialism, etc., admits of a huge number of philosophically interesting variations in a delicate interplay between more specific formal techniques and intuitive, informal considerations. When such techniques are used not only to represent modal notions, but all sorts of intensional contexts (such as tensed or intentional contexts), we are in the domain of intensional logic in general, of which modal logic is only a specific, though the "oldest", and perhaps most controversial branch.

In general, an intensional semantics may be characterized as a formal semantic theory in which the semantic values of some complex expressions in an interpretation in a given (classical) model ("possible world", "possible situation", etc.) are determined not only by the semantic values of the components of that expression in the same (classical) model (by their "extensions" or "factual values"—as they are called, for historical reasons), but also by the values of these components in other (classical) models, that is to say, by the function giving their factual values in all the (classical) models of the given interpretation, that is, their intension. For example, the above system of modal logic is an intensional logic in this sense, because $f_M(\Box(A))$, the "factual value" of $\Box(A)$ in $M$ is not determined by $f_M(A)$, i.e., the factual value of $A$ in $M$ alone, but also by the values $A$ gets in other elements of $M$. But this amounts to saying that $f_M(\Box(A))$ is not a function of $f_M(A)$, but of the intension of $A$, a function telling us the factual values of $A$ in every $M$.

So, formally, if we define $\text{INT}(A)$, the intension of $A$, by specifying that $\text{INT}(A)(M) = f_M(A)$, and if we call $\text{INT}(A)$ "satisfied" iff for some $M$, $\text{INT}(A)(M) = 1$, then we can explicitly state the dependence of $f_M(\Box(A))$ on $\text{INT}(A)$ by stating that $f_M(\Box(A)) = 1$ iff $\text{INT}(A)$ is satisfied.
In general, if we formulate the rule of compositionality thus:
Val₁(α(β)) = Val₁(α)(Val₁(β))

and the "extension" or "factual value" of an expression α in an M is denoted as EXTₘ(α), while its intension in an interpretation I, INT(α), is defined so that INT(α)(M) = EXTₘ(α), then we can say that α is an extensional operator on β (and so it provides an extensional context for β), if
EXTₘ(α(β')) = EXTₘ(α)(EXTₘ(β)),
while α is an intensional operator on β (and so it provides an intensional context for β), if
EXTₘ(α(β')) = EXTₘ(α)(INT(β)).

For example, negation, as defined above, is an extensional operator on formulae, because
EXTₘ('¬(A)') = EXTₘ('¬')(EXTₘ(A)), i.e., fₘ('¬(¬(A))') = fₘ('¬')(fₘ(A)),
where fₘ('¬') may be defined so that fₘ('¬(¬(A))') = 1 iff fₘ(A) = 0 (which, of course, amounts to the same as the relevant clause in the definition of fₘ above).

On the other hand, ◊ is an intensional operator on formulae, because
EXTₘ('◊(A)') = EXTₘ('◊')(INT(A)), i.e., fₘ('◊(¬(A))') = fₘ('◊')(INT(A)),
where fₘ('◊') may be defined so that fₘ('◊')(INT(A)) = 1 iff INT(A) is satisfied, i.e., iff for some M, INT(A)(M) = 1, as stated above. This formulation only makes it explicit that what makes a context intensional is that the semantic value of the expression formed with such a context is a function of the intension (rather than the extension) of the argument-expression.