1. Introduction

“The Verlinde formula for $\mathfrak{sl}_2$ counts the number of lattice points in a polytope.”

I first heard this statement from Bernd Sturmfels in a talk at the Conference on Geometric Invariant Theory in Göttingen, June 2008.

A search of the literature reveals that for $\mathfrak{sl}_2$, this is proven in a paper of Rasmussen and Walton [6]. It is also proven for $n = 3$ points for $\mathfrak{sl}_3$ in [2,7] and conjectured for $\mathfrak{sl}_{r+1}$ for all $r \geq 3$ as well in [7].

In Section 2 we reprove this result using ideas of Boris Alexeev. In Section 3 we extend this to intersection numbers of conformal block divisors $\mathcal{D}(\mathfrak{sl}_2, \ell, \vec{\lambda})$ with $\text{F-curves}$ $\text{F}_{I_1, I_2, I_3, I_4}$.

In Section 4 we describe a Sage worksheet and polymake calculation that were used to test Theorem 3.6 for $n = 5$ and $n = 6$.

Acknowledgements. It is a pleasure to thank Boris Alexeev, who first developed boxed Catalan paths, and Valery Alexeev, who first taught me about them. I would also like to thank Paul Larsen for sharing his proof of Theorem 3.6 for $\ell = 1$. Finally, I would like to thank my undergraduate research student Lauren Vogelstein, who helped me program the inequalities needed to check Theorem 3.6 for $n = 6$.

2. Double sequences and ranks of $\mathfrak{sl}_2$ conformal blocks

Let $n \geq 3$. Fix a level $\ell$ and let $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ be a vector of dominant integral weights in the Weyl alcove $P_\ell$.

Definition 2.1. A double sequence of level $\ell$ and shape $\vec{\lambda}$ is a pair of sequences (which we write as a $2 \times n$ matrix)

\[
\begin{pmatrix}
  x_1 & x_2 & \cdots & x_n \\
  y_1 & y_2 & \cdots & y_n
\end{pmatrix}
\]

(1) Each $x_j$ and $y_j$ is an integer between 0 and $\ell$.

(2) For $j = 1, \ldots, n$, $x_j + y_j = \lambda_j$

(3) For each $k$ in $1, \ldots, n - 1$,

\[
x_k + \sum_{j=1}^{k-1} (x_j - y_j) \leq \ell.
\]

(4) For each $k$ in $1, \ldots, n - 1$,

\[
-y_k + \sum_{j=1}^{k-1} (x_j - y_j) \geq 0.
\]
(5) $\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j$.
(6) $y_1 = 0$ and $x_n = 0$.

We will refer to the equations/inequalities (1)-(6) above as the double sequence inequalities, or DS inequalities for short.

There are some redundancies among the inequalities and equations given above. For instance, we don’t need to require $x_j, y_j \leq \ell$ since $x_j, y_j \geq 0$ and $x_j + y_j = \lambda_j \leq \ell$. Also, $y_1 = 0$ follows from $-y_k + \sum_{j=1}^{k-1}(x_j - y_j) \geq 0$ when $k = 1$.

The definition of double sequences above is a rewording of the definition of boxed Catalan paths due to B. Alexeev [1]. In particular, we can draw a picture in the first quadrant of $\mathbb{R}^2$ associated to a double sequence as follows: Begin at $(0, 0)$. For each $j$, draw two paths: one that makes $x_j$ steps $(1, 1)$ up and to the right followed by $y_j$ steps $(1, -1)$ down and to the right, and a second path given by $y_j$ steps $(1, -1)$ down and to the right followed by $x_j$ steps $(1, 1)$ up and to the right. DS inequality (3) says that neither path ever goes above height $\ell$; DS inequality (4) says that neither path ever goes below height $0$; and DS inequality (5) says that the two paths end at $(\Lambda, 0)$, where $\Lambda = \sum_{j=1}^{n} \lambda_j$.

**Proposition 2.2** (B. Alexeev, 2009).

$$\text{rank } \mathcal{V}(\mathfrak{sl}_2, \ell, \vec{\lambda}) = \#\{\text{double sequences of level } \ell \text{ and shape } \vec{\lambda}\}.$$ 

**Proof.** One may check the proposition for $n = 1, 2, 3$ and that the number of double sequences satisfies factorization. 

Since the entries in a double sequence are integers and the inequalities defining double sequences are all linear in the variables $x_j, y_j$, as a corollary of Proposition 2.2, we obtain a new proof of a result due to Rasmussen and Walton [6]:

**Corollary 2.3.** $\text{rank } \mathcal{V}(\mathfrak{sl}_2, \ell, \vec{\lambda})$ counts the number of lattice points in a polytope.

2.1. **Relationship to the Rasmussen-Walton polytopes.** In general, I don’t know whether the polytope obtained via double sequence inequalities is the same as or different from the polytope Rasmussen and Walton describe in [6]. (I haven’t worked through their definition carefully enough yet to say.) However, for $n = 4$, the polytopes are isometric, though not precisely equal (we will see below that we need to permute the weights by (13)(24)).

To see this, we write their system of inequalities (15) in our notation. Their $\lambda_1^{(i)}$ is our $\lambda_i$, their $k$ is our $\ell$, and they write $S = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)$. Then their polytope for $N = 4$ is one-dimensional and given by the inequalities

- $0 \leq g_1$
- $0 \leq \lambda_2 - g_1$
- $0 \leq \lambda_1 - g_1$
- $0 \leq \ell - \lambda_1 - \lambda_2 + g_1$
- $0 \leq S - \lambda_3 - g_1$
- $0 \leq S - \lambda_4 - g_1$
- $0 \leq -S + \lambda_3 + \lambda_4 + g_1$
- $0 \leq \ell - S + g_1$. 

On the other hand, we can consider the double sequence inequalities for \( n = 4 \). We replace \( x_1 = \lambda_1, y_1 = 0, y_2 = \lambda_2 - x_2, y_3 = \lambda_3 - x_3, x_4 = 0, y_4 = \lambda_4 \). Furthermore, by \( x_1 + x_2 + x_3 + x_4 = y_1 + y_2 + y_3 + y_4 = 0 + \lambda_2 - x_2 + \lambda_3 - x_3 + \lambda_4 \), we may replace \( x_2 = S - \lambda_1 - x_3 \) and hence \( y_2 = x_3 + \lambda_1 + \lambda_2 - S \). Thus we see that the polytope defined by the double sequence inequalities is also one-dimensional.

Next we show that the DS inequalities correspond to the Rasmussen-Walton inequalities above. The DS inequality \( 0 \leq x_3 \) corresponds to \( 0 \leq g_1 \) above. The DS inequality \( x_3 \leq \lambda_3 \) corresponds to \( 0 \leq \lambda_1 - g_1 \) above. The DS inequality \( -y_3 + x_1 - y_4 + x_2 - y_2 \geq 0 \) simplifies to \( x_3 \leq \lambda_4 \), which corresponds to \( 0 \leq \lambda_2 - g_1 \) above. The DS inequality \( x_3 + x_1 - y_1 + x_2 - y_2 \leq \ell \) simplifies to \( 0 \leq \ell - \lambda_3 - \lambda_4 + x_3 \), which corresponds to \( 0 \leq \ell - \lambda_1 - \lambda_2 + g_1 \). The DS inequality \( x_2 \geq 0 \) simplifies to \( 0 \leq S - \lambda_1 - x_3 \), which corresponds to \( 0 \leq S - \lambda_3 - g_1 \). The DS inequality \( -y_2 + x_1 - y_1 \geq 0 \) simplifies to \( 0 \leq S - \lambda_2 - x_3 \), which corresponds to \( 0 \leq S - \lambda_4 - g_1 \). The DS inequality \( y_2 \geq 0 \) may be written \( 0 \leq -S + \lambda_1 + \lambda_2 + x_3 \), which corresponds to \( 0 \leq -S + \lambda_3 + \lambda_4 + g_1 \). Finally, the DS inequality \( -y_2 + x_1 - y_1 \geq 0 \) simplifies to \( 0 \leq \ell - S + x_3 \), which corresponds to \( 0 \leq \ell - S + g_1 \) above.

### 2.2. Relationship to work by Feigin et al.

The definition of double sequences is highly reminiscent of the combinatorial paths defined in [4, Section 3]. However, it seems that their numbers \( d_{k,l}^N \) are not the same as the conformal block ranks \( r(\vec{\lambda}) \) considered here. Recall that their numbers \( d_{k,l}^N \) are defined as follows: the representations of \( \widehat{\mathfrak{s}l}_2 \) of level \( k \) form a ring with addition given by direct sum and multiplication given by the fusion product. The ring thus has as basis the representations the irreducible representations \( \pi_i \) of highest weight \( i \) for \( i = 0, \ldots, k \). Then \( d_{k,l}^N \) is defined by

\[
(\pi_0 + \cdots + \pi_k)^N = \sum_{0 \leq i \leq k} d_{k,l}^N \pi_i.
\]

In this notation, it seems that \( r(\vec{\lambda}) \) would be given by

\[
\pi_{\lambda_1} \pi_{\lambda_2} \cdots \pi_{\lambda_n} = r(\vec{\lambda}) \pi_0 + \sum_{1 \leq i \leq k} c_i \pi_i.
\]

Thus the paper [4] seems to address a different question about \( \widehat{\mathfrak{s}l}_2 \) representations than we do.

### 3. Intersection Numbers of \( \mathfrak{s}l_2 \) Conformal Blocks

In this section we show that the intersection number of a conformal block divisor \( \mathbb{D}(\mathfrak{s}l_2, \ell, \vec{\lambda}) \) with an F-curve \( F_{I_1,I_2,I_3,I_4} \) counts the number of lattice points in a polytope.

**Definition 3.1.** Let \( \{1, \ldots, n\} = I_1 \coprod I_2 \coprod I_3 \coprod I_4 \) be a partition into four nonempty subsets. Let \( \vec{\lambda} = (\lambda_1, \ldots, \lambda_n), \vec{\mu} = (\mu_1, \ldots, \mu_4) \). A 5-tuple of level \( \ell \) and shape \( (\vec{\lambda}, \vec{\mu}, I_1, I_2, I_3, I_4) \) is a 5-tuple of double sequences as shown below. We write \( I_1 = \{i_1, \ldots, i_a\}, I_2 = \{i_{a+1}, \ldots, i_b\}, \ldots \),
Lemma 3.2. Let 
Then there is a bijection
Here $x_{n+i} + y_{n+i} = \mu_i$ for $i = 1, 2, 3, 4$ and $x_{n+i} + y_{n+i} = \mu_{i-4}$ for $i = 5, 6, 7, 8$.

Lemma 3.6. Let \(\{1, \ldots, n\} = I_1 \amalg I_2 \amalg I_3 \amalg I_4\) be a partition into four nonempty subsets. Then there is a bijection
\[
\{\text{double sequences of level } \ell \text{ and shape } \vec{\lambda}\} \\
\leftrightarrow \{\text{5-tuples of double sequences of shape } (\vec{\lambda}, \vec{\mu}, I_1, I_2, I_3, I_4) : \vec{\mu} \in P^4_\ell\}.
\]

Proof. This bijection is a set-theoretic interpretation of the following factorization calculation, which is valid for any \(g\). In the calculation below we write \(r(S)\) for \(\text{rank } \mathcal{V}(g\mathfrak{sl}_2, \ell, (\lambda_i : i \in S))\), and we omit the brackets around singletons, writing for instance \(a\) for \(\{a\}\).
\[
r(\lambda_1, \ldots, \lambda_n) \\
= \sum_{\mu_5 \in P_\ell} r(I_1 \cup I_2 \cup \mu_5) r(I_3 \cup I_4 \cup \mu_5^*) \\
= \sum_{(\mu_1, \mu_5, \mu_5^*) \in P^3_\ell} r(I_1 \cup \mu_1) r(I_2 \cup \mu_5 \cup \mu_1^*) r(I_3 \cup \mu_3) r(I_4 \cup \mu_5^* \cup \mu_3^*) \\
= \sum_{(\mu_1, \ldots, \mu_5) \in P^5_\ell} r(I_1 \cup \mu_1) r(I_2 \cup \mu_2) r(\mu_5, \mu_1^*, \mu_2^*) r(I_3 \cup \mu_3) r(I_4 \cup \mu_4) r(\mu_5^*, \mu_3^*, \mu_4^*) \\
= \sum_{(\mu_1, \ldots, \mu_4) \in P^4_\ell} r(\mu_1^*, \ldots, \mu_4^*) r(I_1 \cup \mu_1) r(I_2 \cup \mu_2) r(I_3 \cup \mu_3) r(I_4 \cup \mu_4),
\]
where to obtain the last line we used
\[
\sum_{\mu_5 \in P_\ell} r(\mu_5, \mu_1^*, \mu_2^*) r(\mu_5^*, \mu_3^*, \mu_4^*) = r(\mu_1^*, \ldots, \mu_4^*).
\]

We also use another result due to Boris Alexeev, which appeared in [9]:

Lemma 3.4 ([1]). Let \(g = g\mathfrak{sl}_2\), let \(n = 4\), and let \(\vec{\mu} = (\mu_1, \mu_2, \mu_3, \mu_4)\). Then
\[
\deg \mathcal{V}(g\mathfrak{sl}_2, \ell, \vec{\mu}) = \text{rank } \mathcal{V}(g\mathfrak{sl}_2, \ell, \vec{\mu}) \cdot \max \left\{0, \frac{1}{2} (\mu_1 + \mu_2 + \mu_3 + \mu_4 - 2\ell) \right\}.
\]

Theorem 3.6. The intersection number \(D(g\mathfrak{sl}_2, \ell, \vec{\lambda}) \cdot F_{I_1, I_2, I_3, I_4}\) is the number of lattice points in a polytope. Specifically, it counts 6-tuples of the form: (5-tuple of level \(\ell\) and shape \((\vec{\lambda}, \vec{\mu}, I_1, I_2, I_3, I_4), k)\) where \(1 \leq k \leq \frac{1}{2} (\mu_1 + \mu_2 + \mu_3 + \mu_4 - 2\ell)\).
Proof. For any simple Lie algebra \( \mathfrak{g} \), Fakhruddin gives the following formula for intersection numbers of conformal block divisors with \( F \)-curves (see [3, Proposition 2.5]):

\[
D(\mathfrak{g}, \ell, \vec{\lambda}) \cdot F_{1,2,3,4} = \sum_{\vec{\mu} \in \mathbb{P}_\ell^4} \deg \mathbb{V}_{\vec{\mu}} r_{\lambda_1 \mu_1} r_{\lambda_2 \mu_2} r_{\lambda_3 \mu_3} r_{\lambda_4 \mu_4}.
\]

Let \( M(\vec{\mu}) = \max \{0, \frac{1}{2} (\mu_1 + \mu_2 + \mu_3 + \mu_4 - 2\ell)\} \). Then (3.5) says that \( \deg \mathbb{V}_{\vec{\mu}} = M(\vec{\mu})r(\mu_1, \ldots, \mu_4) \), and we substitute this in to obtain:

\[
\sum_{\vec{\mu} \in \mathbb{P}_\ell^4} M(\vec{\mu})r(\mu_1, \ldots, \mu_4)r(I_1 \cup \mu_1)r(I_2 \cup \mu_2)r(I_3 \cup \mu_3)r(I_4 \cup \mu_4).
\]

Using the set-theoretic interpretation coming from Lemma 3.2, we see that we need to count each 5-tuple of level \( \ell \) and shape \( (\vec{\lambda}, \vec{\mu}, I_1, I_2, I_3, I_4) \) \( M(\vec{\mu}) \) times. This is accomplished by adding one additional coordinate \( k \) and having it run from 1 to \( M(\vec{\mu}) = \frac{1}{2}(\mu_1 + \mu_2 + \mu_3 + \mu_4 - 2\ell) \), if \( M(\vec{\mu}) \geq 1 \).

3.1. Interpretation via fibered polytopes. Here is a geometric interpretation of the preceding calculations.

Let \( \mathcal{P}(\vec{\lambda}) \) be the polytope defined by the DS inequalities. Then \( r(\vec{\lambda}) \) is the number of lattice points in \( \mathcal{P}(\vec{\lambda}) \).

The factorization calculation

\[
r(\lambda_1, \ldots, \lambda_n) = \sum_{(\mu_1, \ldots, \mu_4) \in \mathbb{P}_\ell^4} r(\mu_1^*, \ldots, \mu_4^*)r(I_1 \cup \mu_1)r(I_2 \cup \mu_2)r(I_3 \cup \mu_3)r(I_4 \cup \mu_4)
\]

allows us to define a map from lattice points of \( \mathcal{P}(\vec{\lambda}) \) to \( \mathbb{P}_\ell^4 \); the map sends lattice point in the original polytope to the unique \( \vec{\mu} \) by which it appears on the right hand side. Thus, \( \mathcal{P}(\vec{\lambda}) \) fibers over \( \mathbb{P}_\ell^4 \).

(In fact, the bijection of Lemma 3.2 contains a little more information than this: each lattice point in \( \mathcal{P}(\vec{\lambda}) \) is assigned a lattice point in \( \mathcal{P}(\vec{\mu}) \).)

Then we may view the piecewise linear function \( M(\vec{\mu}) \) as a height function over \( \mathbb{P}_\ell^4 \). Thus, in the proof of Theorem 3.6, instead of multiplying the volume of a fiber by \( M \) as the degree formula says to do, I introduced an extra coordinate \( k \) and use it to count the volume of the fiber \( M \) times instead.

4. Software

4.1. \( n = 5 \) in Sage. I created a Sage worksheet that computes the lattice points counted by \( \mathbb{D}(\mathfrak{sl}_2, \ell, (\lambda_1, \ldots, \lambda_5)) \cdot F_{1,2,3,4,5} \). It is available at

\texttt{http://sage.ace.fordham.edu/home/pub/28}

The user can enter the level \( \ell \) and weights \( \lambda_1, \ldots, \lambda_5 \) near the top of the cell, hit “evaluate”, and get a 15 tuple as the output, which records \((x_1, y_1, x_2, y_2, x_{11}, y_{11}, x_{12}, y_{12}, x_{13}, y_{13}, x_6, y_6, x_{10}, y_{10}, \mu_1)\).

(I apologize for the funny ordering. I wrote the Sage function before I wrote the paper.)

Note that Definition 3.1 calls for \( x_1, y_1, \ldots, x_{13}, y_{13} \). However, we can simplify this a little; by the fusion rules, we know that there is a nonzero contribution to the intersection number only if \( \mu_2 = \lambda_3, \mu_3 = \lambda_4, \mu_4 = \lambda_5 \). Hence we omit the variables \( x_3, \ldots, y_5 \) and \( x_7, \ldots, y_9 \) that otherwise would have been needed to compute these ranks.
4.2. $n = 6$ in polymake. I also tried to create a Sage worksheet to computes the lattice points counted by $\mathbb{D}(\mathfrak{sl}_2, \ell, (\lambda_1, \ldots, \lambda_6)) \cdot F_{123,4,5,6}$, but unfortunately, I got an error message ''increase POLY_Dmax!'' when I tried to run it. I therefore switched to polymake.

I have posted an input file for a polymake session which was used to compute $\mathbb{D}(\mathfrak{sl}_2, \ell, (\lambda_1, \ldots, \lambda_6)) \cdot F_{123,4,5,6}$ and the session transcript on my UGA website. Input file URL:

http://math.uga.edu/~davids/n6polymake.txt

Transcript URL:

http://math.uga.edu/~davids/n6polymaketranscript19.txt

References

[1] Boris Alexeev, Ranks and degrees of $\mathfrak{sl}_2$ conformal blocks (2009). Private communication. ← 2, 4


Software Packages Referenced


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