The $S_n$ action on $\overline{M}_{0,n}$

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Joint work with Ian Morrison (Fordham University)
The moduli space $\overline{M}_{0,n}$

$$M_{0,n} := \left\{ (P^1, P_1, \ldots, P_n) \mid P_1, \ldots, P_n \in P^1 \text{ distinct} \right\} / \text{PGL}(2)$$

$M_{0,n}$ is not compact.

Deligne-Mumford-Knudsen compactification: add stable curves

$$\overline{M}_{0,n} := \left\{ (C, P_1, \ldots, P_n) \mid \begin{array}{c} \text{C connected projective genus 0 curve} \\ \text{only singularities are nodes} \\ P_1, \ldots, P_n \in C \text{ distinct smooth points} \end{array} \right\} / \cong$$
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C \text{ connected projective genus 0 curve} \\
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\end{cases} 
\end{align*}
The boundary of $\overline{M}_{0,n}$

Let $\Delta := \overline{M}_{0,n} \setminus M_{0,n}$. We call $\Delta$ the boundary divisor.

Points of $\Delta$ correspond to nodal curves.

Irreducible components of $\Delta$ correspond to partitions of $\{1, \ldots, n\}$ into two subsets $I$ and $J$ of size at least 2.

Classes of the boundary components $\delta_{I,J} := [\Delta_{I,J}]$ generate the cohomology ring, and the formula for their intersection is combinatorial.
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Irreducible components of $\Delta \iff$ two subsets $I$ and $J$ of size at least 2

Irreducible components of $\Delta = \{ \{ P_{i_1}, P_{i_2}, \ldots, P_{i_k} \}, \{ P_{j_1}, P_{j_2}, \ldots, P_{j_{n-k}} \} \}$

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\[ \Delta_{I,J} = \left\{ \ldots, P_{i_k}, P_{j_1}, P_{j_2}, \ldots, P_{j_{n-k}} \right\} \]

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State of the art

- Presentations of $H^*(\overline{M}_{0,n}, \mathbb{Z})$ are known (Keel, Tavakol)
  - Seek fast algorithms for certain calculations
- The birational geometry of $\overline{M}_{0,n}$ is still open and known to be combinatorially complicated
  \[
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The $S_n$ action on $\text{Pic}(\overline{M}_{0,n})$

The symmetric group $S_n$ acts on $\overline{M}_{0,n}$ by permuting the marked points.

Get an action on the Chow ring $A^*(\overline{M}_{0,n})$ and $\text{Pic}(\overline{M}_{0,n})$.

Problem Describe the $S_n$-module structure of $\text{Pic}(\overline{M}_{0,n})$. 
Notation

1. $\text{Bdy}_n := \bigoplus_I \mathbb{C} \delta_I$
   - sum over all subsets $I \subset \{1, \ldots, n\}$ such that $2 \leq |I| \leq \lfloor \frac{n}{2} \rfloor$, and if $|I| = \frac{n}{2}$, we insist $1 \in I$.

2. $R_n$ : minimal set of relations (Keel-Rulla).

3. Abbreviate $P_n := \text{Pic}(\overline{M}_{0,n}) \otimes \mathbb{C}$.

Then

$$P_n = \text{Bdy}_n / R_n$$

We’ll describe the $S_n$-module structure of $\text{Bdy}_n$ and $R_n$. 
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We’ll describe the \( S_n \)-module structure of \( Bdy_n \) and \( R_n \).
Representations of $S_n$

Irreducible representations of $S_n$ over $\mathbb{C}$ are indexed by partitions $\lambda = (\lambda_1, \ldots, \lambda_t)$ of the integer $n$.

We assume $\lambda_1 \geq \cdots \geq \lambda_t$. It is standard to associate Young diagrams to these partitions.

We write $V_\lambda$ for the irreducible $S_n$-module associated to the partition $\lambda$. 
Proposition

1. The irreducible $S_n$-modules $V_\lambda$ appearing in $\text{Bdy}_n$ have partitions $\lambda = (\lambda_1, \lambda_2)$ with at most two parts (or, what is the same, Young diagrams with at most two rows). Therefore, the same is true of $R_n$ and $P_n$.

2. The module of relations $R_n$ is irreducible, and $R_n \cong V_{(n-2,2)}$.

3. $P_n \cong \bigoplus_{k=0}^{\lfloor n/2 \rfloor} p_k V_{(n-k,k)}$, where

$$p_k = \begin{cases} 
\lfloor \frac{n+2}{2} \rfloor - 2 & \text{if } k = 0 \\
\lfloor \frac{n-1}{2} \rfloor - 1 & \text{if } k = 1 \\
\lfloor \frac{n-2}{2} \rfloor - 1 & \text{if } k = 2 \\
\lfloor \frac{n-2k+1}{2} \rfloor & \text{if } k \text{ odd, } k \geq 3 \\
\lfloor \frac{n-2k+2}{2} \rfloor & \text{if } k \text{ even, } k \geq 4 
\end{cases}$$
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Fix $n$. For $2 \leq k \leq \lfloor \frac{n}{2} \rfloor$ define a submodule $B_k \subset \text{Bdy}_n$ as follows:

$$B_k := \bigoplus_{|I|=k} \mathbb{C} \delta_I.$$ 

Then $\text{Bdy}_n = B_2 \oplus \cdots \oplus B_{\lfloor \frac{n}{2} \rfloor}$. 

The modules $B_k$
Example: $n = 6$

\[ B_2 \subset \text{Bdy}_6 \]

\[ V_{(6,0)} \quad \delta_{12} + \delta_{13} + \delta_{14} + \cdots + \delta_{46} + \delta_{56} \]

\[ \delta_{13} + \delta_{14} + \delta_{15} + \delta_{16} - \delta_{23} - \delta_{24} - \delta_{25} - \delta_{26} \]
\[ \delta_{12} + \delta_{14} + \delta_{15} + \delta_{16} - \delta_{34} - \delta_{35} - \delta_{36} \]

\[ V_{(5,1)} \quad \delta_{12} + \delta_{13} + \delta_{15} + \delta_{16} - \delta_{24} - \delta_{34} - \delta_{45} - \delta_{46} \]
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\[ \delta_{12} + \delta_{34} - \delta_{13} - \delta_{24} \]
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\[ V_{(4,2)} \quad \delta_{12} + \delta_{46} - \delta_{14} - \delta_{26} \]
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Example: $n = 6$, continued

\[ B_e \subset Bdy_6 \]

\[ V_{(6,0)} = \delta_{123} + \delta_{124} + \delta_{125} + \delta_{126} + \delta_{134} + \delta_{135} + \delta_{136} + \delta_{145} + \delta_{146} + \delta_{156} \]

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Lemma

1. If $k < \frac{n}{2}$ then

$$B_k = V_{(n,0)} \oplus V_{(n-1,1)} \oplus \cdots \oplus V_{(n-k,k)}.$$ 

2. If $k = \frac{n}{2}$, then

$$B_{\frac{n}{2}} = \bigoplus_{h=0, h \text{ even}}^{\frac{n}{2}} V_{(n-h,h)}.$$ 

Morrison-Swinarski (Jan. 2011): proof by character theory.

Then we found this in the literature (G.D. James, 1978).
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Who else cares?

The cosets of the action of $S_{n-k} \times S_k$ on $S_n$ may be identified with partitions of \{1, \ldots, n\} into two parts of sizes $n - k$ and $k$. Thus $B_k = 1^{S_n}_{S_{n-k} \times S_k}$, the induced module from this subgroup.

Representation theorists studied $1^{S_n}_{S_\lambda}$ where $\lambda$ is any partition of $n$ and $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_p}$.

- Useful for constructing *Specht modules* (irreducible $S_n$-modules)
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▶ Useful for constructing Specht modules (irreducible $S_n$-modules)
The module of relations

\[ 0 \rightarrow R_n \rightarrow Bdy_n \rightarrow \text{Pic}(\overline{M}_{0,n}) \rightarrow 0 \]

Know

\[
\begin{align*}
\dim Bdy_n &= 2^{n-1} - n - 1 \\
\dim \text{Pic}(\overline{M}_{0,n}) &= 2^{n-1} - \binom{n}{2} - 1 \\
\Rightarrow \dim R_n &= \frac{n(n-3)}{2}
\end{align*}
\]

Can show that the only dimension \( \frac{n(n-3)}{2} \) submodule of \( Bdy_n \) is an irreducible \( V_{(n-2,2)} \).
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Can show that the only dimension $\frac{n(n-3)}{2}$ submodule of $Bdy_n$ is an irreducible $V_{(n-2,2)}$. 
Bases of the irreducible blocks

Definition
Let $T$ be a standard tableau of shape $(n - h, h)$. For $i = 0, \ldots, h$, let $\tau_{T,i} = (T(1, i), T(2, i))$ be the transposition switching the two entries of $T$ in column $i$. We write

$$S_n(T) := \langle \tau_{T,i} \mid i = 1, \ldots, h \rangle$$

for the subgroup of $S_n$ generated by the transpositions $\tau_{T,i}$. 
Proposition

Suppose given \( n, k, h \) with \( 2 \leq k < \frac{n}{2} \) and \( 0 \leq h \leq k \).

To each standard tableau \( T \) of shape \( (n - h, h) \), we associate the following class:

\[
D_T = \sum_{\sigma \in S_n(T)} \sum_{|I| = k} \text{sgn}(\sigma) \delta_I
\]

\( l \ni \sigma(T(1,i)), i = 1, \ldots, h \)

\( l \not\ni \sigma(T(2,i)), i = 1, \ldots, h \)

Then a basis of the irreducible block \( V_{(n-h,h)} \subset B_k \subset Bdy_n \) is given by the following set:

\[ \{ D_T \mid T \text{ is a standard tableau of shape } (n - h, h) \} . \]
Example: $n = 6$

$$B_2 \subset \text{Bdy}_6$$

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How this fits what we know

Let $B_k := \sum_{|I|=k} \delta_I$.

Then $B_2, \ldots, B_{\lfloor n/2 \rfloor}$ form a basis of $\text{Pic}(\overline{M}_{0,n})^{S_n}$.

There is exactly one standard tableau $T$ of shape $n$, given by the numbers $1, \ldots, n$ in increasing order.

The group $S_n(T) = \langle \text{id} \rangle$.

Then

$$D_T = \sum_{\sigma \in S_n(T)} \sum_{|I|=k} \sigma(T(1,i)), i=1,\ldots,h \quad l \ni \sigma(T(1,i)), i=1,\ldots,h$$

$$= \sum_{|I|=k} \sum_{|I|=k} \text{sgn}(\sigma) \delta_I$$

$$= \sum_{l \ni \sigma(T(1,i)), i=1,\ldots,h} \sum_{l \ni \sigma(T(2,i)), i=1,\ldots,h} \delta_I.$$
How this fits what we know

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Then

$$D_T = \sum_{\sigma \in S_n(T)} \sum_{|I|=k} \text{sgn}(\sigma) \delta_I = \sum_{|I|=k} \delta_I.$$
The standard tableaux of shape \((n - 1, 1)\) are
\[
\begin{array}{ccc}
1 & 2 & i \\
i & & n
\end{array}
\]
for \(i = 2, \ldots, n\).

Thus \(V_{(n-1,1)} \subset B_k\) is spanned by the divisors
\[
\sum_{|I|=k} \delta_I - \sum_{1 \in I} \delta_{I} - \sum_{1 \notin I} \delta_{I}.
\]

We can relate this to the \(\psi\)-classes on \(\overline{M}_{0,n}\).
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We can relate this to the $\psi$-classes on $\overline{M}_{0,n}$. 
$V_{(n-1,1)}$ and the $\psi$ classes, cont.

$\psi_1, \ldots, \psi_n$ form a basis of the defining representation of $S_n$.

Decomposes as a trivial representation spanned by $\sum_{i=1}^{n} \psi_i$ and a $V_{(n-1,1)}$ spanned by $\psi_1 - \psi_i$ for $i = 2, \ldots, n$

$$
\psi_1 - \psi_i = \sum_{k=2}^{n-2} \frac{(n-k)(n-k-1)}{(n-1)(n-2)} \left( \sum_{|I|=k, I \ni 1} \delta_I - \sum_{|I|=k, I \ni i} \delta_I \right)
$$

$$
= \sum_{k=2}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n-2k}{n-2} \left( \sum_{|I|=k, I \ni 1} \delta_I - \sum_{|I|=k, I \ni i} \delta_I \right)
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Decomposes as a trivial representation spanned by $\sum_{i=1}^n \psi_i$ and a $V_{(n-1,1)}$ spanned by $\psi_1 - \psi_i$ for $i = 2, \ldots, n$

$$\psi_1 - \psi_i = \sum_{k=2}^{n-2} \frac{(n-k)(n-k-1)}{(n-1)(n-2)} \left( \sum_{|I|=k} \delta_I - \sum_{|I|=k, I \ni i} \delta_I \right)$$

$$= \sum_{k=2}^{\lfloor n/2 \rfloor} \frac{n-2k}{n-2} \left( \sum_{|I|=k} \delta_I - \sum_{|I|=k, I \ni i} \delta_I \right)$$
$V_{(n-2,2)}$ and the Rulla relations

The Rulla relations are

$$\sum_{1,2\in I, \ i,j\notin I} D_i \equiv \sum_{1,i\in I, \ 2,j\notin I} D_i \quad \text{and} \quad \sum_{1,3\in I, \ 1,p\notin I} D_i \equiv \sum_{1,p\in I, \ 2,3\notin I} D_i$$

where $3 \leq i < j \leq n$ and $4 \leq p \leq n$.

The standard tableaux of shape $(n-2,2)$ are

$$\begin{array}{cccc}
1 & 2 & i & j \\
\text{i} & \text{j}
\end{array}$$

for $3 \leq i < j \leq n$, and

$$\begin{array}{cccc}
1 & 3 & p & n \\
2 & p
\end{array}$$

for $4 \leq p \leq n$.

We can relate the tableaux basis to the “degree k” terms in the Rulla relations.
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Future work

- Compute $H^*(\overline{M}_{0,n}/S_n)$ and develop algorithms for efficient calculations in this ring.
- Which nef divisors are big on $\overline{M}_{0,n}$? Develop algorithms for efficiently calculating top self-intersections in $H^*(\overline{M}_{0,n})$.
- For $n = 6$, Nef($\overline{M}_{0,6}$) has 3190 extremal rays which lie in 28 $S_6$ orbit classes. For $n = 7$, computing the nef cone is intractable, but can we describe the $S_7$ orbits of the rays?
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F-curves

Fix four genus pointed curves $C_1, \ldots, C_4$.

Consider nodal curves which consist of these four tails glued to $P^1$ at four points.

**Definition** An F-curve is the locus $\overline{M}_{0,n}$ corresponding to the curves in this family.

The nodes partition the set $\{1, \ldots, n\}$ into four nonempty subsets:

$$P_C = I_1 \sqcup I_2 \sqcup I_3 \sqcup I_4$$

The homology class of the F-curve depends only on this partition.

F-curves span $H_2(\overline{M}_{0,n}, \mathbb{Q})$. 
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Writing divisors in the tableaux basis

We present a tantalizing example for $V_{(5,1)} \subset B_2$ when $n = 6$:

**Proposition**

Let

$$S = (12, 13, 14, \ldots, 56)$$

be the set of subsets of size 2 of \{1, \ldots, 6\} in lexicographic order.

The projection matrix for $B_2 \to V_{(5,1)}$ is given by intersection numbers. The entry in row $i$, column $j$ is

$$C_{S_i} \cdot \delta_{S_j}$$

where if $S_i = \{a, b\}$,

$$C_{S_i} := \sum_{c \not\in \{a, b\}} F_{a, b, c, \{a, b, c\}} - \sum_{c \not\in \{a, b\}} F_{\{a, b, c\}, d, e, f}.$$